

**MULTIDIMENSIONAL PSEUDO-DYNAMICAL
AND DYNAMICAL SYSTEMS
ON METRIC SPACES I**

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Introduction. Classical dynamical systems based on the qualitative theory of ordinary differential equations have been recently generalized in many directions. First of all, many authors consider very general spaces playing the role of the classical “phase space” (in the case of dynamical systems induced by ordinary differential equations it was \mathbf{R}^n). Secondly, the group $(\mathbf{R}, +)$ is replaced by more general ones (including semi-groups instead of groups). Such a situation produces almost necessarily variety of diversities in the terminology. The author of the present paper used in [2] the following general definitions: if X is a non-empty set (called *space*), $(G, +)$ is an Abelian semi-group with the neutral element 0, π is a mapping from $G \times X$ into X , such that $\pi(0, x) = x$ for all $x \in X$ and $\pi(t, \pi(s, x)) = \pi(t + s, x)$ for $s, t \in G$, $x \in X$, then the triplet $(X, G; \pi)$ is said to be a *pseudo-dynamical semi-system*. If $(G, +)$ is a group, then $(X, G; \pi)$ is a *pseudo-dynamical system*; if G is a topological semi-group (group), X is a topological space and π is continuous, then $(X, G; \pi)$ is a *dynamical semi-system (dynamical system)*. It is known that classical examples and applications motivating many authors are related to the cases $G = \mathbf{R}$, $G = \mathbf{R}_+ (= [0, \infty))$ and $G = \mathbf{Z}$. The first case appears for instance in dynamical systems induced by autonomous differential equations.

The purpose of the present paper is to discuss the case of $(G, +)$ being the additive group $(\mathbf{R}^m, +)$ (with the natural addition: for $t = (t_1, \dots, t_m)$, $s = (s_1, \dots, s_m) \in \mathbf{R}^m$, $t + s := (t_1 + s_1, \dots, t_m + s_m)$); some remarks concerning the case $G = \mathbf{R}_+^m (= [0, \infty)^m)$ will be also given.

All definitions introduced below (in particular, definitions of: trajectories, motions, limit sets) are direct generalizations of the classical ones (see for instance [1]) and – simultaneously – some of them are special cases of definitions proposed in [2] and [3]. In Sec. 3 there is given a direct generalization of the well known theorem of connectedness of compact limit sets in classical dynamical systems $(X, \mathbf{R}; \pi)$ (see for instance [1], Th. 3.6 in Chapter II). Results from the paper [4] are extended in Sec. 5. Asymptotic periodicity of motions, discussed in Sec. 6, is an extension of the notion introduced in [5] (for some generalizations see [6]).

The main results of the paper have been presented during the poster session in Sec. 12 (August 4, 1994) of the International Congress of Mathematicians in Zürich (see [7]).

Examples of systems $(X, \mathbf{R}^m; \pi)$ and applications of the theory presented in this paper will be given separately. Here we will propose only one example of a dynamical system $(X, \mathbf{R}^2; \pi)$ induced by a hiperbolic partial differential equation of the second order and another one being a composition of two systems of the type $(X, \mathbf{R}; \pi)$ (see the last section).

1. Preliminaries. Let (X, ρ) be a metric space. Consider the additive group $(\mathbf{R}^m, +)$ and a mapping

$$(0.1) \quad \pi : \mathbf{R}^m \times X \longrightarrow X.$$

We say (compare Introduction above) that $(X, \mathbf{R}^m; \pi)$ is a *pseudo-dynamical system* if

- (i) $\pi(0, x) = x$ for every $x \in X$,
- (ii) $\pi(t, \pi(s, x)) = \pi(t + s, x)$ for all $t, s \in \mathbf{R}^m$, $x \in X$.

The spaces X and \mathbf{R}^m will be considered as topological spaces with the natural topologies, induced by the metric ρ in X and by the Euclidean norm $\|\cdot\|$ in \mathbf{R}^m ; in $\mathbf{R}^m \times X$ we will consider of course the natural product topology.

If the mapping π is continuous and satisfies (i)–(ii) then the triplet $(X, \mathbf{R}^m; \pi)$ is said to be a *dynamical system*.

For $x \in X$ we denote by π^x the mapping

$$(1.1) \quad \mathbf{R}^m \ni t \longmapsto \pi^x(t) := \pi(t, x) \in X$$

and we call it the *motion of x* .

For $t \in \mathbf{R}^m$ we denote by π_t the mapping

$$(1.2) \quad X \ni x \mapsto \pi_t(x) := \pi(t, x) \in X$$

and we call it the *t-translation*.

For D being a nonempty subset of \mathbf{R}^m and $x \in X$ we will denote by $\pi(D, x)$ the set

$$(1.3) \quad \{\pi(t, x) : t \in D\}$$

and we will call it the *D-trajectory of x*; sometimes we will write $\pi_D(x)$ instead of $\pi(D, x)$. If $D = \mathbf{R}^m$ then we write $\pi(x)$ instead of $\pi(\mathbf{R}^m, x)$ and we call it the *trajectory of x*.

If $D \subseteq \mathbf{R}^m$, $D \neq \emptyset$, $M \subseteq X$, $M \neq \emptyset$, then

$$(1.4) \quad \pi(D, M) := \bigcup \{\pi(D, x) : x \in M\}$$

and for $D = \mathbf{R}^m$ we write $\pi(M)$ instead of $\pi(\mathbf{R}^m, M)$, so

$$(1.5) \quad \pi(M) = \bigcup \{\pi(x) : x \in M\}.$$

We put also

$$(1.6) \quad \pi(D, \emptyset) := \emptyset \text{ for } D \subseteq \mathbf{R}^m \text{ and } \pi(\emptyset, M) := \emptyset \text{ for } M \subseteq X.$$

If C and D are subsets of \mathbf{R}^m , $C \neq \emptyset$, then the set C is said to be *D-invariant* if and only if

$$(1.7) \quad [t \in C, s \in D] \Leftrightarrow (t + s) \in C.$$

Observe that every set is \emptyset -invariant as well as $\{0\}$ -invariant.

