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Coerciveness Property for a Class of Non-Smooth Functionals

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Abstract. The paper establishes a general coerciveness property for a class of non-smooth functionals satisfying an appropriate Palais-Smale condition. This result is obtained by applying an abstract principle supplying qualitative information concerning the asymptotic behaviour of a non-smooth functional. Comparison with other results in this field is provided.

Keywords: *Coerciveness, Palais-Smale condition, variational principle*

AMS subject classification: 58E30, 49J52, 49J40

1. Introduction

An extensive work has been devoted in the setting of differentiable functionals to show the basic property that the Palais-Smale condition implies the coerciveness (see, e.g., [1, 2, 7] and the references therein). The aim of this paper is to establish that this assertion is essentially true for a large class of non-differentiable functionals, too.

The non-smooth functions for which we study this problem are those that can be written as a sum $\Phi + \Psi$ of a locally Lipschitz functional Φ and a proper, convex, lower semicontinuous functional Ψ (see relation (3.1) below). For a detailed study of this class of non-smooth functionals from the point of view of critical point theory we refer to Motreanu and Panagiotopoulos [8: Chapter 3].

Towards our purpose we use a suitable Palais-Smale condition for this class of non-smooth functionals that reduces to the usual concepts in the differentiable situations as well as in all the important non-smooth cases (see Chang [3] and Szulkin [9]). This new formulation for the Palais-Smale condition in our non-smooth setting (see Definition 2.3) can be seen as a unification of the Palais-Smale conditions due to Chang [3] and Szulkin [9] (see Definitions 2.1 and 2.2). The essential tools in our approach are the calculus with generalized gradients developed by Clarke [4] and Ekeland's variational principle [5, 6].

Our coerciveness results stated in Corollaries 3.1 - 3.3 extend the corresponding properties from the differentiable case (see [1, 2, 7]) to the non-smooth framework of functionals of type (3.1) (for a detailed discussion see Remark 3.2). These results are deduced from a general principle, namely Theorem 3.1, involving the asymptotic behaviour of the respective non-smooth functionals. This result extends Proposition 1

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in Brézis and Nirenberg [1] to the general class of functionals of form (3.1). Specifically, our non-smooth coerciveness results are obtained by applying the general principle in Theorem 3.1 in conjunction with the non-smooth version of Palais-Smale condition formulated for the class of non-smooth functionals satisfying the structure hypothesis (3.1).

The rest of the paper is organized as follows. Section 2 deals with three types of Palais-Smale conditions for non-smooth functionals and their relationship. Section 3 contains the statements of the main results and the proofs of our coerciveness properties. Section 4 presents the proof of our main abstract result.

2. Palais-Smale conditions

Throughout the paper X denotes a real Banach space endowed with the norm $\|\cdot\|$. The notation X^* stands for the dual space of X. For the sake of clarity we recall the definition of the generalized directional derivative $\Phi^\circ(u;v)$ of a locally Lipschitz functional $\Phi : X \to \mathrm{R}$ at the point $u \in X$ in the direction $v \in X$:

$$\Phi^\circ(u;v) = \limsup_{\substack{w \to u \\ t \downarrow 0}} \frac{1}{t}\big(\Phi(w+tv) - \Phi(w)\big) \tag{2.1}$$

(see Clarke [4]). We recall three basic definitions of Palais-Smale conditions for non-smooth functionals.

Definition 2.1 (Chang [3]). The locally Lipschitz functional $\Phi : X \to \mathrm{R}$ satisfies the *Palais-Smale condition* (in the *sense of Chang*) if every sequence $(u_n) \subset X$ with $\Phi(u_n)$ bounded and for which there exists a sequence

$$z_n \to 0 \qquad \text{in } X^*,\ z_n \in \partial\Phi(u_n) \tag{2.2}$$

has a (strongly) convergent subsequence in X.

The notation $\partial\Phi$ in (2.2) means the generalized gradient of the locally Lipschitz functional Φ (in the sense of Clarke [4]), that is

$$\partial\Phi(u) = \Big\{x_* \in X^* : \langle x_*, v\rangle \le \Phi^\circ(u;v) \text{ for all } v \in X\Big\} \qquad (u \in X) \tag{2.3}$$

where Φ° is defined in (2.1).

