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DYNAMICAL PERSISTENCE PRINCIPLES AND BIFURCATION

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1. Introduction. The problem considered in this paper concerns the connections between the following two types of phenomena in families of dynamical (or semidynamical) systems, depending on a parameter λ, with respect to an equilibrium point or compact invariant set M independent of λ:

1. Change of the stability behaviour of M as the parameter λ reaches or surpasses a certain "critical" value λ_0;

2. Bifurcation of M in the sense of "splitting" into more than one invariant sets as λ reaches or surpasses a value λ_0. This bifurcation may be *extracritical* (or *ultracritical*, supposing λ is a scalar parameter), i.e., there appear compact invariant sets M_n for values λ_n of λ such that $\lambda_n \to \lambda_0$, $M_n \cap M = \emptyset$, and the maximal distances of M_n from M converge to 0; or it may be *critical* (also called *vertical*) in the sense that for λ_0 the set M is not isolated from compact invariant sets.

Experience has shown that bifurcations (in the above sense) are typically connected with a change ("loss" or "gain") of stability, the most well-known case being the *Poincaré-Andronov-Hopf bifurcation* which occurs when a pair of eigenvalues of the linear parts of the systems considered cross the imaginary axis, for $\lambda = \lambda_0$, causing an abrupt "extracritical" change from asymptotic stability to complete instability. (We are now assuming, for simplicity, that the system considered is two-dimensional.) Then [4, 8], the asymptotically stable equilibrium point (the origin) "splits" into an unstable equilibrium and a stable limit cycle (or several limit cycles, under weaker conditions) for values of λ beyond λ_0. These limit cycles recede into the origin as $\lambda \to \lambda_0$. {The simplest case is the so-called pitchfork, given by the equation $\dot{x} = x(\lambda - x^2)$. Here the limit cycles reduce to critical points.}

It has first been observed by Marchetti, Negrini, Salvadori and Scalia [7], that the occurrence of bifurcation is a general consequence of the change of asymptotic stability to complete instability, independent of what happens with the linear part of the system. (We are leaving aside here the question of whether the invariant sets emerging from the bifurcation are periodic orbits.) For this type of bifurcations we will use the term *ASCI-bifurcations.*

The authors of the paper [7] also observed that the deeper reason for the connection between loss of (asymptotic) stability and bifurcation is to be found in what we call *persistence principle for asymptotic stability*, which states that when a dynamical system containing an asymptotically stable invariant set, or stable attractor, is subjected to a sufficiently small perturbation, then the perturbed system exhibits a stable attractor arbitrarily close to the one of the original system. {In its complete form, this principle was first given by T. Yoshizawa ([13], theorem 25.3); in a preliminary, but more general form, it was proved by the author in [10].}

In [11], J.S. Florio and the author extended the results of [7] mainly in two directions:

1. Eliminating the condition of local compactness of the state space and assuming, instead, that the system be asymptotically compact. Also, for most of the results, the system need only be assumed semidynamical.

2. Weakening the requirement of complete instability, as in the *ASCI-bifurcations*, by assuming, for instance, that M be a saddle set [the trivial prototype being the node-saddle transition $\dot{x} = -x(x^2 - \lambda)$, $\dot{y} = -y$].

More recently, the following variant of the problem outlined above has been considered:

Instead of relaxing, with respect to the class of *ASCI-bifurcations*, the complete instability condition for extracritical values, we now relax, after inverting the time scale, the complete instability assumption for critical λ to simple instability, while preserving the requirement of stability for extracritical values [the trivial prototype (saddle-node transition) being $\dot{x} = x(x^2 - \lambda)$, $\dot{y} = -y$]. In this case, the existence of a bifurcation is in general less easy to prove than in the first case, because no such convenient tool as the persistence principle for asymptotic stability is available. Instead, we have a different kind of persistence principle, which we may call *persistence of instability*: If a system exhibits a certain degree of instability, this situation cannot be substantially improved by arbitrary small changes of the system parameters, though it may be worsened. This fact, which is a simple consequence of continuity, serves as basis for the proof of the existence of ("extracritical") bifurcations, provided a certain condition of equistability (*CRES*) is satisfied. If this condition fails, it can be proved by arguments of the kind used in Ważewski's topological method, that M is not isolated from compact invariant sets for $\lambda = \lambda_0$ ("weak critical bifurcation").