If $Y \subseteq X$, $D \subseteq \mathbf{R}^m$ then the set Y is said to be *D-invariant* if and only if

$$(1.8) \quad \pi(D, Y) \subseteq Y.$$

It is obvious that every subset Y of X is trivially \emptyset -invariant as well as $\{0\}$ -invariant; the empty set is trivially *D-invariant* for every $D \subseteq \mathbf{R}^m$.

CONDITION (A). Let G be a nonempty and unbounded subset of \mathbf{R}^m . We say that G satisfies the Condition (A) if and only if for every two sequences $\{t^n\}$ and $\{s^n\}$ of elements of G , such that

$$(*) \quad \|t^n\| \rightarrow \infty \text{ and } \|s^n\| \rightarrow \infty \text{ as } n \rightarrow \infty$$

there is a sequence $\{\phi^n\}$ of continuous functions

$$\phi^n: [0, 1] \rightarrow G$$

such that

$$\phi^n(0) = t^n, \quad \phi^n(1) = s^n \quad \text{for every } n$$

(shortly: there is a sequence $\{\phi^n\}$ of arcs joining corresponding points of $\{t^n\}$ and $\{s^n\}$, contained in G), and

$$\|\phi^n(\tau)\| \rightarrow \infty \quad \text{as } n \rightarrow \infty, \quad \text{uniformly in } \tau \in [0, 1].$$

CONDITION (C). We say that a nonempty subset G of \mathbf{R}^n satisfies the Condition (C) if and only if :

$$\{t = (t_1, \dots, t_m), s = (s_1, \dots, s_m) \in G, r = (r_1, \dots, r_m) \in \mathbf{R}^m, t_i \leq r_i \leq s_i\} \\ \implies r \in G.$$

EXAMPLES OF SETS SATISFYING THE CONDITIONS (A) AND (C).

The sets

- (j) $G = \mathbf{R}^m, G_o = [0, \infty)^m,$
- (jj) $G_1 = [a_1, b_1] \times \dots \times [a_k, b_k] \times (-\infty, b_{k+1}] \times \dots \times (-\infty, b_m]$ (with $k < m$),
- (jjj) $G_2 = [a_1, b_1] \times \dots \times [a_k, b_k] \times [a_{k+1}, \infty) \times \dots \times [a_m, \infty)$ (with $k < m$),
- (jv) $G_3 = \{(\xi_1, \xi_2) : \xi_2 > \xi_1^2, \xi_1 < 0\}$ for $m = 2$,
- (v) $G_4 = \{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 \leq r^2, z \geq a\}$ (with some fixed $r > 0$ and $a \in \mathbf{R}$),
- (j) $G_5 = [0, \infty)^m \setminus [0, 1]^m$

satisfy the Conditions (A) and (C).

The set

$$(vj) \mathbf{R}^m \setminus \{(x_1, \dots, x_m) : x_1^2 + \dots + x_m^2 < 1\}$$

satisfies the Condition (A) but it does not satisfy the Condition (C), while in \mathbf{R}^2 the set

$$(vjij) (-\infty, 0] \times [0, \infty) \cup [0, \infty) \times (-\infty, 0]$$

satisfies the Condition (C) but it does not satisfy the Condition (A).

REMARK 1.1. Replacing the group $(\mathbf{R}^m, +)$ by the semi-group $(\mathbf{R}_+^m, +)$ and keeping the assumptions (i)–(ii) for the mapping π we will get a *pseudo-dynamical semi-system* (compare the general terminology mentioned in the Introduction). Natural modifications of all definitions introduced above for sytems can be introduced with respect to semi-systems; suitable modifications of theorems given below are true with respect to semi-systems.

PROPOSITION 1.1. Assume that $(X, \mathbf{R}^m; \pi)$ is a dynamical system. If W is a closed subset of X , $x \in X$, $\phi: [0, 1] \rightarrow \mathbf{R}^m$ is continuous and such that $\pi(\phi(0), x) \in X \setminus W$, $\pi(\phi(1), x) \in W$, then there is a $\tau \in (0, 1)$ such that $\pi(\tau, x) \in \partial W$ (=the boundary of W).

An elementary proof will be omitted.

2. Limit sets. Let $(X, \mathbf{R}^m; \pi)$ (with (X, ρ) being a metric space) be a pseudo-dynamical system fixed throughout this section.

Suppose that G is a nonempty and unbounded subset of \mathbf{R}^m . For $x \in X$ we put

$$(2.1) \quad \Lambda_G(x) := \left\{ y \in X : \text{there is a sequence } \{t^n\} \text{ of elements of } G \text{ such} \right. \\ \left. \text{that } \|t^n\| \rightarrow \infty \text{ and } \pi(t^n, x) \rightarrow y \text{ as } n \rightarrow \infty. \right\}$$

The set (2.1) is called the G -limit set of x .

Let now K and P be two disjoint subsets of the set of integers $\{1, \dots, m\}$, such that $K \cup P \neq \emptyset$ (so: $K = \{i_1, \dots, i_k\}$, $P = \{j_1, \dots, j_p\}$ with $k + p \leq m$, $i_r \neq j_s$ for every $r \in \{1, \dots, k\}$, $s \in \{1, \dots, p\}$ or $K = \emptyset$ or $P = \emptyset$ but at least one of these sets is not empty).

For $G \subseteq \mathbf{R}^m$, $G \neq \emptyset$ and $x \in X$ we put

$$(2.2) \quad \Lambda_G[(+; K), (-; P)](x) := \left\{ y \in X : \text{there is a sequence } \{t^n\} \right. \\ \left. \text{of elements of } G \text{ such that } t_j^n \rightarrow \infty \text{ for } j \in K, \right. \\ \left. t_j^n \rightarrow -\infty \text{ for } j \in P, \pi(t^n, x) \rightarrow y \text{ as } n \rightarrow \infty \right\}.$$

The set (2.2) is called $(K$ -positive)- $(P$ -negative)- G -limit set of y .

