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On a Simple System of Discrete Two-Scale Difference Equations

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Abstract. A special system of two discrete two-scale difference equations with polynomial solutions is investigated. For the solutions, addition and subtraction theorems are established showing in particular the behaviour of the solutions for a great argument, as well as further relations and inequalities. Also, corresponding generating functions are constructed which imply explicit representations for the solutions.

Keywords: *Discrete two-scale difference equations, polynomials, addition theorems, generating functions, Collatz graph, Fibonacci numbers*

AMS subject classification: 39A10, 39A12, 26C05, 30C10

1. Introduction

In this paper we consider the special system

$$\left.\begin{aligned} Z_{2k} &= p\,Z_k \\ Z_{2k+1} &= q\,Z_k + r\,Z_{k+1} \end{aligned}\right\} \qquad (k \in \mathrm{N}) \tag{1.1}$$

of two discrete two-scale difference equations under the initial condition

$$Z_1 = 1\,. \tag{1.2}$$

The coefficients are assumed to be non-vanishing complex numbers, and the solution is obviously a polynomial $Z_n = Z_n(p,q,r)$ of the coefficients. In a forthcoming paper [3] the solution of system (1.1) shall be used for an explicit representation of solutions of continuous two-scale difference equations at dyadic points. Such equations appear in wavelet theory and subdivision schemes, cf. [4, 7]. The special case $S_n = Z_n(q+1,q,1)$ was already considered in [2] in connection with de Rham's singular function. After replacement $Z_{n+1} = x_n$, the second equation of system (1.1) with $q = \frac{1}{c}$ and $r = -\frac{1}{c}$ for $c > 0$ appeared also in [1, 5], however in another context and without its first equation.

It is very simple to calculate the first polynomials Z_n (cf. Table 1) as well as $Z_{2^\ell} = p^\ell$ for $\ell \in \mathrm{N}_0$, but our aim is to analyze the general structure of Z_n which becomes visible in addition and subtraction theorems. We establish further relations and calculate infinite series. For $p = q = r = 1$ some Z_n are the Fibonacci numbers which here have an

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extremal property. Moreover, we construct generating functions of Z_n and of related polynomials and derive different explicit representations for Z_n.

Table 1: The first polynomials $Z_n = Z_n(p,q,r)$

In Table 1 it is conspicuous that the non-vanishing coefficients of the polynomials Z_n are all equal to 1. However, this is not a general property as the example

$$Z_{21} = pr^2\big(pq + pr + qr\big) + \big(p^2q + 2pqr + pr^2 + qr^2\big)q$$

shows. It is possible to use the second equation of system (1.1) also for $k = 0$ and to introduce

$$Z_0 = \frac{1-r}{q}\,. \tag{1.3}$$

However, only in the case $p = 1$ or $r = 1$, i.e.

$$(p-1)(r-1) = 0 \tag{1.4}$$

the first equation of system (1.1) is compatible with value (1.3) so that we shall use (1.3) only in these two cases.

2. Addition theorems

We begin with the construction of addition theorems, i.e. of formulas for Z_n where n is a certain sum of two terms.

Proposition 2.1. *Under conditions* (1.2) *and $p \neq r$ the solution of system* (1.1) *with initial condition* (1.2) *has the structure*

$$Z_{2^\ell k+j} = \tfrac{q}{p-r}\big(p^\ell X_j - r^\ell Y_j\big)Z_k + r^\ell\, Y_j Z_{k+1} \tag{2.1}$$

for $0 \le j \le 2^\ell$ $\;(k \in \mathrm{N};\ j, \ell \in \mathrm{N}_0)$ *where* $X_j = Z_j\big(1, \frac{q}{p}, \frac{r}{p}\big)$ *and* $Y_j = Z_j\big(\frac{p}{r}, \frac{q}{r}, 1\big)$.

Proof. In the case $\ell = 0$ equation (2.1) is satisfied for $j \in \{0,1\}$ in view of $X_0 = \frac{p-r}{q}$, $Y_0 = 0$ and $X_1 = Y_1 = 1$. Hence for $\ell = 0$ and $0 \le j \le 1$ the polynomials Z_n have the structure

$$Z_{2^\ell k+j} = X_{\ell j} Z_k + Y_{\ell j} Z_{k+1}\,. \tag{2.2}$$

We assume that (2.2) is satisfied for a fixed ℓ and $0 \le j \le 2^\ell$. Replacing k by $2k$ und using (1.1) we obtain

$$Z_{2^{\ell+1}k+j} = (pX_{\ell j} + qY_{\ell j})Z_k + rY_{\ell j}Z_{k+1}$$

and therefore

$$\left.\begin{aligned} X_{\ell+1,j} &= pX_{\ell j} + qY_{\ell j} \\ Y_{\ell+1,j} &= rY_{\ell j} \end{aligned}\right\}. \tag{2.3}$$

Analogously, replacing k in (2.2) by $2k+1$ we find

$$\left.\begin{aligned} X_{\ell+1,2^\ell+j} &= qX_{\ell j} \\ Y_{\ell+1,2^\ell+j} &= rX_{\ell j} + pY_{\ell j} \end{aligned}\right\}, \tag{2.4}$$

both equations for $0 \le j \le 2^\ell$. This shows that (2.2) is satisfied for $\ell+1$ instead of ℓ and for $0 \le j \le 2^{\ell+1}$. Hence by induction (2.2) is proved for all $\ell \in \mathrm{N}_0$.

Equations (2.3) have the general solutions

$$\left.\begin{aligned} X_{\ell j} &= \tfrac{q}{p-r}\big(p^\ell X_j - r^\ell Y_j\big) \\ Y_{\ell j} &= r^\ell Y_j \end{aligned}\right\} \tag{2.5}$$

for every fixed j and $j \le 2^\ell$ so that (2.2) implies (2.1). Replacing j in (2.1) by $2j$ and using $Z_{2^\ell k+2j} = pZ_{2^{\ell-1}k+j}$ for $\ell \ge 1$ we obtain by comparison of coefficients

$$\begin{aligned} X_{2j} &= X_j \\ Y_{2j} &= \tfrac{p}{r}Y_j. \end{aligned}$$

Analogously, replacing j in (2.1) by $2j+1$ and using $Z_{2^\ell k+2j+1} = qZ_{2^{\ell-1}k+j} + rZ_{2^{\ell-1}k+j+1}$ we obtain

$$\begin{aligned} X_{2j+1} &= \tfrac{q}{p}X_j + rpX_{j+1} \\ Y_{2j+1} &= \tfrac{q}{r}Y_j + Y_{j+1}. \end{aligned}$$

In view of the initial conditions the proposition is proved ∎

Remark 2.2.