Definition 2.2 (Szulkin [9]). Let $\Phi : X \to \mathrm{R}$ be a differentiable functional of class C^1 and let $\Psi : X \to \mathrm{R} \cup \{+\infty\}$ be a proper (i.e. $\not\equiv +\infty$) convex and lower semicontinuous function. The functional $I = \Phi + \Psi : X \to \mathrm{R} \cup \{+\infty\}$ satisfies the *Palais-Smale condition* (in the *sense of Szulkin*) if every sequence $(u_n) \subset X$ with $I(u_n)$ bounded and for which there exists a sequence $(\varepsilon_n) \subset \mathrm{R}^+$ with $\varepsilon_n \downarrow 0$ such that

$$\Phi'(u_n)(v-u_n) + \Psi(v) - \Psi(u_n) \ge -\varepsilon_n\|v-u_n\| \qquad (v \in X) \tag{2.4}$$

contains a (strongly) convergent subsequence in X.

Definition 2.3 (Motreanu and Panagiotopoulos [8]). Let $\Phi : X \to \mathrm{R}$ be a locally Lipschitz functional and let $\Psi : X \to \mathrm{R}\cup\{+\infty\}$ be a proper, convex and lower semicontinuous function. The functional $I = \Phi + \Psi : X \to \mathrm{R}\cup\{+\infty\}$ satisfies the *Palais-Smale condition* (in the *sense of Motreanu and Panagiotopoulos*) if every sequence $(u_n) \subset X$ with $I(u_n)$ bounded and for which there exists a sequence $(\varepsilon_n) \subset \mathrm{R}^+$ with $\varepsilon_n \downarrow 0$ such that

$$\Phi^\circ(u_n; v - u_n) + \Psi(v) - \Psi(u_n) \geq -\varepsilon_n \|v - u_n\| \qquad (v \in X) \tag{2.5}$$

contains a (strongly) convergent subsequence in X.

In order to establish a relationship between the foregoing definitions, we need the following result.

Lemma 2.1 (Szulkin [9]). *Let X be a real Banach space and let $\chi : X \to \mathrm{R}\cup\{+\infty\}$ be a lower semicontinuous convex function with $\chi(0) = 0$. If $\chi(x) \geq -\|x\|$ for all $x \in X$, then there exists some $z \in X^*$ such that $\|z\|_{X^*} \leq 1$ and $\chi(x) \geq \langle z, x\rangle$ for all $x \in X$.*

The result below points out a relationship between Definitions 2.1 - 2.3.

Proposition 2.1.

(i) *If $\Psi = 0$, Definition 2.3 reduces to Definition 2.1.*

(ii) *If $\Phi \in C^1(X, \mathrm{R})$, Definition 2.3 coincides with Definition 2.2.*

Proof. (i) Let $\Psi = 0$ in Definition 2.3. It is sufficient to show the equivalence between relations (2.2) and (2.5). Suppose that property (2.2) holds. By relation (2.3) it follows that

$$\Phi^\circ(u_n; v) \geq \langle z_n, v\rangle \geq -\|z_n\| \, \|v\| \qquad (v \in X).$$

Therefore inequality (2.5) (with $\Psi = 0$) is verified for $\varepsilon_n = \|z_n\|$.

Conversely, we admit that (2.5) is satisfied. We apply Lemma 2.1 to $\chi = \frac{1}{\varepsilon_n}\Phi^\circ(u_n; \cdot)$. Since χ is continuous, convex and (2.5) is satisfied (with $\Psi = 0$), the assumptions of Lemma 2.1 are verified. Lemma 2.1 yields an element $w_n \in X^*$ with $\|w_n\|_{X^*} \leq 1$ and $\frac{1}{\varepsilon_n}\Phi^\circ(u_n; x) \geq \langle w_n, x\rangle$ for all $x \in X$. Choosing $z_n = \varepsilon_n w_n$ we arrive at (2.2).

(ii) This assertion follows from the fact that Φ° is equal to the Fréchet differential Φ' if the functional $\Phi : X \to \mathrm{R}$ is of class C^1. Therefore, in this case inequalities (2.4) and (2.5) coincide. The proof of Proposition 2.1 is complete ∎

3. Main results

Our main result is stated below.