If $P = \emptyset$ or $K = \emptyset$ then we write $\Lambda_G[(+; K)](x)$ or $\Lambda_G[(-; P)](x)$ instead of $\Lambda_G[(+; K), (-; \emptyset)](x)$ or $\Lambda_G[(+; \emptyset), (-; P)](x)$, calling these sets $(K$ -positive)- G -limit set of x and $(P$ -negative)- G -limit set of x respectively.

REMARK 2.0. If $G = \mathbf{R}^m$ and $K, P \subseteq \{1, \dots, m\}$ are given, then considering the set (2.2) we assume implicitly that if $K \neq \emptyset$ ($P \neq \emptyset$) then there is at least one sequence $\{t^n\}$ of elements of G such that $t_k^n \rightarrow \infty$ for $k \in K$ ($t_p^n \rightarrow \infty$ for $p \in P$) as $n \rightarrow \infty$, and so G must be unbounded. This assumption will be understood as supposed automatically when the set (2.2) is considered; otherwise the definition (2.2) would be in some cases without any sense (this however does not exclude such cases in which for some x the set (2.2) can be empty).

It is not difficult to observe that instead of $\Lambda_G[(+; K), (-; P)](x)$ one can consider $\Lambda_H(x)$ with a suitable subset H of G and so all reasoning presented below with respect to limit sets of the type (2.2) can be reduced to corresponding reasoning applied to limit sets of the type (2.1). There are however some conditions, having sense only with respect to limit sets of the type (2.2); they seem to be useful as direct generalizations of classical ones considered in dynamical systems $(X, \mathbf{R}; \pi)$, with natural geometrical interpretation. Further comments and observations will be collected in the following

REMARK 2.1. If $m = 1$, then we have an ordinary pseudo-dynamical system $(X, \mathbf{R}; \pi)$. For $G = \mathbf{R}_+$ the set $\Lambda_G[(+; \{1\})](x)$ is the positive limit set $\Lambda^+(x)$ (called by some authors the ω -limit set of x); the negative limit set $\Lambda^-(x)$ (called also the α -limit set of x) is in our case equal to $\Lambda_H[(-; \{1\})](x)$ with $H := \{t \in \mathbf{R} : t \leq 0\}$. It is clear that in this case for every $x \in X$: $\Lambda_G[(+; \{1\})](x) = \Lambda_{\mathbf{R}}[(+; \{1\})](x) = \Lambda_G(x)$ and $\Lambda_H[(-; \{1\})](x) = \Lambda_{\mathbf{R}}[(-; \{1\})](x) = \Lambda_H(x)$. It is also clear that in this case $\Lambda_{\mathbf{R}}(x) = \Lambda_G(x) \cup \Lambda_H(x)$.

PROPOSITION 2.1. Assume that G and D are subsets of \mathbf{R}^m , $G \neq \emptyset$, G is unbounded, G is D -invariant. Let $x \in X$ be given. Suppose that for every $t \in D$ the t -translation π_t is continuous at every $y \in \Lambda_G(x)$.

Then the G -limit set $\Lambda_G(x)$ is D -invariant.

PROOF. If D is empty then the assertion is trivially true. Assume that $D \neq \emptyset$ and take a $t \in D$. Let $y \in \Lambda_G(x)$ be given. There exists a sequence $\{t^n\}$ of elements of G such that

$$\|t^n\| \longrightarrow \infty \quad \text{and} \quad \pi(t^n, x) \longrightarrow y, \quad \text{as } n \longrightarrow \infty.$$

We have

$$(2.3) \quad \pi(t, \pi(t^n, x)) = \pi(t + t^n, x) \quad \text{and} \quad t + t^n \in G \quad \text{for every } n$$

and

$$(2.4) \quad \|t + t^n\| \longrightarrow \infty \quad \text{as } n \longrightarrow \infty.$$

Moreover

$$(2.5) \quad \pi(t, \pi(t^n, x)) \longrightarrow \pi(t, y)$$

since π_t is continuous at every point of $\Lambda_G(x)$.

Thus (compare (2.3))

$$\pi(t + t^n, x) \longrightarrow \pi(t, y) \quad \text{as } n \longrightarrow \infty,$$

which means that $\pi(t, y)$ belongs to $\Lambda_G(x)$. □

PROPOSITION 2.2. *Let $G \subseteq \mathbf{R}^m$ be nonempty and unbounded. For every $x \in X$ the G -limit set $\Lambda_G(x)$ is closed.*

PROOF. Suppose that $y = \lim y_n$, $y_n \in \Lambda_G(x)$. We have to show that y belongs to $\Lambda_G(x)$.

For every n there is a sequence $\{t_n^k\}$ of elements of G such that

$$(2.6) \quad y_n = \lim \pi(t_n^k, x) \text{ as } k \rightarrow \infty$$

and

$$(2.7) \quad \|t_n^k\| \rightarrow \infty \text{ as } k \rightarrow \infty.$$

Thus for every n there is $k(n)$ such that

$$(2.8) \quad \rho(y_n, \pi(t_n^k, x)) < \frac{1}{n} \text{ for } k \geq k(n)$$

and

$$(2.9) \quad \|t_n^k\| > n \text{ for } k \geq k(n).$$

Put now

$$s^n := t_n^{k(n)} \quad \text{for } n = 1, 2, \dots$$

We have (compare (2.8) and (2.9))

$$\rho(y_n, \pi(s^n, x)) < \frac{1}{n} \text{ and } \|s^n\| > n \text{ for every } n.$$