1. In the case (1.4) at most two of the sequences X_n, Y_n, Z_n are different since $X_n = Z_n$ for $p = 1$ and $Y_n = Z_n$ for $r = 1$. Obviously, $X_n = Y_n = Z_n$ for $p = r = 1$.

2. For $k = 1$ equation (2.1) specializes to

$$Z_{2^\ell+j} = \tfrac{q}{p-r}p^\ell X_j + \big(p - \tfrac{q}{p-r}\big)r^\ell Y_j \tag{2.6}$$

with $0 \le j \le 2^\ell$ ($\ell \in \mathrm{N}_0$). In view of $Y_j = Z_j\big(\frac{p}{r}, \frac{q}{r}, 1\big)$ and $X_j = Z_j\big(1, \frac{q}{p}, \frac{r}{p}\big)$ equation (2.6) immediately implies

$$Y_{2^\ell+j} = \tfrac{q}{p-r}\big(\tfrac{p}{r}\big)^\ell X_j + \big(\tfrac{p}{r} - \tfrac{q}{p-r}\big)Y_j \tag{2.7}$$

$$X_{2^\ell+j} = \tfrac{q}{p-r}X_j + \big(1 - \tfrac{q}{p-r}\big)\big(\tfrac{r}{p}\big)^\ell Y_j \tag{2.8}$$

and the three equations (2.6) - (2.8) can be used to calculate Z_n for $n = 2^{\gamma_k}+...+2^{\gamma_1}+2^{\gamma_0}$ with integers $\gamma_k > ... > \gamma_1 > \gamma_0 \geq 0$. In Section 6 we shall come back to this question in a special case. Equations (2.5), (2.7) - (2.8) can also be used to check equations (2.4).

3. Eliminating X_j and Y_j out of (2.6) - (2.8) we obtain the relation

$$Z_n = \tfrac{1}{p-r}\big((p-1)r^{\ell+1}Y_n - (r-1)p^{\ell+1}X_n\big) \tag{2.9}$$

for $2^\ell \leq n \leq 2^{\ell+1}$ $(\ell \in \mathrm{N}_0)$.

The excluded case $p = r$ in Proposition 2.1 can be treated in an analogous way or by means of the limit process $r \to p$. For convenience, we consider the case $p \to r$ and write afterwards once more p instead of r. The appearing derivatives with respect to p shall be labelled by means of a dash.

Proposition 2.3. *For $p = r$ the solution of system* (1.1) *with initial condition* (1.2) *has the structure*

$$Z_{2^\ell+j} = p^{\ell-1}\left[\left(\ell q + p^2\right)Y_j - qw_j\right] \tag{2.10}$$

where

$$w_j = Z_j'(p, xp, p)|_{p=1} \tag{2.11}$$

satisfies

$$w_{2^\ell+j}(x) = \left(x\ell^2 + \ell + 1\right)Y_j - \ell x w_j(x) \tag{2.12}$$

for $0 \leq j \leq 2^\ell$ $(j, \ell \in \mathrm{N}_0)$, $Y_j = Z_j(1, x, 1)$, $x = \frac{q}{p}$, and $w_0 = -\frac{1}{x}$, $w_1 = 0$.

Proof. Since Z_n is a polynomial in p, q, r it is differentiable and so are X_n and Y_n in view of $p \neq 0$ and $r \neq 0$. Equation (2.6) can be written in the form

$$Z_{2^\ell+j} = \Big(pr^\ell + q\frac{p^\ell - r^\ell}{p-r}\Big)Y_j - p^\ell q\frac{Y_j - X_j}{p-r}\,.$$

For $p \to r$ both $X_j = Z_j\big(1, \frac{q}{p}, \frac{r}{p}\big)$ and $Y_j = Z_j\big(\frac{p}{r}, \frac{q}{r}, 1\big)$ converge to $Z_j(1, \frac{q}{p}, 1)$ and, by means of de l'Hospital's rule (which is also applicable to holomorphic functions), we obtain

$$Z_{2^\ell+j} = p^{\ell-1}(p^2 + \ell q)Y_j - p^\ell q(Y_j' - X_j')$$

and therefore (2.10) with

$$w_j = p\big[Z_j'\big(\tfrac{p}{r}, \tfrac{q}{r}, 1\big) - Z_j'\big(1, \tfrac{q}{p}, \tfrac{r}{p}\big)\big]\big|_{r=p}\,. \tag{2.13}$$

Obviously, w_j depends on $x = \frac{q}{p}$ alone and it can be represented as (2.11). In particular, (2.11) yields the initial values of w_j for $j = 0$ and $j = 1$. Substituting $q = px$ in (2.10) we obtain

$$Z_{2^\ell+j} = p^\ell\left[(\ell x + p)Y_j - xw_j\right]$$

and by differentiation with respect to p, choosing $p = 1$ and considering (2.11), we also have proved (2.12) ∎

Remark 2.4. More generally, it follows from (2.1) for $p \to r$

$$Z_{2^\ell k+j} = p^{\ell-1} q(\ell Y_j - w_j) Z_k + p^\ell Y_j Z_{k+1} \tag{2.14}$$

and in view of (2.11)

$$w_{2^\ell k+j} = x(\ell^2 Y_j - \ell w_j) Y_k + \ell Y_j Y_{k+1} + x(\ell Y_j - w_j) w_k + Y_j w_{k+1} \tag{2.15}$$

for $0 \le j \le 2^\ell$ $(k \in \mathrm{N}, \ell \in \mathrm{N}_0)$, $Y_j = Z_j(1, x, 1)$. For $j = 0$ this implies $w_{2^\ell k} = \ell Y_k + w_k$ and for $\ell \ge 0$ in particular $w_{2^\ell} = \ell$. Moreover, for $\ell = 1$ and $j = 0$ resp. $j = 1$ we easily see:

Corollary 2.5. *The polynomials* w_j $(j \in \mathrm{N})$ *are uniquely determined by the initial value* $w_1 = 0$ *and the system*

$$\left.\begin{aligned} w_{2j} &= w_j + Y_j \\ w_{2j+1} &= x w_j + w_{j+1} + Y_{2j+1} \end{aligned}\right\}, \tag{2.16}$$

$Y_j = Z_j(1, x, 1)$, *which is the inhomogeneous counterpart to the homogeneous system* (1.1) *with* $p = r = 1$ *and* $q = x$.