We may summarize the results as follows: *Whenever a compact invariant set M undergoes a gain or loss of stability as a parameter reaches or surpasses a critical value for which M is isolated from compact invariant sets, a bifurcation of M occurs, except in the very special case of an extracritical transition from a stable attractor to a saddle set surrounded entirely by homoclinic orbits.*

2. Persistence of asymptotic stability and generalized Hopf bifurcation.

2.1. Persistence of asymptotic stability.

2.1.1. *A family of semidynamical systems* (X, T, Λ, F), consists of a metric space (X, d) (*state space*), an ordered topological semigroup (*time scale* T, in particular, $\mathbf{R}^+$), a metric space Λ (*parameter space*), and a mapping

$$F : X \times T \times \Lambda \to X$$

(the *dynamics*).

We introduce the following notations:

$$B_r(x) := \{y \in X | d(x, y) < r\} \quad (r > 0,\ x \in X),$$

$$B_r(A) := \{y \in X | d(x, A) < r\} \quad (A \subset X).$$

Here $d(x, A) := inf\{d(x, y) | y \in A\}$.

By $\mathcal{V}_x$ we denote the neighbourhood filter of x and by $\mathcal{V}_A$ $(A \subset X)$ the filter generated by the neighbourhoods $B_r(A)$, $r > 0$, of A. Finally, we denote the neighbourhood filter of the point $\lambda_0 \in \Lambda$ by $\mathcal{N}_{\lambda_0}$.

By $F_\lambda : X \times T \to X$, we denote the λ-*system* defined by

$$F_\lambda(x, t) := F(x, t, \lambda),$$

by $F_\lambda^t : X \to X$, the transformation of the space X defined by

$$F_\lambda^t(x) := F(x, t, \lambda),$$

and by $F_{t,\lambda} : X \to X$, the *truncated* λ-*orbit*,

$$F_{t,\lambda}(x) := \left\{ F_\lambda^{t'}(x) \middle| t' \geq t \right\}.$$

In particular, the *positive* λ-*semiorbit* will be denoted by $\gamma_\lambda^+(x)$:

$$\gamma_\lambda^+(x) := F_{0,\lambda}(x).$$

We assume that the family F satisfies the following **axioms:**

(I) F_λ^0 is the identity mapping for every $\lambda \in \Lambda$.

(II) $F_\lambda^t F_\lambda^{t'} = F_\lambda^{t+t'} (t, t' \in T,\ \lambda \in \Lambda)$.

(III) F_λ is continuous for every λ.

(IV) The motions depend in a uniformly continuous manner on the parameter λ:

$$(\forall \tau \in T,\ \epsilon > 0)(\exists N \in \mathcal{N}_{\lambda_0})(\forall x \in X,\ t \in [0, \tau],\ \lambda \in N)$$
$$d(F_\lambda^t(x),\ F_{\lambda_0}^t(x)) < \epsilon.$$

2.1.2. In what follows, the metric space X will be assumed to be complete.

DEFINITION 2.1.1. A system F_λ is said to be *asymptotically compact* (abbreviated AC) on a set $A \subset X$, if every sequence $\{F_\lambda^{t_k}(x_k)\}_{k=1}^\infty$, where $\{x_k\} \subset A$ and $t_k \to +\infty$, is relatively compact.

The class of AC systems is of great importance in the study of evolution equations [5].

DEFINITION 2.1.2. The λ-*limit set* $L_\lambda^+(A)$ of a set $A \subset X$ is the set of the limits of all convergent sequences of the form $\{F_\lambda^{t_k}(x_k)\}$, where $x_k \in A$ and $t_k \to +\infty$.