The triangle inequality gives

$$\rho(y, \pi(s^n, x)) \leq \rho(y, y_n) + \rho(y_n, \pi(s^n, x)) \leq \rho(y, y_n) + \frac{1}{n}$$

and so

$$\pi(y, \pi(s^n, x)) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Since $\|s^n\| \rightarrow \infty$, we get $y \in \Lambda_G(x)$. □

PROPOSITION 2.3. *Assume that $G, D \subseteq \mathbf{R}^m$, $G \neq \emptyset$, G is unbounded, G is D -invariant, x is a given point of X . Let K and P be two disjoint subsets of $\{1, \dots, m\}$ such that $K \cup P \neq \emptyset$. Assume that π_t is continuous at every $y \in \Lambda_G[(+; K), (-; P)](x)$. Then the set $\Lambda_G[(+; K), (-; P)](x)$ is D -invariant.*

PROOF. Similarly as in the proof of Proposition 2.1 we fix a $t \in D$ ($t = (t_1, \dots, t_m)$) and taking for a given $y \in \Lambda_G[(+; K), (-; P)](x)$ a suitable sequence $\{t^n\} = \{(t_1^n, \dots, t_m^n)\}$, we repeat the reasoning presented in that proof, with the only one change: in the place of (2.4) we observe that

$$t_k + t_k^n \rightarrow \infty \text{ if } k \in K \quad \text{and} \quad t_p + t_p^n \rightarrow -\infty \text{ if } p \in P.$$

□

PROPOSITION 2.4. Assume that $G \subseteq \mathbf{R}^m$, $G \neq \emptyset$, $K, P \subseteq \{1, \dots, m\}$, $K \cup P \neq \emptyset$, $K \cap P = \emptyset$. For every $x \in X$ the set $\Lambda_G[(+; K), (-; P)](x)$ is closed.

PROOF. We repeat the reasoning used in the proof of Proposition 2.2 with the only one change: instead of (2.9) we use the inequalities

$$(2.11) \quad (t_n^k)_j > n \text{ for } j \in K, k \geq k(n)$$

and

$$(2.12) \quad (t_n^k)_p < -n \text{ for } p \in P, k \geq k(n).$$

□

REMARK 2.2. It is obvious that if $G \subseteq H$ then $\Lambda_G(x) \subseteq \Lambda_H(x)$ and $\Lambda_G[(+; K), (-; P)](x) \subseteq \Lambda_H[(+; K), (-; P)](x)$ for every $K, P \subseteq \{1, \dots, m\}$ such that $K \cup P \neq \emptyset$, $K \cap P = \emptyset$ and each $x \in X$.

PROPOSITION 2.5. Let G and H be two nonempty and unbounded subsets of \mathbf{R}^m . Then for every $x \in X$

$$(2.13) \quad \Lambda_{G \cup H}(x) \subseteq \Lambda_G(x) \cup \Lambda_H(x).$$

An elementary proof will be omitted.

PROPOSITION 2.6. Assume that $x \in X$ is such that the motion π^x is continuous. Suppose that $G \subseteq \mathbf{R}^m$, $G \neq \emptyset$, G is unbounded. Then

$$(2.14) \quad \pi(G, x) \cup \Lambda_G(x) \subseteq \overline{\pi(\bar{G}, x)} \subseteq \pi(\bar{G}, x) \cup \Lambda_G(x).$$

PROOF. The first inclusion is obvious. In order to prove the second one take a $y \in \overline{\pi(\bar{G}, x)}$. There is a sequence $\{y_n\}$ of elements of $\pi(G, x)$ convergent to y . Every y_n is equal to $\pi(t^n, x)$ with some t^n belonging to G . If $\{\|t^n\|\}$ is bounded then without loss of generality we may assume that $\{t^n\}$ tends to some $t^* \in \bar{G}$ and so $\{\pi(t^n, x)\}$ tends to $\pi(t^*, x) \in \pi(\bar{G}, x)$. If $\{\|t^n\|\}$ is unbounded, then without loss of generality we may assume that $\|t^n\| \rightarrow \infty$; thus $y \in \Lambda_G(x)$. □

COROLLARY 2.1. If $G \subseteq \mathbf{R}^m$, $G \neq \emptyset$, G is unbounded and closed, then

$$(2.15) \quad \pi_G(x) \cup \Lambda_G(x) = \overline{\pi_G(x)}$$

for every $x \in X$ such that the motion π^x is continuous.

COROLLARY 2.2. *If $(X, \mathbf{R}^m; \pi)$ is a dynamical system, $G \subseteq \mathbf{R}^m$, $G \neq \emptyset$, G is unbounded and closed, then the equality (2.15) is true for every $x \in X$.*

The above Corollary 2.2 generalizes directly well known results concerning closures of trajectories and semi-trajectories in the classical theory of dynamical systems $(X, \mathbf{R}; \pi)$:

$$\overline{\pi(x)} = \pi(x) \cup \Lambda(x), \quad \overline{\pi_+(x)} = \pi_+(x) \cup \Lambda^+(x), \quad \overline{\pi_-(x)} = \pi_-(x) \cup \Lambda^-(x)$$

(see for instance [1]).

PROPOSITION 2.7. *Let G and H be two nonempty and unbounded subsets of \mathbf{R}^m , K and P be two subsets of $\{1, \dots, m\}$ such that $K \cap P = \emptyset$, $K \cup P \neq \emptyset$. Then for every $x \in X$*

(2.16)

$$\Lambda_{G \cup H}[(+; K), (-; P)](x) = \Lambda_G[(+; K), (-; P)](x) \cup \Lambda_H[(+; K), (-; P)](x).$$

An elementary proof will be omitted.