By elimination of Y_j in (2.16), using $Y_{2j+1} = xY_j + Y_{j+1}$, we obtain the further relation

$$w_{2j+1} = x w_{2j} + w_{2j+2} \tag{2.17}$$

which is also satisfied by Y_n instead of w_n, and from (2.10) and (2.12) with $p = 1$ and $q = x$ we get

$$w_{2^\ell + j} = \ell\, Y_{2^\ell + j} + Y_j, \tag{2.18}$$

$Y_j = Z_j(1, x, 1)$. All these relations can be checked for the first indices by means of Table 2.

Table 2: The first polynomials $Y_n = Z_n(1, x, 1)$ and $w_n(x)$

3. Subtraction theorems

There exist analogous formulas for negative j, i.e. corresponding subtraction theorems.

Proposition 3.1. *In the case $p \neq q$ the solution of system* (1.1) *with initial condition* (1.2) *has the property*

$$Z_{2^\ell - j} = \tfrac{r}{p-q} p^\ell U_j + \big(\tfrac{1-r}{q} - \tfrac{r}{p-q}\big) q^\ell V_j \tag{3.1}$$

for $0 \leq j \leq 2^\ell - 1$ ($\ell \in \mathrm{N}_0$) where $U_j = Z_j\big(1, \frac{r}{p}, \frac{q}{p}\big)$ and $V_j = Z_j\big(\frac{p}{q}, \frac{r}{q}, 1\big)$.

The proof can easily be carried out inductively using the initial values $Z_{2^\ell} = p^\ell$, $U_0 = \frac{p-q}{r}$, $V_0 = 0$ and the recursions

$$\left.\begin{aligned} U_{2j} &= U_j \\ U_{2j+1} &= \tfrac{r}{p} U_j + \tfrac{q}{p} U_{j+1} \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} V_{2j} &= \tfrac{p}{q} V_j \\ V_{2j+1} &= \tfrac{r}{q} V_j + V_{j+1} \end{aligned}\right\}$$

so that it shall be omitted here. For $j = 2^\ell$ the right-hand side of (3.1) is equal to $\frac{1-r}{q} p^\ell$, and is equal to (1.3) for all ℓ if and only if $p = 1$ or $r = 1$.

As a consequence of (2.6) and (3.1) we find

$$Z_{2^\ell + j} = p r^\ell Z_j\big(\tfrac{p}{r}, \tfrac{q}{r}, 1\big) + q^\ell Z_{2^\ell - j}\big(\tfrac{p}{q}, \tfrac{r}{q}, 1\big) \tag{3.2}$$

and this equation is not only valid for $0 \leq j < 2^\ell$ but also for $j = 2^\ell$. Owing to continuity, equation (3.2) remains valid in the limit case $p = r$. Since both terms on the right-hand side of (3.2) are homogeneous polynomials we can conclude (cf. Table 1):

Corollary 3.2. *For $1 \leq j \leq 2^\ell - 1$ every polynomial $Z_{2^\ell + j}$ is a sum of a homogeneous polynomial of degree $\ell + 1$ plus such a polynomial of degree ℓ.*

It is also possible to consider the limit case $q \to p$ in (3.1) where we proceed analogously as before.

Proposition 3.3. *For $p = q$ the solution of system* (1.1) *with initial condition* (1.2) *has the property*

$$Z_{2^\ell - j} = p^{\ell - 1}\big[(r(\ell - 1) + 1) U_j - r w_j\big] \tag{3.3}$$

for $0 \leq j \leq 2^\ell - 1$ ($\ell \in \mathrm{N}_0$) where $U_j = Z_j\big(1, \frac{r}{p}, 1\big)$ and $w_j = w_j\big(\frac{r}{p}\big)$ is determined by (2.16) *with $x = \frac{r}{p}$ and $w_1 = 0$.*

Proof. By means of de l'Hospital's rule we obtain from (3.1) for $p \to q$

$$Z_{2^\ell - j} = (r\ell + 1 - r) p^{\ell - 1} Z_j\big(1, \tfrac{r}{p}, 1\big) - r p^\ell \big[Z_j'\big(\tfrac{p}{q}, \tfrac{r}{q}, 1\big) - Z_j'\big(1, \tfrac{r}{p}, \tfrac{q}{p}\big)\big]\big|_{q=p}$$

where in view of (2.11)

$$p\big[Z_j'\big(\tfrac{p}{q}, \tfrac{r}{q}, 1\big) - Z_j'\big(1, \tfrac{r}{p}, \tfrac{q}{p}\big)\big]\big|_{q=p} = w_j\big(\tfrac{r}{p}\big)$$

so that (3.3) is proved ∎

Analogously, in the case $x \neq 1$ we can derive

$$w_{2^\ell - j}(x) = \tfrac{\ell}{1-x}(U_j - x^\ell V_j) - x^{\ell-1} V_j \tag{3.4}$$

for $0 \leq j \leq 2^\ell$ $(\ell \in \mathrm{N}_0)$ with $U_j = Z_j(1,1,x)$ and $V_j = Z_j\big(\tfrac{1}{x},\tfrac{1}{x},1\big)$, and in the case $x = 1$

$$w_{2^\ell - j}(1) = (\ell^2 - 1)Y_j - \ell w_j \tag{3.5}$$

with $Y_j = Z_j(1,1,1)$. Moreover, a simple consequence of (2.10) with $p = q = 1$ as well as $\ell - 1$ instead of ℓ and (3.3) with $p = r = 1$ is

$$Y_{2^\ell - j} = Y_{2^{\ell-1}+j} \tag{3.6}$$

for $0 \leq j \leq 2^{\ell-1}$ with $Y_n = Z_n(1,1,1)$. This equation shows a local symmetry of Y_n with respect to the points $n = 3 \cdot 2^{\ell-2}$ $(\ell \geq 2)$ (cf. the later Table 3).