PROPOSITION 2.1.3. *[5, 11]. If F_λ is AC on the bounded set A, $L_\lambda^+(A)$ is a nonempty, compact, positively λ-invariant set attracting A uniformly for λ [in the sense that, given $\epsilon > 0$, there exists a $t > 0$ such that $F_{t,\lambda}A := \{F_{t,\lambda}(x) | x \in A\} \subset B_\epsilon(L_\lambda^+(A))$].*

If, in particular, A is a neighbourhood of $L_\lambda^+(A)$, the latter is λ-stable. (The terms λ-invariant and λ-stable refer to the system F_λ).

The *persistence principle for asymptotic stability* may then be stated as follows.

THEOREM 2.1.4. *[11] Suppose each member of a family $\{F_\lambda\}$ of semidynamical systems is AC on a bounded set A, and that A is uniformly attracted for $\lambda = \lambda_0$ to a set M of which A is a neigbourhood. Then, for any neigbourhood $U \in \mathcal{V}_M$, there exists a neighbourhood $N \in \mathcal{N}_{\lambda_0}$ such that, for all*

$\lambda \in N$, U contains a compact, λ-stable set M_λ which attracts A uniformly for λ. Moreover, the smallest sets M_λ with this property are the limit sets $L^+_\lambda(A)$.

The proof given in [11] is based on an induction argument which first appears in [10] (see also [12]). Other proofs, along more traditional lines using Lyapunov functions and limited to systems in locally compact spaces, were given in [7] and [13].

The theorem also says that, in a sense, the region of attraction cannot suddenly shrink; on the other hand, it may "explode", as exemplified by the equation

$$\dot{x} = -x((x-1)^2 + \lambda).$$

Here, for $\lambda = 0$, the region of attraction of the origin is $(-\infty, 1)$; for $\lambda > 0$ it is $\mathbf{R}$.

2.2. Generalized ASCI-bifurcations. In what follows, it will always be assumed that there exists a compact set M which is λ-invariant for all λ. By *invariance of M*, in the case of a semidynamical system, we mean that both M and its complement are positively invariant.

DEFINITION 2.2.1. Let the compact set M be λ-invariant for all $\lambda \in \Lambda$. We say, M undergoes an *extracritical bifurcation* at λ_0 if, for any pair of neigbourhoods, $U \in \mathcal{V}_M$ and $N \in \mathcal{N}_{\lambda_0}$, there exist a $\lambda \in N$, $\lambda \neq \lambda_0$, and a compact positively λ-invariant set M'_λ such that $M'_\lambda \cap M = \emptyset$ and $\emptyset \neq M'_\lambda \subset U$. In the remainder of this section, we will omit the adjective "extracritical", because no critical bifurcations will be considered.

The principle theorem, of which all the others are consequences, is the following:

THEOREM 2.2.2. *[11] Let M be a compact subset of the complete metric space X, which is λ-invariant for all λ, attracts a neighbourhood $A \in \mathcal{V}_M$ uniformly for $\lambda = \lambda_0$, and suppose that the systems F_λ have the property AC on A. Furthermore, assume that there exist a sequence of values λ_n such that $\lambda_n \to \lambda_0$, and a sequence of points $x_n \in A$ such that $L^+_{\lambda_n}(x_n) \cap M = \emptyset$.*

Then M undergoes a bifurcation at λ_0, and the sets M'_λ of Definition 2.2.1 exist for $\lambda = \lambda_n$, and they may be defined by

$$M'_{\lambda_n} = L^+_{\lambda_n}(x_n).$$

These last sets are obviously contained in the limit sets $L^+_{\lambda_n}(A)$ which have all the properties stated in proposition 2.1.3.

If the set A is compact, the property AC is redundant.

The idea of the proof is the following:

We take a fundamental system of neighbourhoods U_n of M and a corresponding sequence of neighbourhoods N_n of λ_0, according to Theorem 2.1.4. Replacing $\{x_n\}$ and $\{\lambda_n\}$ by subsequences if neccesary, we may assume $\lambda_n \in N_n$. Denote by M_{λ_n} the λ_n-attractor corresponding to M_λ, for $\lambda = \lambda_n$, in the same theorem. It is contained in U_n. Then the limit sets $L^+_{\lambda_n}(x_n)$ are contained in M_{λ_n}, hence in U_n, but (by hypothesis) not in M. They are compact and positively λ_n-invariant (λ_n-invariant if the system is dynamical), and thus have the properties of the sets M'_λ in definition 2.2.1.