REMARK 2.3. It is easy to see that if $G \subseteq \mathbf{R}^m$, $G \neq \emptyset$, G is unbounded and $K \subseteq L \subseteq \{1, \dots, m\}$, $P \subseteq R \subseteq \{1, \dots, m\}$, $L \cap R = \emptyset$, $K \cup P \neq \emptyset$, then for every $x \in X$

$$\Lambda_G[(+; L), (-; R)](x) \subseteq \Lambda_G[(+; K), (-; P)](x).$$

In particular, if $K = \{i_1, \dots, i_k\}$, $P = \{j_1, \dots, j_p\}$, then

$$\Lambda_G[(+; K), (-; P)](x) \subseteq \Lambda_G[(+; K)](x) \subseteq \Lambda_G[(+; \{i_r\})](x) \quad \text{for } r \in \{1, \dots, k\}$$

and

$$\Lambda_G[(+; K), (-; P)](x) \subseteq \Lambda_G[(-; P)](x) \subseteq \Lambda_G[(-; \{j_s\})](x) \quad \text{for } s \in \{1, \dots, p\}.$$

Let G be a nonempty and unbounded subset of \mathbf{R}^m and let K, P be nonempty subsets of $\{1, \dots, m\}$. We say that K belongs to the class \mathcal{A}_G^+ (P belongs to the class \mathcal{A}_G^-) if and only if there is a sequence $\{t^n\} = \{(t_1^n, \dots, t_m^n)\}$ of elements of G such that $t_k^n \rightarrow \infty$ for $k \in K$ ($t_p^n \rightarrow -\infty$ for $p \in P$) as $n \rightarrow \infty$.

REMARK 2.4. Using this terminology we can say that Remark 2.0 requires that considering any set of the type (2.2) we assume that K belongs to \mathcal{A}_G^+ (or $K = \emptyset$) and P belongs to \mathcal{A}_G^- (or $P = \emptyset$).

PROPOSITION 2.8. Assume that $G \subseteq \mathbf{R}^m$, $G \neq \emptyset$, G is unbounded and $x \in X$ is such that π^x is continuous. Then

$$(2.17) \quad \left\{ \begin{array}{l} \pi(G, x) \cup \bigcup \{ \Lambda_G[(+; K), (-; P)](x) : \\ \quad K \in \mathcal{A}_G^+, P \in \mathcal{A}_G^-, K \cap P = \emptyset, K \cup P \neq \emptyset \} \subseteq \\ \pi(G, x) \cup \bigcup \{ \Lambda_G[(+; K)](x) : K \in \mathcal{A}_G^+, K \neq \emptyset \} \\ \cup \bigcup \{ \Lambda_G[(-; P)](x) : P \in \mathcal{A}_G^-, P \neq \emptyset \} \subseteq \\ \pi(G, x) \cup \bigcup \{ \Lambda_G[(+; \{k\})](x) : \{k\} \in \mathcal{A}_G^+ \} \\ \cup \bigcup \{ \Lambda_G[(-; \{p\})](x) : \{p\} \in \mathcal{A}_G^- \} \subseteq \\ \overline{\pi(G, x)} \subseteq \\ \pi(\overline{G}, x) \cup \bigcup \{ \Lambda_G[(+; \{k\})](x) : \{k\} \in \mathcal{A}_G^+ \} \\ \cup \bigcup \{ \Lambda_G[(-; \{p\})](x) : \{p\} \in \mathcal{A}_G^- \}. \end{array} \right.$$

PROOF. The first two inclusions are obvious because of Remark 2.3. The third one is also clear since $\pi(G, x) \subseteq \overline{\pi(G, x)}$ and if $y \in \Lambda_G[(+; \{k\})](x)$ for some k such that $\{k\} \in \mathcal{A}_G^+$ or $y \in \Lambda_G[(-; \{p\})](x)$ for some p such that $\{p\} \in \mathcal{A}_G^-$ then $y = \lim \pi(t^n, x)$ with $t^n \in G$, and so $y \in \overline{\pi(G, x)}$. Assume now that $y \in \overline{\pi(G, x)}$; then $y = \lim y_n$, with y_n in $\pi(G, x)$, which means that $y_n = \pi(t^n, x)$ where $t^n \in G$. If $\{\|t^n\|\}$ is bounded then $y \in \pi(\overline{G}, x)$, if this sequence is unbounded then we may assume that there is at least one $j \in \{1, \dots, m\}$ such that $|t_j^n| \rightarrow \infty$, so (taking suitable subsequences, if necessary) we may say that for some $j \in \{1, \dots, m\}$ $t_j^n \rightarrow \infty$ or $t_j^n \rightarrow -\infty$; in the first case y belongs to $\Lambda_G[(+; \{j\})](x)$ in the second one to $\Lambda_G[(-; \{j\})](x)$. \square

COROLLARY 2.3. If $G \subseteq \mathbf{R}^m$, $G \neq \emptyset$, G is unbounded and closed, $x \in X$ is such that π^x is continuous then in (2.17) we have equalities instead of inclusions. In particular if $(X, \mathbf{R}^m; \pi)$ is a dynamical system, $G = \overline{G} \neq \emptyset$ is unbounded, then (2.17) with equalities instead of inclusions holds for every $x \in X$.

REMARK 2.5. Corollaries 2.3 and 2.4 of Proposition 2.8 are in fact also consequences of the following simple observation: if $G \subseteq \mathbf{R}^m$, $G \neq \emptyset$, G is

unbounded, then for every $x \in X$

$$\begin{aligned}
& \bigcup \{ \Lambda_G[(+; K), (-; P)](x) : K \in \mathcal{A}_G^+, P \in \mathcal{A}_G^-, K \cap P = \emptyset, K \cup P \neq \emptyset \} \\
&= \bigcup \{ \Lambda_G[(+; K)](x) : K \in \mathcal{A}_G^+, K \neq \emptyset \} \\
&\quad \cup \bigcup \{ \Lambda_G[(-; P)](x) : P \in \mathcal{A}_G^-, P \neq \emptyset \} \\
&= \bigcup \{ \Lambda_G[(+; \{k\})](x) : \{k\} \in \mathcal{A}_G^+ \} \\
&\quad \cup \bigcup \{ \Lambda_G[(-; \{p\})](x) : \{p\} \in \mathcal{A}_G^- \} = \Lambda_G(x)
\end{aligned}$$

3. Connectedness of compact limit sets. Let $(X, \mathbf{R}^m; \pi)$ be a dynamical system (with (X, ρ) being a metric space) fixed throughout this section.

