

On a Generalization of Fueter's Theorem

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Abstract. In this paper we discuss a generalization of Fueter's theorem which states that whenever $f(x_0, \underline{x})$ is holomorphic in $x_0 + \underline{x}$, then it satisfies $D \square f = 0$, $D = \partial_{x_0} + i\partial_{x_1} + j\partial_{x_2} + k\partial_{x_3}$ being the Fueter operator.

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1. Introduction

In the quaternion setting we consider the paravector $x_0 + \underline{x}$ whereby the vector $\underline{x} = x_1i + x_2j + x_3k$ may be written in spherical coordinates as $\underline{x} = \underline{w}r$ with $r = |\underline{x}|$ and $\underline{w}^2 = -1$. This means that for fixed \underline{w} the variable $x_0 + \underline{w}r$ behaves like the complex variable $z = x + iy$, making the identification $x \rightarrow x_0$, $y \rightarrow r$, $i \rightarrow \underline{w}$. Hence for any complex holomorphic function $f(z)$ one may consider the function $f(x_0 + \underline{w}r)$ for any \underline{w} fixed and hence give a well defined meaning to $f(x_0 + \underline{x})$ as a function in 4-space. In [1], it was established that the function $\square f = (\partial_{x_0}^2 + \sum \partial_{x_j}^2)f(x_0, \underline{x})$ is left monogenic, i.e. $D \square f = 0$, $D = \partial_{x_0} + i\partial_{x_1} + j\partial_{x_2} + k\partial_{x_3}$ being the Fueter operator.

In [4] the generalization of this result to the even-dimensional Clifford algebra setting was obtained. Hereby one replaces the quaternion paravector by $x_0 + \underline{x}$ whereby $\underline{x} = \sum e_j x_j$ and e_1, \dots, e_m satisfy the generating relations $e_i e_j + e_j e_i = -2\delta_{ij}$. The result says that for m odd and $D = \partial_{x_0} + \partial_{\underline{x}}$, $\partial_{\underline{x}} = \sum e_j \partial_{x_j}$ one may for any holomorphic function $f(z)$ define the function $f(x) = f(x_0 + \underline{x})$, and it turns out that with $\square = \partial_{x_0}^2 - \partial_{\underline{x}}^2$ the function $g(x) = \square^{\frac{m-1}{2}} f(x_0 + \underline{x})$ is left monogenic, i.e. $(\partial_{x_0} + \partial_{\underline{x}})g(x) = 0$.

Later on in [7] the result was extended to the case m even by considering a suitable pseudodifferential operator giving a meaning to $\square^{\frac{m-1}{2}}$. Recently there is new interest in this simple but very elegant result which leads to the explicit construction of various special functions. We hereby refer to [2, 6].

In this paper we are going to construct another generalization of Fueter's theorem in which we are going to use axially monogenic functions. It is namely such that monogenic functions of the form

$$A(x_0, r) + \underline{w} B(x_0, r) \quad (r = |\underline{x}|, r\underline{w} = \underline{x})$$

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satisfy the Vekua system (see also [3])

$$\left. \begin{array}{l} \partial_{x_0} A - \partial_r B = \frac{m-1}{r} B \\ \partial_{x_0} B + \partial_r A = 0 \end{array} \right\}.$$

Later on in [5] we constructed generalized axial monogenics of the form

$$(A(x_0, r) + \underline{w} B(x_0, r)) P_k(\underline{w}),$$

whereb $P_k(\underline{w})$ is spherical monogenic of degree k on S^{m-1} , i.e.

$$\partial_{\underline{x}} P_k(\underline{x}) = \partial_{\underline{x}} (r^k P_k(\underline{w})) = 0.$$

The condition of left monogenicity may then be expressed by the Vekua system

$$\left. \begin{array}{l} \partial_{x_0} A - \partial_r B = \frac{k+m-1}{r} B \\ \partial_{x_0} B + \partial_r A = \frac{k}{r} A \end{array} \right\}.$$

In this paper we establish the following

Theorem 1. *Let m be odd and $f(x_0 + \underline{x})$ be a holomorphic function of the paravector $x_0 + \underline{x}$. Then the function*

$$\square^{k+\frac{m-1}{2}} [f(x_0 + \underline{x}) P_k(\underline{x})]$$

is left monogenic, whereby $P_k(\underline{x})$ is spherical monogenic of degree k .

2. Proof of the theorem

First we write the holomorphic function $f(z)$ in the form $f(z) = a(x, y) + i b(x, y)$ whereby a, b are real-valued. Then it is clear that

$$f(x_0 + \underline{x}) = a(x_0, r) + \underline{w} b(x_0, r).$$

Next, it is easy to see that

$$\square^{k+\frac{m-1}{2}} [f(x_0 + \underline{x}) P_k(\underline{x})]$$

has the form

$$(A(x_0, r) + \underline{w} B(x_0, r)) P_k(\underline{w}).$$

Although the computations are far from trivial one may verify that A and B satisfy the Vekua system for axial monogenics of degree k , from which the theorem follows.

We also expect the result to hold in the case m even.

3. A more constructive proof

The proof in the previous section may be applied on examples, but in general it is kind of short. But a more direct argument will clarify things.