The theorem clearly covers the case of the *ASCI-bifurcations*, if we define *complete instability* of M for semidynamical systems as the existence of a neighbourhood of M devoid of positive semiorbits, except those contained in M. If M is completely unstable for a certain value λ of the parameter, we say also that M is a *λ-repeller*, and denote by R_λ the *region of λ-repulsion*, defined as the union of all the neighbourhoods which figure in the above definition of complete instability.

As a direct generalization of the principal theorem of [7], we have the following:

THEOREM 2.2.3. (GENERALIZED POINCARÉ-ANDRONOV-HOPF BIFURCATION.). *[11] The hypotheses are the same as in the preceding theorem, except that the last one is replaced by the condition that for a certain sequence of values of $\lambda_n \in \Lambda$, with $\lambda_n \to \lambda_0$, M is a λ_n-repeller. Then M undergoes a bifurcation at λ_0; in particular, there appears a set of compact positively invariant λ_n-attractors, M'_{λ_n}, which are disjoint from M, and attract $A\backslash M$ for $\lambda = \lambda_n$. The sets M'_{λ_n} may be chosen as*

$$M'_{\lambda_n} = \overline{L^+_{\lambda_n}(A)\backslash R_{\lambda_n}},$$

where R_{λ_n} are the regions of λ_n-repulsion of M, and they are contained in any neighbourhood of M for λ_n sufficiently near λ_0.

If the systems F_λ are dynamical, the sets M'_{λ_n} are λ_n-stable and separate M from the complement of $L^+_{\lambda_n}(A)$.

(The question whether the last part of the theorem also holds for semidynamical systems remains open.)

In another paper [6], Marchetti extended the main result of [7] in two directions: a) by developing his theory within the context of a continuous mapping of an arbitrary topological space onto itself; b) by replacing the hypothesis of asymptotic stability at the critical value by a weaker one which is equivalent to total stability. A similar result was also obtained by Bertotti and Moauro in [2], in the context of dynamical systems in $\mathbf{R}^n$.

Another direct consequence of Theorem 2.2.2, covering a wider class of systems, is the following, involving the concept of weak attractor:

M is a *weak attractor* if, for some neighbourhood U, every $x \in U$ has the property

$$L^+(x) \cap M \neq \emptyset.$$

COROLLARY 2.2.4. *The hypotheses are those of Theorem 2.2.2, except for the last one which is replaced by the condition that M is not a weak λ_n-attractor for any n. Then the conclusions of Theorem 2.2.2 hold unchanged if the points x_n are chosen in A, and such that the sets $L^+_{\lambda_n}(x_n)$ do not intersect M.*

One of the most important potential consequences of Theorem 2.2.2 concerns extracritical transitions from stable attractors to saddle sets. These produce bifurcations unless all outside orbits of the saddle set M have positive limit sets on M.

We conclude this section by the following example concerning a family of differential systems in the plane.

EXAMPLE. Consider the family of dynamical systems in $\mathbf{R}^2$ given by the equations

$$\dot{x} = y, \; \dot{y} = -x + y^3(\lambda - x^2).$$

Here the eigenvalues of the linear part are constant and equal to $\pm i$. (The parameter λ acts only on the nonlinear terms). The linear part therefore gives no information about the behaviour of the system for any value of λ. Use of the Lyapunov function $V(x, y) = \frac{1}{2}(x^2 + y^2)$ however yields

$$\dot{V} = y^4(\lambda - x^2).$$

For $\lambda \leq 0$, applying the LaSalle invariance principle (or the Barbashin-Krasovskii theorem), we find that the origin is globally asymptotically stable.

In the case where $\lambda > 0$, we have $\dot{V} > 0$ for $|x| < \sqrt{\lambda}$, $y \neq 0$. Using again the invariance principle, we find that the origin is completely unstable. Considering that there are no equilibrium points except the origin, we conclude from Theorem 2.2.3 and the Poincaré-Bendixson theorem that, for every $\lambda > 0$, there exists at least one stable limit cycle, and that these shrink to the origin as $\lambda \to 0$.