NOTATION. Let M be a nonempty subset of X . We put for $x \in X$

$$(3.1) \quad \rho(x, M) := \inf \{ \rho(x, y) : y \in M \}$$

and for any fixed positive number η

$$(3.2) \quad B(M, \eta) := \{ y \in X : \rho(y, M) < \eta \}.$$

If $M = \{x\}$ then we write $B(x, \eta)$ instead of $B(\{x\}, \eta)$; it is the usual (open) ball centered at x with the radius η .

Let G be a nonempty and unbounded subset of \mathbf{R}^m .

THEOREM 3.1. *Assume that (X, ρ) is locally compact and that the set G satisfies the Condition (A) (see Sec. 1).*

If $x \in X$ is such that $\Lambda_G(x)$ is not empty and compact, then $\Lambda_G(x)$ is connected.

PROOF. Suppose that $\Lambda_G(x)$ is compact but not connected. There are two sets $C, D \subseteq \Lambda_G(x)$ such that $C \cap D = \emptyset$, $C \cup D = \Lambda_G(x)$, C and D are compact. The space is locally compact, so there is a $\delta > 0$ such that $\overline{B(C, \delta)}$ and $\overline{B(D, \delta)}$ are compact; we may require that these sets are disjoint. Take a $y \in C$ and a $z \in D$. There are sequences $\{s^n\}$ and $\{t^n\}$ of elements of G such that $\|s^n\| \rightarrow \infty$, $\|t^n\| \rightarrow \infty$, $\pi(s^n, x) \rightarrow y$ and $\pi(t^n, x) \rightarrow z$ as $n \rightarrow \infty$. Since the Condition (A) is satisfied, we can find a sequence $\{\phi^n\}$ of arcs joining s^n and t^n , such that $\phi^n(\tau) \in G$ for every τ and every n , and $\|\phi^n(\tau)\| \rightarrow \infty$ uniformly in $\tau \in [0, 1]$. For n sufficiently large

$$(3.3) \quad \pi(s^n, x) \in B(C, \delta) \quad \text{and} \quad \pi(t^n, x) \in B(D, \delta).$$

Without loss of generality we may assume that the relations (3.3) are true for every n . Applying Proposition 1.1 we can find for every n a number τ^n belonging to the interval $(0, 1)$ such that

$$(3.4) \quad \pi(\tau^n, x) \in \partial B(C, \delta)$$

for every n .

The uniform convergence of $\{\|\phi^n\|\}$ to the infinity gives in particular

$$(3.5) \quad \|\phi^n(\tau^n)\| \longrightarrow \infty \text{ as } n \longrightarrow \infty.$$

Since $\partial B(C, \delta)$ is compact we may assume that $\{\pi(\phi^n(\tau^n), x)\}$ converges to some $w \in \partial B(C, \delta)$. It is clear that w belongs to $\Lambda_G(x)$. On the other hand $\partial B(C, \delta) \cap \Lambda_G(x) \neq \emptyset$. This contradiction finishes the proof. \square

THEOREM 3.2. *Assume that (X, ρ) is locally compact, $G \subseteq \mathbf{R}^m$, $G \neq \emptyset$, G is unbounded, G satisfied the Condition (C), $K = \{1, \dots, m\}$.*

If $\Lambda_G[(+; K)](x)$ is nonempty and compact then it is connected.

PROOF. Assume that $\Lambda_G[(+; K)](x)$ is compact but not connected. So there are two disjoint compact sets Y and Z such that their union is equal to $\Lambda_G[(+; K)](x)$. Take a $y \in Y$ and a $z \in Z$. There are sequences $\{t^n\}$ and $\{s^n\}$ of elements of G such that $t_j^n \rightarrow \infty$, $s_j^n \rightarrow \infty$ for all $j \in K$ and $\pi(t^n, x) \rightarrow y$, $\pi(s^n, x) \rightarrow z$ as $n \rightarrow \infty$.

There is an $\varepsilon > 0$ such that the sets $\overline{B(Y, \varepsilon)}$ and $\overline{B(Z, \varepsilon)}$ are compact and disjoint. Without loss of generality we may assume that

$$\pi(t^n, x) \in B(Y, \varepsilon) \text{ and } \pi(s^n, x) \in B(Z, \varepsilon) \text{ for all } n.$$

We may also assume (selecting suitable subsequences, if necessary) that

$$t_j^n \leq s_j^n \leq t_j^{n+1} \text{ for every } j \text{ and every } n$$

requiring moreover that at least for some indices the inequalities are strict. Because of the Condition (C) the set

$$T_n := \left\{ t : t = \{t_1, \dots, t_m\}, t_j^n \leq t_j \leq s_j^n \text{ for all } j \right\}$$

is contained in G for every n .

It is easy to see that for every n there is an $r^n \in T_n$ such that

$$\pi(r^n, x) \in \partial B(Y, \varepsilon).$$

It is clear that $r_j^n \rightarrow \infty$ for every j as $n \rightarrow \infty$. The set $\partial B(Y, \varepsilon)$ is compact, so we may assume that the sequence $\{\pi(r^n, x)\}$ is convergent to some $w \in \partial B(Y, \varepsilon)$. This point w should be also in the set $\Lambda_G[(+; K)](x)$, but $\partial B(Y, \varepsilon) \cap \Lambda_G[(+; K)](x) = \emptyset$. \square

THEOREM 3.3. *Assume that (X, ρ) is locally compact, G is a nonempty and unbounded subset of \mathbf{R}^m , G satisfies the Condition (C), $P = \{1, \dots, m\}$. If $\Lambda_G[(-; P)](x)$ is nonempty and compact then it is connected.*

The proof is similar ("symmetric") to that of Theorem 3.2. \square

4. Semi-stability and stability of motions. Let $(X, \mathbf{R}^m; \pi)$ be a pseudodynamical system (with (X, ρ) being a metric space) fixed throughout this section.