4. Further relations and inequalities

In the following we also admit vanishing coefficients in system (1.1). In order to establish new relations between different solutions Z_n we need the definition of a k-sequence.

Definition 4.1. Let $k \in \mathrm{N}$ and $\ell \in \mathrm{N}_0$.

(i) A finite sequence $\mu_1, \mu_2, ..., \mu_k$ is called a *k-sequence* if $\mu_1 \in \{1,3\}$, $\mu_j \in \{8\ell + 1, 8\ell + 3\}$ for $\mu_{j-1} = 4\ell + 3$ and $\mu_j \in \{8\ell + 5, 8\ell + 7\}$ for $\mu_{j-1} = 4\ell + 1$ $(2 \leq j \leq k)$.

(ii) A finite sequence $\mu_1, \mu_2, ..., \mu_k, \mu_k^*$ is called an *extended k-sequence* if $\mu_1, ..., \mu_k$ is a k-sequence, $\mu_k^* = 4\ell + 3$ for $\mu_k = 4\ell + 1$ and $\mu_k^* = 4\ell + 1$ for $\mu_k = 4\ell + 3$.

The foregoing definitions can be visualized by means of a so-called Collatz graph (cf. [8]). We begin with the directed Collatz graph in Figure 1 for the function g defined by

$$g(4\ell + 1) = g(4\ell + 3) = 2\ell + 1 \qquad (\ell \in \mathrm{N}_0)\,.$$

Inverting the directions and interchanging the neighbouring numbers $4\ell + 1$ and $4\ell + 3$ for all $\ell \in \mathrm{N}_0$, we obtain the inversely directed Collatz graph in Figure 2 for the function f defined by

$$\left.\begin{aligned} f(8\ell + 1) = f(8\ell + 3) = 4\ell + 3 \\ f(8\ell + 5) = f(8\ell + 7) = 4\ell + 1 \end{aligned}\right\} \qquad (\ell \in \mathrm{N}_0)\,.$$

After these preparations, the numbers of k consecutive vertices in a directed path of Figure 2 beginning with 1 or 3, where in the last case the loop at the vertex 3 can be passed several times, yield always terms of a k-sequence. The term μ_k^* of the corresponding extended k-sequence is fixed by the demand that $\mu_k^* \neq \mu_k$ and that an interchange of

μ_k and μ_k^* again yields an extended k-sequence. Note that for all j we have $\mu_j < 2^{j+1}$.

Figure 1: The directed Collatz graph of the function g

Figure 2: The inversely directed Collatz graph of the function f

Proposition 4.2. *For every extended k-sequence the polynomials Z_n satisfy the relations*

$$\begin{aligned}\lambda^k Z_{2n+1} = \lambda^{k-1} Z_{4n+\mu_1} + \lambda^{k-2} Z_{8n+\mu_2} + \dots \\ + \lambda Z_{2^k n+\mu_{k-1}} + Z_{2^{k+1}n+\mu_k} + Z_{2^{k+1}n+\mu_k^*}\end{aligned} \tag{4.1}$$

for arbitrary $k \in \mathbb{N}$, $n \in \mathbb{N}_0$ and $\lambda = p + q + r$.

Proof. From system (1.1) we easily derive

$$\left.\begin{aligned} Z_{4k+1} &= pqZ_k + rZ_{2k+1} \\ Z_{4k+3} &= qZ_{2k+1} + prZ_{k+1} \end{aligned}\right\}. \tag{4.2}$$

By addition we obtain

$$(p+q+r)Z_{2n+1} = Z_{4n+1} + Z_{4n+3} \tag{4.3}$$

and therefore (4.1) for $k = 1$. If (4.1) is satisfied for a fixed k-sequence, we multiply this equation by λ and regard that

$$\begin{aligned}\lambda Z_{2^{k+1}n+4\ell+1} &= Z_{2^{k+2}n+8\ell+1} + Z_{2^{k+2}n+8\ell+3} \\ \lambda Z_{2^{k+1}n+4\ell+3} &= Z_{2^{k+2}n+8\ell+5} + Z_{2^{k+2}n+8\ell+7}.\end{aligned}$$

in view of (4.3). Hence we obtain (4.1) with $k+1$ instead of k and two extended $(k+1)$-sequences, one with the old μ_j for $j \le k$ and one with the old μ_j for $j \le k-1$ and μ_k^* instead of μ_k, and both with suitable μ_{k+1}, μ_{k+1}^* ∎

Remark 4.3.

1. Further special cases of relations (4.1) besides of (4.3) are

$$\begin{aligned}\lambda^2 Z_{2n+1} &= \lambda Z_{4n+1} + Z_{8n+5} + Z_{8n+7} \\ \lambda^2 Z_{2n+1} &= \lambda Z_{4n+3} + Z_{8n+1} + Z_{8n+3}.\end{aligned}$$

2. Dividing (4.1) by λ^k and considering the case $k \to \infty$ we obtain the expansion

$$Z_{2n+1} = \sum_{\ell=1}^{\infty} \tfrac{1}{\lambda^\ell} Z_{2^{\ell+1}n+\mu_\ell} \tag{4.4}$$

so long as the series is converging. This is always the case for positive p, q, r but also for some complex coefficients:

Proposition 4.4. *The series* (4.4) *converges for complex p, q, r provided that*

$$C = \max\big\{|p|, |q| + |r|, 1\big\} < |\lambda| \tag{4.5}$$

where $\lambda = p + q + r$.

Proof. 1. In order to show the convergence of series (4.4) first we shall prove that

$$|Z_k| \le C^\ell \tag{4.6}$$

for $1 \le k \le 2^\ell \quad (\ell \in \mathrm{N}_0)$. For this reason we shall show by induction that

$$|Z_{2^\ell+j}| \le C^{\ell+1} \tag{4.7}$$

for $0 \le j \le 2^\ell$. This inequality is true for $\ell = 0, j \in \{0,1\}$ according to $Z_1 = 1 \le C$ and $|Z_2| = |p| \le C$. Assume that (4.7) is valid for a fixed ℓ. Then we have

$$|Z_{2^{\ell+1}+2j}| = |p|\,|Z_{2^\ell+j}| \le |p|C^{\ell+1} \le C^{\ell+2}$$

$$|Z_{2^{\ell+1}+2j+1}| \le |q|\,|Z_{2^\ell+j}| + |r|\,|Z_{2^\ell+j+1}| \le (|q|+|r|)C^{\ell+1} \le C^{\ell+2}$$

for $j \le 2^\ell$ and $j < 2^\ell$, respectively, i.e. (4.7) with $\ell+1$ instead of ℓ so that (4.7) is proved. This implies inequality (4.6) in view of $C \ge 1$.