3. Persistence of instability and bifurcations arising from unstable invariant sets.

3.1. Persistence of instability. We will first consider the case of a dynamical system, defined on a locally compact metric space X, and a compact invariant set M which we assume to be unstable. In order to specify the type of instability, we use the prolongation $D^+(M)$ of M defined as follows:

$$\begin{aligned} D^+(x) :=& \{y \in X \mid \exists x_n \to x, \quad y_n \to y : y_n \in \gamma^+(x_n)\} \\ =& \cap \{\overline{\gamma^+(U)} \mid U \in \mathcal{V}_x\}, \\ D^+(M) :=& \cup \{D^+(x) \mid x \in M\} \end{aligned}$$

and the set

$$I^+(M) := D^+(M) \backslash M,$$

which we will call *set of instability* of M (*region* of instability [3], though in general it is not a region; *unstable manifold* in the case of hyperbolic saddle points or sets, although it is actually stable). $I^+(M)$ is the set of all points which can be approximated from points arbitrarily close to M.

PROPOSITION 3.1.1. *[Ura] The set M is stable if and only if*

$$I^+(M) = \emptyset.$$

We consider a family of dynamical systems depending on a parameter λ and denote the corresponding sets by a subscript λ. It will be assumed that M is compact and λ-invarant for all λ. Under these assumptions we have the following:

PERSISTENCE PRINCIPLE FOR INSTABILITY. *For any compact set C, $\lambda_0 \in \Lambda$, and $\epsilon > 0$, there exists a neighbourhood N of λ_0 such that*

$$\gamma_\lambda^+(B_\epsilon(M)) \quad \textit{is } \epsilon\textit{-dense in } I_{\lambda_0}^+(M) \cap C, \quad \textit{for any } \lambda \in N.$$

An inmediate consequence of the ϵ-*density* property is the following inclusion:

$$B_\epsilon(I_\lambda^+(B_\epsilon(M)) \supset I_{\lambda_0}^+(M) \cap C. \tag{3.1}$$

This gives rise to a succinct formulation of the persistence principle. Associating to every ϵ a $\lambda(\epsilon)$ such that (3.1) holds, *the set-valued function of ϵ, $I_{\lambda(\epsilon)}^+(B_\epsilon(M))$, is lower semicontinuous at $\epsilon = 0$.*

EXAMPLE 1. $\dot{x} = x(x - \lambda)(1 - \lambda - x), \; M = \{0\}$.

Here, $I_0^+(0) = (0,1]$ and $I_\lambda^+(B_\lambda(0)) = (\lambda, 1-\lambda]$ $(\lambda > 0)$, which is a continuous set-valued function, while $I_\lambda^+(0) = (0,\lambda]$, for $\lambda > 0$, which fails to be lower semicontinuous at $\lambda = 0$. In this example, one may take $\lambda(\epsilon) = \epsilon$.

EXAMPLE 2. $\dot{x} = x^2((x-1)^2 + \lambda)$, $M = \{0\}$.
Here $I_\lambda^+(0) = (0,1]$, for $\lambda = 0$, and $\mathbf{R}^+$ for $\lambda > 0$. The set of instability "explodes" when passing from $\lambda = 0$ to $\lambda > 0$.

We now consider the case of a *semidynamical system on an arbitrary metric space*. Here one cannot work with the set of instability of an unstable invariant set M, because it may be empty. Instead, one has to use the semiorbits themselves. The corresponding persistence principle can then be formulated as follows.

General persistence principle for instability. *Let M be an unstable compact invariant set and C an arbitrary compact set. Then*

$$(\forall \epsilon > 0)(\exists N \in \mathcal{N}_{\lambda_0})(\forall \lambda \in N) \;\; B_\epsilon(\gamma_\lambda^+(B_\epsilon(M))) \supset \gamma_{\lambda_0}^+(B_\epsilon(M)) \cap C; \qquad (3.2)$$

in other words, $\gamma_\lambda^+(B_\epsilon(M))$ is ϵ-dense in $\gamma_{\lambda_0}^+(B_\epsilon(M)) \cap C$.