Theorem 3.1. *Let $\Phi : X \to \mathrm{R}$ be a locally Lipschitz functional and let $\Psi : X \to \mathrm{R}\cup\{+\infty\}$ be a proper, convex, lower semicontinuous function. For the function*

$$I = \Phi + \Psi \tag{3.1}$$

we suppose that

$$\alpha := \liminf_{\|v\|\to\infty} I(v) \in \mathrm{R}. \tag{3.2}$$

Then for every sequence $(\varepsilon_n) \subset \mathrm{R}^+$ *with* $\varepsilon_n \downarrow 0$ *there exists a sequence* $(u_n) \subset X$ *satisfying*

$$\|u_n\| \to \infty \qquad as\ n \to \infty \tag{3.3}$$

$$I(u_n) \to \alpha \qquad as\ n \to \infty \tag{3.4}$$

and (2.5)*.*

The proof of Theorem 3.1 is given in Section 4.

Corollary 3.1. *Assume that the functional* $I : X \to \mathrm{R} \cup \{+\infty\}$ *satisfies the structure hypothesis* (3.1)*, with* Φ *and* Ψ *as in the statement of Theorem 3.1, together with*

$$\alpha > -\infty \tag{3.5}$$

where α *is defined in* (3.2)*, and*

$$I \text{ verifies the Palais-Smale condition of Definition } 2.3. \tag{3.6}$$

Then I *is coercive on* X*, i.e.*

$$I(u) \to +\infty \qquad as\ \|u\| \to \infty. \tag{3.7}$$

Proof. Arguing by contradiction we admit that the functional I in (3.1) is not coercive. Since (3.7) does not hold there exists a sequence $(v_n) \subset X$ satisfying $\|v_n\| \to \infty$ and

$$\alpha \le \liminf_{n\to\infty} I(v_n) < +\infty. \tag{3.8}$$

From (3.5) and (3.8) one obtains that $\alpha = \liminf_{\|v\|\to\infty} I(v) \in \mathrm{R}$. Consequently, we may apply Theorem 3.1 to the functional $I : X \to \mathrm{R} \cup \{+\infty\}$ for a fixed sequence $(\varepsilon_n) \subset \mathrm{R}^+$ with $\varepsilon_n \downarrow 0$. In this way a sequence $(u_n) \subset X$ is found fulfilling properties (3.3), (3.4) and (2.5). According to assumption (3.6) it results that (u_n) possesses a convergent subsequence denoted again by (u_n), say $u_n \to u$ as $n \to \infty$, for some $u \in X$. This contradicts assertion (3.3), which accomplishes the proof ∎

Corollary 3.2. *Let* $\Phi : X \to \mathrm{R}$ *be a locally Lipschitz functional which satisfies the Palais-Smale condition of Definition* 2.1 *and* $\liminf_{\|v\|\to\infty} \Phi(v) > -\infty$*. Then* Φ *is coercive on* X*, i.e.* $\Phi(u) \to +\infty$ *as* $\|u\| \to \infty$*.*

Proof. Let us apply Corollary 3.1 with $\Psi = 0$. Then condition (3.5) with $\Psi = 0$ is satisfied (for α introduced in (3.2)). By part (i) in Proposition 2.1 requirement (3.6) is satisfied for $I = \Phi$. Then Corollary 3.1 leads to the desired result ∎

Corollary 3.3. *Let* $\Phi : X \to \mathrm{R}$ *be a function of class* C^1 *and let* $\Psi : X \to \mathrm{R} \cup \{+\infty\}$ *be a proper, convex, lower semicontinuous function. Assume that the functional* $I = \Phi + \Psi : X \to \mathrm{R} \cup \{+\infty\}$ *satisfies the Palais-Smale condition in the sense of Definition* 2.2 *and fulfils also* (3.5) *where* α *is introduced in* (3.2)*. Then* I *is coercive on* X*.*

Proof. Let us apply Corollary 3.1 for $I = \Phi + \Psi : X \to \mathrm{R} \cup \{+\infty\}$, with Φ and Ψ as in Corollary 3.3. Since we supposed that property (3.5) holds, it remains to check (3.6). This follows from Proposition 2.1/(ii). The proof is thus complete ∎

Remark 3.1. If $\Phi \in C^1(X, \mathrm{R})$ and $\Psi = 0$ in (3.1), Theorem 3.1 reduces to Proposition 1 of Brézis and Nirenberg [1].