2. Now, from (4.6) and $\mu_\ell < 2^{\ell+1}$ we obtain $|Z_{2^{\ell+1}n+\mu_\ell}| \le C^{\ell+m+1}$ for $n+1 \le 2^m$ in view of $2^{\ell+1}n + \mu_\ell < 2^{\ell+1}(n+1) \le 2^{\ell+m+1}$. This yields $\left|\frac{1}{\lambda^\ell} Z_{2^{\ell+1}n+\mu_\ell}\right| \le C^{m+1}\left(\frac{C}{|\lambda|}\right)^\ell$ so that according to (4.5) the series in (4.4) converges ∎

For $k = 2^\ell + j$ we immediately obtain from (4.7) and $2^\ell \le k \le 2^{\ell+1}$:

Corollary 4.5. *The polynomials $Z_k \quad (k \in \mathrm{N})$ can be estimated by*

$$|Z_k| \le Ck^c \tag{4.8}$$

with $c = \frac{\ln C}{\ln 2}$.

In the case $p = q = r = 1$ we can state the following curious connection between the numbers $Y_n = Z_n(1,1,1)$ and the Fibonacci numbers $F_k \quad (k \in \mathrm{N}_0)$:

Proposition 4.6. *With the notation $m_k = \frac{1}{3}\left(2^{k+1} + (-1)^k\right) \quad (k \in \mathrm{N}_0)$ the numbers $Y_{m_k} = Z_{m_k}(1,1,1)$ are the Fibonacci numbers F_k. These have the extremal property $Y_n < Y_{m_k}$ for $n < m_k$ and $k \ge 2$.*

Proof. In view of $m_0 = m_1 = 1$ and (1.2) the first assertion is valid for $k = 0$ and $k = 1$. According to

$$2^{k+1} + (-1)^k = 2^k + (-1)^{k-1} + 2\left(2^{k-1} + (-1)^{k-2}\right)$$

and (1.1) with $p = q = r = 1$ the numbers Y_{m_k} satisfy the difference equation

$$Y_{m_k} = Y_{m_{k-1}} + Y_{m_{k-2}} \tag{4.9}$$

for $k \ge 2$ which proves the first assertion.

In order to prove the second assertion it suffices to consider odd indices since $Y_{2n} = Y_n$ and to consider (4.2) in the specialization

$$\left.\begin{aligned} Y_{4n+1} &= Y_n + Y_{2n+1} \\ Y_{4n+3} &= Y_{n+1} + Y_{2n+1} \end{aligned}\right\}. \tag{4.10}$$

The assertion is valid for $n < m_2 = 3$ where $Y_3 = 2$ (cf. Table 3). We assume that it is valid for $n < m_{k-1}$ with $k \geq 3$. In the case that k is even we choose $\ell = \frac{1}{3}(2^{k-1} - 2)$ and have

$$\begin{aligned} m_{k-2} &= \ell + 1 \\ m_{k-1} &= 2\ell + 1 \\ m_k &= 4\ell + 3. \end{aligned}$$

Hence $4n + 1 < 4\ell + 3$ implies $n \leq \ell$, i.e. $n < \ell + 1$ as well as $2n + 1 \leq 2\ell + 1$, and $4n + 3 < 4\ell + 3$ implies $n < \ell$, i.e. $n + 1 < \ell + 1$ as well as $2n + 1 < 2\ell + 1$. In the case that k is odd we choose $\ell = \frac{1}{3}(2^{k-1} - 1)$ and obtain

$$\begin{aligned} m_{k-2} &= \ell \\ m_{k-1} &= 2\ell + 1 \\ m_k &= 4\ell + 1. \end{aligned}$$

Hence $4n+1 < 4\ell+1$ implies $n < \ell$ as well as $2n+1 < 2\ell+1$, and $4n+3 < 4\ell+1$ implies $n+1 \leq \ell$ as well as $2n+1 < 2\ell+1$. In both cases equations (4.9) and (4.10) show that the second assertion is also valid for $n < m_k$ so that it is proved by induction ∎

It can be shown analogously by induction that $Y_n < Y_{m_k}$ for $m_k < n \leq 3 \cdot 2^{k-2}$ and $k \geq 3$, but we can extend this inequality a second time by means of (3.6). Introducing numbers $\overline{m}_k$ $(k \in \mathrm{N})$ by $2^k - \overline{m}_k = 2^{k-1} + m_k$, i.e. by

$$\overline{m}_k = \tfrac{1}{3}\big(5 \cdot 2^{k-1} - (-1)^k\big)$$

we have $Y_{m_k} = Y_{\overline{m}_k}$ according to (3.6). Obviously, $2^{k-1} \leq m_k \leq 3 \cdot 2^{k-2} \leq \overline{m}_k \leq 2^k$ and $m_k = \overline{m}_k$ if and only if $k = 2$. Now, the foregoing remarks and equation (3.6) imply:

Corollary 4.7 *For a fixed $k \in \mathrm{N}$ the Fibonacci number F_k is equal to the maximum of Y_n for $1 \leq n \leq 2^k$ which is attained in this interval exactly for both $n = m_k$ and $n = \overline{m}_k$.*

The extremal properties in Proposition 4.6 and in Corollary 4.7 can be checked for the first indices by means of Table 3 where the Fibonacci numbers Y_{m_k} are underlined and the Fibonacci numbers $Y_{\overline{m}_k}$ are labelled by an overhead bar.