Suppose that G is a nonempty and unbounded subset of \mathbf{R}^n and Y is a subset of X . Let $x \in \bar{Y}$ be given.

We say that the motion π^x is *G-semi-stable in Y* if and only if for every $\varepsilon > 0$ there are a $\delta > 0$ and an $r > 0$ such that

$$(4.1) \quad \left\{ y \in B(x, \delta) \cap Y, t \in G, \|t\| > r \right\} \implies \left\{ \rho(\pi(t, x), \pi(t, y)) < \varepsilon \right\}.$$

We say that π^x is *G-stable in Y* if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$(4.2) \quad \left\{ y \in B(x, \delta) \cap Y, t \in G \right\} \implies \left\{ \rho(\pi(t, x), \pi(t, y)) < \varepsilon \right\}.$$

If $K, P \subseteq \{1, \dots, m\}$, $K \cap P = \emptyset$, $K \cup P \neq \emptyset$, then π^x is said to be *(K-positively)-(P-negatively)-G-semi-stable in Y* if and only if for every $\varepsilon > 0$ there are a $\delta > 0$ and an $r > 0$ such that

$$(4.3) \quad \left\{ y \in B(x, \delta) \cap Y, t_k \geq r \text{ for } k \in K, t_p \leq -r \text{ for } p \in P \right\} \\ \implies \left\{ \rho(\pi(t, x), \pi(t, y)) < \varepsilon \right\}$$

and is said to be *(K-positively)-(P-negatively)-G-stable in Y* if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$(4.4) \quad \left\{ y \in B(x, \delta) \cap Y, t_k \geq 0 \text{ for } k \in K, t_p \leq 0 \text{ for } p \in P \right\} \\ \implies \left\{ \rho(\pi(t, x), \pi(t, y)) < \varepsilon \right\}.$$

In case $Y = X$ we will omit Y in the names of the above conditions, saying for instance that π^x is *G-semi-stable*, instead of: π^x is *G-semi-stable in X*, etc. If $K = \emptyset$ ($P = \emptyset$) then we will say that π^x is *(P-negatively)-G-stable in Y* (*(K-positively)-G-stable in Y*) instead of: π^x is *(\emptyset -positively)-(P-negatively)-G-stable in Y*, etc.

REMARK 4.1. If (X, ρ) is locally compact, $(X, \mathbf{R}^m; \pi)$ is a dynamical system, $x \in \text{int}(Y)$ then every semi-stability condition is equivalent to the corresponding stability condition, which means in particular that in that case π^x is *G-semi-stable in Y* if and only if it is *G-stable in Y*, etc.

5. Upper semi-continuous dependence of limit sets on points.

Let Z be a topological space. Denote by $\mathcal{P}(Z)$ the family of all nonempty subsets of Z . Let W be a subset of Z . A mapping

$$F: W \ni x \mapsto F(x) \in \mathcal{P}(Z)$$

is said to be *(H)-upper semi-continuous* (upper semi-continuous in the sense of Heine) at a point $w \in W$ if and only if the following implication holds:

$$(5.0) \quad \begin{array}{l} \text{If } \{w_n\}, \{z_n\} \text{ are sequences of elements of } Z, w_n \in W \text{ and} \\ z_n \in F(w_n) \text{ for every } n, z \text{ is an element of } Z, w_n \rightarrow w \\ \text{and } z_n \rightarrow z \text{ as } n \rightarrow \infty, \text{ then } z \in F(w). \end{array}$$

Let $(X, \mathbf{R}^m; \pi)$ be a pseudo-dynamical system (with (X, ρ) being a metric space) fixed throughout this section.

For a nonempty and unbounded subset G of \mathbf{R}^m we put:

$$(5.1) \quad L_G := \left\{ \{x \in X : \Lambda_G(x) \neq \emptyset\} \right\}.$$

THEOREM 5.1. *Suppose that G is nonempty and unbounded in \mathbf{R}^m , Y is a nonempty subset of X .*

Then

- 1). *Every point x belonging to $\overline{Y \cap L_G}$ such that π^x is G -semi-stable in Y , belongs to L_G .*
- 2). *The mapping*

$$L_G \ni x \mapsto \Lambda_G(x) \in \mathcal{P}(X)$$

is (H)-upper semi-continuous at every point $x \in \overline{Y \cap L_G}$ such that the motion π^x is G -semi-stable in Y .

PROOF. In order to prove the both parts of the assertion it is enough to show that the following implication holds

if: $x \in \overline{Y \cap L_G}$, π^x is G -semi-stable in Y , $\{x_n\}$ is a sequence of elements of $\overline{Y \cap L_G}$ tending to x , $\{y_n\}$ is a sequence of elements of X convergent to some $y \in X$ and such that $y_n \in \Lambda_G(x_n)$,

then :

$$y \in \Lambda_G(x).$$

So assume that $x \in \overline{Y \cap L_G}$ is such that π^x is G -semi-stable in Y , $x_n \in \overline{Y \cap L_G}$, $y_n \in X$, $x_n \rightarrow x$, $y_n \rightarrow y$ as $n \rightarrow \infty$ and for all n :

$$(5.2) \quad y_n \in \Lambda_G(x_n).$$

For every n

$$y_n = \lim \pi(t_n^k, x_n) \quad (\text{as } k \rightarrow \infty)$$

where $\{t_n^k\}$ is such a sequence of elements of G that

$$\|t_n^k\| \rightarrow \infty \quad \text{as } k \rightarrow \infty \quad (n = 1, 2, \dots).$$