Considering that instability of a small neighbourhood, from a practical point of view, is like instability of the set itself, (3.2) means that *instability cannot suddenly diminish, though it can suddenly increase*, as the second above example shows.

3.2. Bifurcations arising from unstable compact invariant sets. We consider a family F_Λ of semidynamical systems on an arbitrary metric space X endowed with a compact set M which is λ-invariant for all $\lambda \in \Lambda$. Throughout this section we will assume that M is λ_0-unstable, but not necessarily completely unstable, and that for certain values $\lambda_n (\neq \lambda_0)$, $\lambda_n \to \lambda_0$, M is λ_n-stable. We want to find out under what further conditions an extracritical bifurcation of M arises at λ_0.

It will be assumed that the systems F_{λ_n} are asymptotically compact (AC) on certain neighbourhoods of M (which is reasonable because M is λ_n-stable). Then we may limit ourselves to the case where M is λ_n-asymptotically stable, since in the opposite case, every neighbourhood of M would contain λ_n-limit points outside of M. Actually, every neighbourhood would contain entire λ_n-limit sets disjoint from M, because the case of weak attraction (as defined in section 2.2) without attraction is excluded by λ_n-stability of M. (Stable weak attractors are attractors [9].) We may then choose neighbourhoods U_n such that each U_n contains a λ_n-limit set disjoint from M and shrinks to M as $n \to \infty$. Since these limit sets are compact (because of the AC property) and positively λ_n-invariant [9], they constitute an extracritical bifurcation of M (as defined in section 2.2).

So the only dubious case, on which we may therefore concentrate our attention, is the transition

$$\lambda_0 - \textit{unstable} \quad \rightarrow \quad \lambda_n - \textit{asymptotically stable.}$$

In order to be able to formulate a criterion for the occurrence of an extracritical bifurcation, we have to introduce the following concepts:

DEFINITION 3.2.1. The set M is *equistable* (ES) for the family $\{F_\lambda \mid \lambda \in \Lambda' \subset \Lambda\}$ if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall \lambda' \in \Lambda'$, $x \in B_\delta(M) \Rightarrow \gamma^+_{\lambda'}(x) \subset B_\epsilon(M)$.

DEFINITION 3.2.2. The family of attractors $(M, \Lambda', F_{\Lambda'}, A_{\Lambda'})$, where $\Lambda' \subset \Lambda$, M is a λ'-attractor $(\forall \lambda' \in \Lambda')$, $A_{\lambda'}$ is the region of λ'-attraction, is *relatively equistable (RES)* if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall \lambda' \in \Lambda') \quad \gamma^+_{\lambda'}(B_\delta(M) \cap A_{\lambda'}) \subset B_\epsilon(M).$$

A simple example where the condition RES, but not ES, holds is $\dot{x} = x(x^2 - \lambda)$, $\dot{y} = -y$. The origin is a saddle point for $\lambda = 0$ and a stable node for $\lambda > 0$. We refer to this situation as *saddle-node transition.*

DEFINITION 3.2.3. The family of attractors $(M, \Lambda', F_{\lambda'}, A_{\Lambda'})$ is *connected relatively equistable (CRES)* if

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall \lambda' \in \Lambda') \ \gamma^+_{\lambda'}(A^*_{\lambda',\delta}) \subset B_\epsilon(M),$$

where $A^*_{\lambda',\delta}$ denotes the component of $B_\delta(M) \cap A_{\lambda'}$ which contains M (assumed to be connected).

We give the following example of a family of attractors which is CRES but not RES (in polar coordinates):

$$\dot{r} = r(1-r), \ \dot{\theta} = \sin\frac{\theta}{2}\sin\frac{\theta - \lambda}{2} \quad (\textit{for } r > 0).$$

The point $r = 1$, $\theta = 0$ is asymptotically stable and attracts the whole plane except the ray $\theta = \lambda$ (including the origin). We denote the attractor by p_0 and fix two constants $\epsilon > 0$ and $\delta > 0$. For λ small, some of the semiorbits issuing from $B_\delta(p_0) \cap A_\lambda$ do not remain within $B_\epsilon(p_0)$. However, the semiorbits starting in $A^*_{\lambda,\delta}$ (which lies below the ray $\theta = \lambda$) do remain within $B_\epsilon(p_0)$.