Table 3: The first numbers $Y_n = Z_n(1, 1, 1)$

5. Generating functions

It is useful to construct the generating function

$$G(t)=\sum_{n=1}^{\infty} Z_n t^{n-1} \tag{5.1}$$

of the sequence Z_n. In view of (4.8) the series converges for $|t|<1$. The recursions (1.1) easily imply the functional equation

$$G(t)=1-r+\left(r+pt+qt^2\right)G(t^2) \tag{5.2}$$

and therefore by iteration for arbitrary $n\in \mathrm{N}_0$

$$G(t)=(1-r)\sum_{k=0}^{n-1}\prod_{j=0}^{k-1}\left(r+pt^{2^j}+qt^{2^{j+1}}\right)+G(t^{2^n})\prod_{j=0}^{n-1}\left(r+pt^{2^j}+qt^{2^{j+1}}\right).$$

As usual the products are defined by 1 in the cases $k=0$ and $n=0$. For $|t|<1$ we have $G(t^{2^n})\to G(0)=1$ as $n\to\infty$. Hence for $|t|<1$ we get in the case $|r|<1$

$$G(t)=(1-r)\sum_{k=0}^{\infty}\prod_{j=0}^{k-1}\left(r+pt^{2^j}+qt^{2^{j+1}}\right) \tag{5.3}$$

and in the case $r=1$

$$G(t)=\prod_{j=0}^{\infty}\left(1+pt^{2^j}+qt^{2^{j+1}}\right). \tag{5.4}$$

However, if we write (5.2) in the form

$$G(t)-1=pt+qt^2+\left(r+pt+qt^2\right)\left(G(t^2)-1\right)$$

we get

$$G(t)=1+\sum_{k=0}^{\infty}\left(pt^{2^k}+qt^{2^{k+1}}\right)\prod_{j=0}^{k-1}\left(r+pt^{2^j}+qt^{2^{j+1}}\right) \tag{5.5}$$

for arbitrary r and again for $|t|<1$. Summarizing these results we have proved:

Proposition 5.1. *For $|t|<1$ the generating function* (5.1) *has the representation* (5.5)*. In the case $|r|<1$ it can also be represented by* (5.3) *and in the case $r=1$ by* (5.4)*.*

Concerning the different representations (5.4) and (5.5) in the case $r=1$ cf. [6: p. 233].

Moreover, we consider the generating functions

$$F(t)=\sum_{n=1}^{\infty} Y_n t^{n-1} \qquad \text{and} \qquad H(t)=\sum_{n=1}^{\infty} w_n t^{n-1} \tag{5.6}$$

where $Y_n=Z_n(1,q,1)$ and $w_n=w_n(q)$ so that $F(t)=G(t)$ from (5.1) with $p=r=1$ and (5.2) specializes to

$$F(t)=\left(1+t+qt^2\right)F(t^2). \tag{5.7}$$

According to (2.18) and (4.8) the series for $H(t)$ converges for $|t|<1$, too.

Proposition 5.2. *The generating function H from* (5.6) *satisfies the equation*

$$H(t) = F(t) - 1 + (1 + t + qt^2)\, H(t^2) \tag{5.8}$$

and it can be represented by the series

$$H(t) = F(t) \sum_{k=0}^{\infty} \Big(1 - \frac{1}{F(t^{2^k})}\Big) \tag{5.9}$$

which converges for $|t| < 1$.

Proof. Equation (5.8) follows from (2.16) and (5.6) by straitforward calculations, and (5.9) follows from (5.7) and (5.8) in view of $H(0) = 0$ and $F(0) = 1$ ∎

Let us mention that series (5.9) can be written in the form

$$H(t) = \sum_{k=0}^{\infty} \Big(F(t) - \frac{F(t)}{F(t^{2^k})}\Big) \tag{5.10}$$

where the quotients

$$\frac{F(t)}{F(t^{2^k})} = \prod_{j=0}^{k-1} \big(1 + t^{2^j} + qt^{2^{j+1}}\big)$$

are polynomials. It is also possible to eliminate $F(t)$ out of (5.7) and (5.8), but then $H(t^4)$ appears in the equation.

6. Explicit representations

We begin with very special representations. In our representations we need the dyadic sum-of-digits function $\nu(j)$ and its complement $\mu(k) = \ell - \nu(k)$ for $2^{\ell-1} \le k < 2^\ell$ with $j \in \mathrm{N}_0$ and $\ell, k \in \mathrm{N}$, i.e. $\nu(j)$ denotes the number of 1*s* and $\mu(k)$ the number of 0*s* in the dyadic representation of j resp. k. Obviously, we have the initial values $\nu(0) = \mu(1) = 0$ and the recursions

$$\left.\begin{aligned} \nu(2j) &= \nu(j) \\ \nu(2j+1) &= \nu(j) + 1 \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} \mu(2k) &= \mu(k) + 1 \\ \mu(2k+1) &= \mu(k) \end{aligned}\right\}. \tag{6.1}$$

Moreover, we put $\mu(0) = 0$ which is compatible with the last equation of (6.1). Since $Z_{2^\ell k} = p^\ell Z_k$ it suffices to consider odd k only.

Proposition 6.1. *For $k \in \mathrm{N}$ the polynomials Z_{2k+1} have the representations*

$$Z_{2k+1} = \begin{cases} q^{\nu(k)} r^{\mu(k)} & \textit{for } p = 0 \\ p^{\nu(k)} r^{\mu(k)+1} & \textit{for } q = 0 \\ p^{\mu(k)} q^{\nu(k)} & \textit{for } r = 0. \end{cases} \tag{6.2}$$

Proof. From (1.1) and (4.2) we immediately obtain

$$Z_{4k+1} = \begin{cases} rZ_{2k+1} \\ rZ_{2k+1} \\ pZ_{2k+1} \end{cases} \quad \text{and} \quad Z_{4k+3} = \begin{cases} qZ_{2k+1} & \text{for } p = 0 \\ pZ_{2k+1} & \text{for } q = 0 \\ qZ_{2k+1} & \text{for } r = 0. \end{cases} \tag{6.3}$$

In view of $Z_3 = pr + q$ equations (6.2) are valid for $k = 1$. If they are valid for a fixed $k \in \mathrm{N}$, then also for $2k$ resp. $2k + 1$ instead of k in view of (6.1) and (6.3). Hence the proposition is proved by induction ∎

Remark 6.2. In view of (1.2) the first and last equations of (6.2) remain valid for $k=0$. In the case $r=0$ we have $Z_n = p^{\mu(n)}q^{\nu(n)-1}$ for all $n \in \mathrm{N}$. We even can use Proposition 2.1 for $q=0$ and Proposition 3.1 for $r=0$ if for $j=0$ we interpret $qX_0 = p-r$ resp. $rU_0 = p-q$.