Remark 3.2. The case in (3.1) where Φ is Gâteaux differentiable and lower semicontinuous has been studied in Caklovic, Li and Willem [2] (with $\Psi = 0$) and in Goeleven [7]. Our Corollary 3.1 provides, in particular, non-differentiable versions of these results. Precisely, Corollary 3.1 covers the non-differentiable situation where, in (3.1), $\Phi : X \to \mathrm{R}$ is locally Lipschitz and $\Psi : X \to \mathrm{R} \cup \{+\infty\}$ is proper, convex and lower semicontinuous. Therefore Corollary 3.1 deals with different situations with respect to [2] and [7]. Corollary 3.2 treats the purely locally Lipschitz case, i.e. $\Psi = 0$ in (3.1). It extends Corollary 1 in [1] and allows to extend the main result in [2] to locally Lipschitz functionals. It overlaps with the main result in [2] if $\Phi \in C^1(X, \mathrm{R})$ and Φ is bounded from below. Corollary 3.3 represents the version of Corollary 3.1 in the case where $\Phi \in C^1(X, \mathrm{R})$. Under the assumption that $\Phi \in C^1(X, \mathrm{R})$ is bounded from below, Corollary 3.3 has been obtained in [7].

Remark 3.3. Corollaries 3.1 - 3.3 correspond to the three concepts of Palais-Smale conditions in Definitions 2.3, 2.1 and 2.2, respectively.

4. Proof of Theorem 3.1

The proof of Theorem 3.1 relies on the following version of Ekeland's Variational Principle.

Theorem 4.1 (Ekeland [5,6]). *Let M be a complete metric space endowed with distance d and let $f : M \to \mathrm{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and bounded from below function. Then for every number $\varepsilon > 0$ and every point $x_0 \in M$ there exists $v_0 \in M$ such that*

$$f(v_0) \le f(x_0) - \varepsilon d(v_0, x_0) \tag{4.1}$$

$$f(x) > f(v_0) - \varepsilon d(v_0, x) \quad (x \in M \setminus \{v_0\}). \tag{4.2}$$

Proof of Theorem 3.1. Suggested by the argument in the proof of Proposition 1 in [1], for each $r > 0$, we denote

$$m(r) = \inf_{\|u\| \ge r} I(u). \tag{4.3}$$

Assumption (3.2) in conjunction with (4.3) leads to

$$\alpha = \lim_{r \to \infty} m(r) \in \mathrm{R}. \tag{4.4}$$

Assersion (4.4) ensures that for each $\varepsilon > 0$ there exists $r_\varepsilon > 0$ satisfying

$$\alpha - \varepsilon^2 \le m(r) \qquad \forall\, r \ge r_\varepsilon. \tag{4.5}$$

For any fixed $\varepsilon > 0$ let us choose a number $\overline{r}_\varepsilon$ with

$$\overline{r}_\varepsilon \ge \max\{r_\varepsilon, 2\varepsilon\}. \tag{4.6}$$

Using assumption (3.2), we can fix some $u_0 = u_0(\varepsilon) \in X$ such that

$$\|u_0\| \geq 2\overline{r}_\varepsilon \qquad \text{and} \qquad I(u_0) < \alpha + \varepsilon^2. \tag{4.7}$$

The set $M = M(\varepsilon) \subset X$ given by

$$M = \{x \in X : \|x\| \geq \overline{r}_\varepsilon\} \tag{4.8}$$

is a closed subset of X, so M is a complete metric space with respect to the metric induced on M by the norm $\|\cdot\|$. The function $I : X \to \mathrm{R} \cup \{+\infty\}$ expressed in (3.1) is lower semicontinuous on X, thus on M. By (4.3), (4.5) and (4.6) we derive that

$$I(u) \geq m(\|u\|) \geq \alpha - \varepsilon^2 \qquad \forall\, u \in X \text{ with } \|u\| \geq \overline{r}_\varepsilon. \tag{4.9}$$

This estimate ensures that the function I is bounded from below on M. From (4.8) and the first inequality in (4.7) it is seen that $u_0 \in M$. Hence by the second relation in (4.7) we know that the function I is proper on M. Since all the assumptions of Theorem 4.1 are fulfilled for the functional $f = I\big|_M : M \to \mathrm{R} \cup \{+\infty\}$, it is allowed to apply Theorem 4.1, where the fixed number $\varepsilon > 0$ and the point $x_0 = u_0$ are the data entering relations (4.5) - (4.7). Consequently, we find some $v_\varepsilon \in M$ such that

$$I(v_\varepsilon) \leq I(u_0) - \varepsilon\|v_\varepsilon - u_0\| \tag{4.10}$$

$$I(x) > I(v_\varepsilon) - \varepsilon\|v_\varepsilon - x\| \;\forall\, x \neq v_\varepsilon \text{ with } \|x\| \geq \overline{r}_\varepsilon \tag{4.11}$$

(see (4.1) and (4.2)).