We finally give an example of a family of attractors which fails to be CRES. Consider the family of linear systems

$$\dot{y} = z, \ \dot{z} = -\lambda y - \lambda z \quad (y, z \in \mathbf{R};\ \lambda \in \mathbf{R}^+).$$

For small positive values of λ, the origin is a stable focus. Moreover, routine calculations reveal that as $\lambda \to 0$, the spirals flatten out in the direction of the z-axis, while being elongated in the direction of the y-axis. As a consequence, for any given $\delta > 0$, there exist, for λ sufficiently small, positive λ-semiorbits originating in $B_\delta(o)$ and leaving $B_\epsilon(o)$. Since, for all small $\lambda > 0$, the region of attraction is $\mathbf{R}^2$, which is connected, the sets $A^*_{\lambda,\delta}$ are the balls $B_\delta(o)$. It follows that the origin is not CRES.

We now formulate a sufficient condition for the occurrence of a bifurcation.

THEOREM 3.2.4. *[1] Given a family F_Λ of semidynamical systems defined on a metric space X and suppose the compact connected set $M \subset X$ is λ-invariant for all $\lambda \in \Lambda$. Let the following hypotheses be satisfied:*

(i) The space X is locally connected.

(ii) The set M is λ_0-unstable.

(iii) M is a λ-attractor for all λ in a certain set $\Lambda' \subset \Lambda$ such that $\lambda_0 \in \bar{\Lambda}'$. The regions of λ-attraction will be denoted by A_λ.

(iv) The family of attractors $(M, \Lambda', F_{\Lambda'}, A_{\Lambda'})$ is CRES.

(v) For every $\lambda' \in \Lambda'$, $F_{\lambda'}$ is AC on $\bar{A}_{\lambda'}$.

Then M undergoes an extracritical bifurcation at λ_0.

The proof is a direct consequence of the persistence principle for instability.

If the sets $A_{\lambda'}$ are relatively compact, the AC property is redundant.

In the case of the saddle-node transition, the conditions (i) through (v) are obviously satisfied and the existence of the bifurcation is also obvious.

In the example following Definition 3.2.3, the hypotheses of the theorem are satisfied if M is taken as the point $r = 1$, $\theta = 0$, and indeed, the equilibrium points $r = 1$, $\theta = \lambda$ "split off" from M.

In the last example, condition (iv) is not satisfied. Actually, no extracritical bifurcation occurs, because all nontrivial positive semiorbits have diameters which are unbounded as $\lambda \to \lambda_0$.

Theorem 3.2.4. is complemented by the following one.

THEOREM 3.2.5. *In the same general context as in the preceding theorem, we make the following assumptions: (ii), (iii) and (v) [allowing also variation of λ' in the sequence] as before; (iv) is replaced by its opposite:*

(iv') The family of attractors which figures in condition (iv) fails to be CRES.

Then every neighbourhood of M contains a weakly λ_0-invariant (in the case of a dynamical system, invariant) set not contained in M.

(A set is called *weakly invariant* if through each of its points there exists an orbit which lies entirely in the set; see [9].)

In this case we say that there occurs a *critical weak bifurcation*.

The proof of the theorem is based on an argument which forms the core of Ważewski's topological method.

The last example given above falls into the range of this theorem: All the hypotheses are satisfied, and actually, for $\lambda = 0$, a critical (or "vertical") bifurcation occurs, because the system reduces to $\dot{x} = y$, $\dot{y} = 0$.

The last two theorems, together with the introductory remark of this section, can be summarized as follows.

COROLLARY. *In the general context of Theorem 3.2.4, including conditions (i) and (v) and assuming that M be isolated from (weakly) λ_0-invariant sets, an extracritical gain of stability of M at λ_0 results in an extracritical bifurcation of M at λ_0.*

An application of the theory presented here to a semidynamical system on an infinite-dimensional space (argument-delayed differential system) can be found at the end of the paper [11].

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