In order to deal with the general case we need some preparations. It can easily be seen that

$$\prod_{j=0}^{\ell-1}\left(1+pt^{2^j}\right) = \sum_{k=0}^{2^\ell-1} p^{\nu(k)}t^k$$

and, more generally,

$$\prod_{j=0}^{\ell-1}\left(r+p_jt^{2^j}\right) = \sum_{k=0}^{2^\ell-1} r^{\ell-\nu(k)}\left(\prod_{m=0}^{\nu(k)-1} p_{\gamma_{km}}\right)t^k \tag{6.4}$$

where the indices $\gamma_{km} \in \mathrm{N}_0$ are defined by

$$k = 2^{\gamma_{k,\nu(k)-1}} + \dots + 2^{\gamma_{k1}} + 2^{\gamma_{k0}} \tag{6.5}$$

with $\gamma_{k0} < \gamma_{k1} < \gamma_{k2} < \dots$. For another generalization we need

Definition 6.3. We say that the ordered pair $(i,k) \in \mathrm{N}_0 \times \mathrm{N}_0$ *belongs to the relation* $\omega(i,k)$, if $i=0$ or if $\{\gamma_{i0}, \gamma_{i1}, \dots, \gamma_{i,\nu(i)-1}\} \subset \{\gamma_{k0}, \gamma_{k1}, \dots, \gamma_{k,\nu(k)-1}\}$.

By means of this definition we find that

$$\prod_{m=0}^{\nu(k)-1}\left(p+qt^{2^{\gamma_{km}}}\right) = \sum_{\omega(i,k)} p^{\nu(k)-\nu(i)}q^{\nu(i)}t^i . \tag{6.6}$$

Choosing $p_j = p + qt^{2^j}$ we obtain

$$\prod_{j=0}^{\ell-1}\left(r+pt^{2^j}+qt^{2^{j+1}}\right) = \prod_{j=0}^{\ell-1}\left(r+p_jt^{2^j}\right) = \sum_{k=0}^{2^\ell-1}\sum_{\omega(i,k)} s_{ik\ell}t^{i+k} \tag{6.7}$$

according to (6.4) and (6.5) where we have used the abbreviation

$$s_{ik\ell} = r^{\ell-\nu(k)}p^{\nu(k)-\nu(i)}q^{\nu(i)} . \tag{6.8}$$

Proposition 6.4. *For* $n \in \mathrm{N}$ *the solution of problem* (1.1) – (1.2) *has the representation*

$$Z_{n+1} = {\sum_{i+k=n}}' s_{ik\ell} + {\sum_{i+k=n-2^\ell}}' p\, s_{ik\ell} \tag{6.9}$$

where $2^\ell \le n < 2^{\ell+1}$ $(i,k \in \mathrm{N}_0)$ *and a prime at sums shall mean that* (i,k) *must belong to* $\omega(i,k)$ *and that* $k \le 2^\ell - 1$.

Proof. Comparing (5.1) with (5.5) we see that Z_{n+1} is the coefficient of t^n in the polynomial

$$\left(pt^{2^{\ell-1}} + qt^{2^\ell}\right) \prod_{j=0}^{\ell-2} \left(r + pt^{2^j} + qt^{2^{j+1}}\right) + t^{2^\ell} \prod_{j=0}^{\ell-1} \left(r + pt^{2^j} + qt^{2^{j+1}}\right) \tag{6.10}$$

since the product in (5.5) is a polynomial in t of degree $\sum_{j=1}^{k} 2^j = 2^{k+1} - 2$. It is possible to replace (6.10) by

$$\left(1 + pt^{2^\ell}\right) \prod_{j=0}^{\ell-1} \left(r + pt^{2^j} + qt^{2^{j+1}}\right)$$

because the difference is a polynomial in t of degree $2^\ell - 2$ which gives no contribution to the coefficient in question. Now, (6.7) immediately implies (6.9) ∎

Remark 6.5.

1. In accordance with Corollary 3.2 the first sum of (6.9) is a homogeneous polynomial of degree ℓ and the last sum is such a polynomial of degree $\ell + 1$.

2. In view of $i + k = n - 2^\ell$ and $n < 2^{\ell+1}$, the restriction $k \le 2^\ell - 1$ is automatically satisfied in the second sum of (6.9).

By means of (2.11), it follows from (6.9) with $q = px$:

Corollary 6.6. *For $n \in \mathrm{N}$ the polynomial w_{n+1} has the representation*

$$w_{n+1}(x) = \ell \sum_{i+k=n}{}' x^{\nu(i)} + (\ell + 1) \sum_{i+k=n-2^\ell}{}' x^{\nu(i)} \tag{6.11}$$

with the same restrictions as in Proposition 6.4.

Comparing (6.9) and (6.11) with (2.18) in the case $p = r = 1$, $x = q$ and $n = 2^\ell + j - 1$ we obtain the simplification

$$Y_j = \sum_{i+k=j-1}{}' q^{\nu(i)} \tag{6.12}$$

where $j \in \mathrm{N}$, $(i, k) \in \omega(i, k)$ and $Y_j = Z_j(1, q, 1)$, but a further restriction with respect to k is not required.

In the special case $r = 1$ we can derive another type of representations. For convenience we use the notation $z_n = z_n(p, q) = Z_n(p, q, 1)$ for $n \in \mathrm{N}_0$. If we introduce new parameters α and β as solutions of $\xi^2 - p\xi + q = 0$ so that

$$\left.\begin{aligned} p &= \alpha + \beta \\ q &= \alpha\beta \end{aligned}\right\} \tag{6.13}$$

we can write system (1.1) with $r = 1$ in the form

$$\begin{aligned} z_{2k} &= (\alpha + \beta) z_k \\ z_{2k+1} &= \alpha\beta\, z_k + z_{k+1} \end{aligned}$$

and every z_n is a symmetric polynomial with respect to α and β. The generating function (5.4) supplies a representation for z_n:

Proposition 6.7. *The polynomial z_n has the representation*

$$z_n = \sum_{j=0}^{n-1} \alpha^{\nu(j)} \beta^{\nu(n-1-j)} \tag{6.14}$$

where α and β are determined by (6.13) *and $\nu(j)$ by* (6.1).