First, for any scalar function $h(x_0, r)$,

$$\partial_{\underline{x}}[h(x_0, r)P_k(\underline{x})] = \underline{w} \partial_r h P_k(x),$$

Moreover, locally every holomorphic function $f(x+iy)$ may be written as $(\partial_x - i\partial_y)h(x, y)$ for some scalar harmonic function $h(x, y)$, whence one also has that in this region

$$(\partial_{x_0} - \partial_{\underline{x}})[h(x_0, r)P_k(\underline{x})] = f(x_0 + \underline{x})P_k(\underline{x}).$$

The theorem will hence be true if one can prove that

$$[(\partial_{x_0} + \partial_{\underline{x}})(\partial_{x_0} - \partial_{\underline{x}})]^{k+\frac{m+1}{2}} \{h(x_0, r)P_k(\underline{x})\} = 0.$$

To that end, let us write

$$\square^s \{h(x_0, r)P_k(x)\} = h_s(x_0, r)P_k(\underline{x}).$$

That this is indeed possible follows by induction from the fact that, for any scalar function $g(x_0, r)$,

$$\partial_{\underline{x}}(g(x_0, r)P_k(\underline{x})) = \underline{w} \partial_r g P_k(\underline{x})$$

and

$$\begin{aligned} \partial_{\underline{x}}(\underline{w} g(x_0, r)P_k(\underline{x})) &= \partial_{\underline{x}}\left(g(x_0, r)r^{2k+m-1}\left\{\frac{\underline{x}}{|\underline{x}|^m}P_k\left(\frac{\underline{x}}{r^2}\right)\right\}\right) \\ &= -\left(\partial_r g + \frac{2k+m-1}{r}g\right)P_k(\underline{x}) \end{aligned}$$

from which it follows that

$$\begin{aligned} \square \{g(x_0, r)P_k(\underline{x})\} &= (\partial_{x_0} - \partial_{\underline{x}})(\partial_{x_0} + \partial_{\underline{x}})\{g(x_0, r)P_k(\underline{x})\} \\ &= (\partial_{x_0} - \partial_{\underline{x}})\{\partial_{x_0}g P_k(\underline{x}) + \underline{w} \partial_r g P_k(\underline{x})\} \\ &= \partial_{x_0}^2 g P_k(\underline{x}) - \partial_{\underline{x}}\{\partial_r g \underline{w} P_k(\underline{x})\} \\ &= \{\partial_{x_0}^2 g + \partial_r^2 g + \frac{2k+m-1}{r}\partial_r g\}P_k(\underline{x}). \end{aligned}$$

Hence the Laplace operator in $m+1$ dimensions acting on $g(x_0, r)P_k(\underline{x})$ is formally the same as the Laplace operator in $2k+m+1$ dimensions acting on the axial function $g(x_0, r)$ and this multiplied with $P_k(\underline{x})$. This rest is similar to the proof of [4] (compare also with the lemma in [2]); it follows that $h_{(k+(m+1)/2)} = 0$. Indeed, assuming that $h(x_0, r)$ is harmonic we obtain

$$\begin{aligned} h_1(x_0, r)P_k(\underline{x}) &= \square(h(x_0, r)P_k(\underline{x})) \\ &= (2k+m-1)\left(\frac{1}{r}\partial_r\right)\{h(x_0, r)\}P_k(\underline{x}) \\ h_2(x_0, r)P_k(\underline{x}) &= \square(h_1(x_0, r)P_k(\underline{x})) \\ &= (2k+m-1)(2k+m-3)\left(\frac{1}{r}\partial_r\right)^2\{h(x_0, r)\}P_k(\underline{x}) \end{aligned}$$

and so on. After $k+\frac{m+1}{2}$ iterations one obtains a factor in front of the form

$$(2k+m-1)(2k+m-3)\cdots(2k+m+1-2\left(k+\frac{m+1}{2}\right)) = 0.$$

References

- [1] Fueter, R.: *Die Funktionentheorie der Differentialgleichungen $\Delta u = 0$ und $\Delta\Delta u = 0$ mit vier reellen Variablen*. Comm. Math. Helv. 7 (1935), 307 – 330.
- [2] Laville, G. and I. Ramadanoff: *Holomorphic Cliffordian functions*. Adv. Appl. Clifford Alg. 8 (1998), 323 – 340.
- [3] Lounesto, P. and P. Bergh: *Axially symmetric vector fields and their complex potentials*. Compl. Variables: Theory & Appl. 2 (1983), 139 – 150.
- [4] Sce, M.: *Osservazioni sulle serie di potenze nei moduli quadratici*. Atti Acc. Lincei Rend. Fis. (Serie 8) 23 (1957), 220 – 225.
- [5] Sommen, F.: *Plane elliptic systems and monogenic functions in symmetric domains*. Suppl. Rend. Circ. Mat. Palermo (2) 6 (1984), 259 – 269.
- [6] Sproessig, W.: *On operators and elementary functions in Clifford analysis*. Z. Anal. Anw. 18 (1999), 349 – 360.
- [7] Qian, T.: *Generalization of Fueter's result to \mathbf{R}^{n+1}* . Rend. Mat. Acc. Lincei (Serie 9) 8 (1997), 111 – 117.

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