Since $v_\varepsilon \in M$, using relations (4.5), (4.6), (4.8), (4.3), (4.10) and the second inequality in (4.7), we have

$$\alpha - \varepsilon^2 \leq m(\overline{r}_\varepsilon) \leq I(v_\varepsilon) \leq I(u_0) - \varepsilon\|v_\varepsilon - u_0\| < \alpha + \varepsilon^2 - \varepsilon\|v_\varepsilon - u_0\|.$$

This implies that

$$\|v_\varepsilon - u_0\| < 2\varepsilon. \tag{4.12}$$

Combining (4.12), the first inequality in (4.7) and (4.6) we deduce that

$$\|v_\varepsilon\| \geq \|u_0\| - \|v_\varepsilon - u_0\| > 2\overline{r}_\varepsilon - 2\varepsilon \geq \overline{r}_\varepsilon. \tag{4.13}$$

From here it is clear that v_ε is an interior point of M defined in (4.8). This guaranties that for an arbitrary $v \in X$ with $v \neq v_\varepsilon$ it is true that $x = v_\varepsilon + t(v - v_\varepsilon)$ belongs to the interior of M in (4.8) whenever $t > 0$ is sufficiently small. It is thus permitted to use such a point x above in (4.11). By means of (3.1) and (4.11) we can write

$$\Phi(v_\varepsilon + t(v - v_\varepsilon)) + \Psi(v_\varepsilon + t(v - v_\varepsilon)) > \Phi(v_\varepsilon) + \Psi(v_\varepsilon) - \varepsilon t\|v - v_\varepsilon\| \tag{4.14}$$

for all $v \in X \setminus \{v_\varepsilon\}$ and all $t > 0$ sufficiently small. On the other hand, we observe from inequality (4.10) and the second relation in (4.7) that $\Psi(v_\varepsilon) < +\infty$. On the basis of the convexity of $\Psi : X \to \mathrm{R} \cup \{+\infty\}$, inequality (4.14) yields

$$\Phi(v_\varepsilon + t(v - v_\varepsilon)) - t\Psi(v_\varepsilon) + t\Psi(v) > \Phi(v_\varepsilon) - \varepsilon t\|v - v_\varepsilon\|$$

for all $v \in X \setminus \{v_\varepsilon\}$ and all $t > 0$ small enough. Passing to the limit one obtains that

$$\limsup_{t \downarrow 0} \frac{1}{t}(\Phi(v_\varepsilon + t(v - v_\varepsilon)) - \Phi(v_\varepsilon)) + \Psi(v) - \Psi(v_\varepsilon) \geq -\varepsilon \|v - v_\varepsilon\|$$

for all $v \in X \setminus \{v_\varepsilon\}$. Taking into account formula (2.1) we deduce that

$$\Phi^\circ(v_\varepsilon; v - v_\varepsilon) + \Psi(v) - \Psi(v_\varepsilon) \geq -\varepsilon \|v - v_\varepsilon\| \tag{4.15}$$

for all $v \in X \setminus \{v_\varepsilon\}$. Consider now a sequence $(\varepsilon_n) \subset \mathrm{R}^+$ with $\varepsilon_n \downarrow 0$. Corresponding to it we may choose a sequence of positive numbers $r_{\varepsilon_n} \to +\infty$ as $n \to \infty$ satisfying (4.5) with $\varepsilon = \varepsilon_n$. We denote $u_n = v_{\varepsilon_n}$ where we recall that $v_{\varepsilon_n} \in M = M(\varepsilon_n)$ is the point satisfying (4.15) with $\varepsilon = \varepsilon_n$, i.e., property (2.5) holds true. Since $\|u_n\| \geq \overline{r}_{\varepsilon_n} \geq r_{\varepsilon_n}$ (cf. (4.8) and (4.6)), we obtain that property (3.3) is satisfied. In order to check relation (3.4) we notice that (4.10) and the second inequality in (4.7) imply

$$I(u_n) \leq I(u_0) - \varepsilon_n \|u_n - u_0\| \leq I(u_0) < \alpha + \varepsilon_n^2.$$

This combined with (3.3) and (3.2) expresses that

$$\alpha \leq \liminf_{n \to \infty} I(u_n) \leq \limsup_{n \to \infty} I(u_n) \leq \alpha$$

which establishes (3.4). The proof of Theorem 3.1 is complete ∎

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