Proof. In view of (6.13) we have $1 + pt + qt^2 = (1 + \alpha\, t)\,(1 + \beta\, t)$ so that the generating function (5.4) has the form

$$G(t) = \prod_{j=0}^{\infty} \left(1 + \alpha\, t^{2^j}\right) \prod_{j=0}^{\infty} \left(1 + \beta\, t^{2^j}\right)$$

for $|t| < 1$. Owing to

$$\prod_{j=0}^{\infty} \left(1 + \xi\, t^{2^j}\right) = \sum_{k=0}^{\infty} \xi^{\nu(k)}\, t^k \tag{6.15}$$

we obtain

$$G(t) = \sum_{j=0}^{\infty} \alpha^{\nu(j)}\, t^j \sum_{k=0}^{\infty} \beta^{\nu(k)}\, t^k$$

and hence, by means of the Cauchy product and (5.1), representation (6.14) ∎

Solving (6.13) with respect to p and q it is possible in (6.14) to express z_n explicitly by means of the parameters p and q.

Examples 6.8.

1. In the special case $\beta = 1$ and therefore $\alpha = q$, $p = q + 1$ formula (6.14) reduces to a representation of $S_n = z_n(q + 1, q)$ in [2].

2. In the special case $q = 1$, i.e. $\beta = \frac{1}{\alpha}$, formula (6.14) simplifies to

$$z_n(p, 1) = \sum_{j=0}^{n-1} \alpha^{\nu(j) - \nu(n-1-j)} \tag{6.16}$$

where

$$\alpha = \tfrac{p}{2} \pm \sqrt{\tfrac{p^2}{4} - 1} \tag{6.17}$$

and, in particular for $p = 2$, i.e. $\alpha = 1$, (6.16) implies $z_n(2, 1) = n$ which also follows immediately from (1.1) with $p = 2$, $q = r = 1$ and (1.2). For $p \geq 2$ we can put $p = 2 \cosh r$ with real r so that $\alpha = e^{\pm r}$ and

$$z_n(2 \cosh r, 1) = \sum_{j=0}^{n-1} \cosh \big[r(\nu(j) - \nu(n-1-j)) \big]. \tag{6.18}$$

For $-2 \leq p \leq 2$ we can put $p = 2 \cos \varrho$ with real ϱ so that $\alpha = e^{\pm i \varrho}$ and

$$z_n(2 \cos \rho, 1) = \sum_{j=0}^{n-1} \cos \big[\varrho(\nu(j) - \nu(n-1-j)) \big]. \tag{6.19}$$

Of course, representations (6.18) and (6.19) are also valid for complex r resp. ϱ.

3. A last special case is $p = 1$ which concerns the polynomials $Y_n = Z_n(1, q, 1) = z_n(1, q)$. Formula (6.14) yields the representation

$$Y_n = \sum_{j=0}^{n-1} \alpha^{\nu(j)}(1-\alpha)^{\nu(n-1-j)} \tag{6.20}$$

where

$$\alpha = \tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} - q}. \tag{6.21}$$

From (2.18) and (6.20) also a representation for w_n can be obtained, but we do not deal with that case.

Finally, we want to give a third type of representation for $Y_n = Z_n(1, q, 1) = z_n(1, q)$ where once more it suffices to consider odd n only. From (2.10) with $p = r = 1$ and (2.18) with $x = q$ we obtain

$$Y_{2^\ell + 2^\lambda + j} = \big(q(\ell - \lambda) + 1\big) Y_{2^\lambda + j} - q Y_j \tag{6.22}$$

for $0 \le j \le 2^\lambda < 2^\ell$. As in (6.5), an arbitrary positive odd integer can be written in the form

$$n_k = 2^{\gamma_k} + 2^{\gamma_{k-1}} + ... + 2^{\gamma_1} + 2^{\gamma_0}$$

with $\gamma_0 = 1 < \gamma_1 < \gamma_2 < ...$ $(k \in \mathrm{N}_0)$ and $\gamma_j \in \mathrm{N}$. For a fixed sequence γ_j we introduce the notation

$$\eta_j = q(\gamma_j - \gamma_{j-1}) + 1 \qquad (j \in \mathrm{N}).$$

Then, with $\ell = \gamma_k$, $\lambda = \gamma_{k-1}$ and $j = n_{k-2}$, (6.22) can be written as

$$Y_{n_k} = \eta_k Y_{n_{k-1}} - q Y_{n_{k-2}} \tag{6.23}$$

for $k \ge 2$. Since $n_0 = 1$ and $n_1 = 2^{\gamma_1} + 1$ we have the initial values $Y_{n_0} = 1$ and $Y_{n_1} = q\gamma_1 + 1 = \eta_1$; cf. (2.10) with $p = j = 1$ and $\ell = \gamma_1$.

Proposition 6.9. *For $k \in \mathrm{N}_0$ the polynomial $Y_n = Z_n(1, q, 1)$ has the representation*

$$Y_{n_k} = \sum_{j=0}^{[k/2]} (-q)^j \eta_{i_1}\eta_{i_2}\cdots\eta_{i_{k-2j}} \tag{6.24}$$

where the sum runs over all indices with $1 \le i_1 < i_2 < ... \le k$ under the condition that i_j is odd or even for j odd or even, respectively.

Proof. In view of the initial values, (6.24) is valid for $k = 0$ and $k = 1$. If (6.24) is valid up to a fixed k, then we obtain from (6.23)

$$Y_{n_{k+1}} = \sum_{j=0}^{[k/2]} (-q)^j \eta_{i_1}\eta_{i_2}\cdots\eta_{i_{k-2j}}\eta_{k+1} + \sum_{j=1}^{[(k+1)/2]} (-q)^j \eta_{i_1}\eta_{i_2}\cdots\eta_{i_{k+1-2j}}$$

and these sums can be gathered up as one single sum (6.24) with $k+1$ instead of k since $i_{k+1-2j} \le k-1$ in the second sum ∎

The first sums (6.24) with $k \geq 2$ read

$$Y_{n_2} = \eta_1\eta_2 - q$$
$$Y_{n_3} = \eta_1\eta_2\eta_3 - q(\eta_1 + \eta_3)$$
$$Y_{n_4} = \eta_1\eta_2\eta_3\eta_4 - q(\eta_1\eta_2 + \eta_1\eta_4 + \eta_3\eta_4) + q^2.$$

By means of (2.18) and (6.24) it is possible to derive also a representation for w_{n_k} but we are not concerned with that.

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