For every n there is a $k(n)$ such that for $k \geq k(n)$

$$(5.3) \quad \rho(y_n, \pi(t_n^k, x_n)) < \frac{1}{n}$$

and

$$(5.4) \quad \|t_n^k\| \geq n.$$

Let $\varepsilon > 0$ be arbitrarily fixed. We have

$$(5.5) \quad \begin{aligned} \rho(y, \pi(t_n^{k(n)}, x)) &\leq \rho(y_n, y) + \rho(y_n, \pi(t_n^{k(n)}, x_n)) + \rho(\pi(t_n^{k(n)}, x_n), \pi(t_n^{k(n)}, x)) \\ &\leq \rho(y, y_n) + \frac{1}{n} + \rho(\pi(t_n^{k(n)}, x_n), \pi(t_n^{k(n)}, x)). \end{aligned}$$

If n is large enough, say $n \geq n^*$, then

$$\rho(y, y_n) < \frac{1}{3}\varepsilon \quad \text{and} \quad \frac{1}{n} < \frac{1}{3}\varepsilon.$$

Convergence of $\{x_n\}$ to x and G -semi-stability of the motion π^x in Y permit us to find n^{**} such that for $n > n^{**}$

$$\rho(\pi(t_n^{k(n)}, x_n), \pi(t_n^{k(n)}, x)) < \frac{1}{3}\varepsilon.$$

Thus (see (5.5))

$$\rho(y, \pi(t_n^{k(n)}, x)) < \varepsilon \quad \text{for } n \geq \max(n^*, n^{**}).$$

This (by virtue of (5.4)) proves that $y \in \Lambda_G(x)$. □

Let now $P, K \subseteq \{1, \dots, m\}$ be such that $P \cap K = \emptyset$, $P \cup K \neq \emptyset$ and let $G \subseteq \mathbf{R}^m$ be nonempty and unbounded. We put

$$(5.6) \quad L_G[(+; K), (-; P)] := \left\{ x \in X : \Lambda_G[(+; K), (-; P)](x) \neq \emptyset \right\}.$$

THEOREM 5.2. *Suppose that G is a nonempty and unbounded subset of \mathbf{R}^m and Y is a nonempty subset of X .*

Then

- 1). *Every point $x \in \overline{Y \cap L_G[(+; K), (-; P)]}$ such that π^x is $(K\text{-positively})\text{-}(P\text{-negatively})\text{-}G\text{-semi-stable}$ in Y , belongs to $L_G[(+; K), (-; P)]$.*
- 2). *The mapping*

$$L_G[(+; K), (-; P)] \ni x \longmapsto \Lambda_G[(+; K), (-; P)](x) \in \mathcal{P}(X)$$

is (H) -upper semi-continuous at every point x belonging to the set $\overline{Y \cap L_G[(+; K), (-; P)]}$ such that the motion π^x is $(K\text{-positively})\text{-}(P\text{-negatively})\text{-}G\text{-semi-stable}$ in Y .

The proof is based on the same idea as that of Theorem 5.1 and will be omitted. \square

6. Periodic and asymptotically periodic motions. Let $(X, \mathbf{R}^m; \pi)$ (with (X, ρ) being a metric space) be a pseudo-dynamical system fixed throughout this section.

Let $\alpha \in \mathbf{R}^m \setminus \{0\}$ be fixed. We say that a motion π^x is α -periodic if

$$(6.1) \quad \pi(t, x) = \pi(t + \alpha, x) \quad \text{for every } t \in \mathbf{R}^m.$$

It is clear that (6.1) is equivalent to

$$x = \pi(\alpha, x).$$

Let G be a nonempty and unbounded subset of \mathbf{R}^m . We say that π^x is G -asymptotically α -periodic if and only if:

$$(6.2) \quad \begin{cases} \text{for every } \varepsilon > 0 \text{ there is } aT > 0 \text{ such that} \\ \rho(\pi(t + \alpha, x), \pi(t, x)) < \varepsilon \text{ for } t \in G, \|t\| \geq T. \end{cases}$$

Let $K, P \subseteq \{1, \dots, m\}$ be such that $K \cap P = \emptyset$, $K \cup P \neq \emptyset$. We say that π^x is $(K\text{-positively})\text{-}(P\text{-negatively})\text{-}G\text{-asymptotically } \alpha\text{-periodic}$ if and only if:

$$(6.3) \quad \begin{cases} \text{for every } \varepsilon > 0 \text{ there is a } T > 0 \text{ such that} \\ \rho(\pi(t + \alpha, x), \pi(t, x)) < \varepsilon \text{ for } t = (t_1, \dots, t_m) \in G, t_k \geq T \\ \text{for } t \in K \text{ and } t_p \leq -T \text{ for } p \in P. \end{cases}$$

REMARK 6.0. Observations analogous to that presented in Remark 2.0 are applicable here with respect to G -asymptotic periodicity and $(K$ -positive)- $(P$ -negative)- G -asymptotic periodicity.

REMARK 6.1. If π^x is G -asymptotically α -periodic then

$$(6.4) \quad \rho(\pi(t^n + \alpha, x), \pi(t^n, x)) \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

for every sequence $\{t^n\}$ of elements of G such that

$$\|t^n\| \longrightarrow \infty \text{ as } n \longrightarrow \infty$$

and vice versa.

If π^x is $(K$ -positively)- $(P$ -negatively)- G -asymptotically α -periodic then (6.4) holds for every sequence $\{t^n\}$ of elements of G such that

$$t_k^n \longrightarrow \infty \text{ for } k \in K, \quad t_p^n \longrightarrow -\infty \text{ for } p \in P, \text{ as } n \longrightarrow \infty$$

and vice versa.

REMARK 6.2. The above definitions of the G -asymptotic periodicity and $(K$ -positive)- $(P$ -negative)- G -asymptotic periodicity generalize directly the positive and negative asymptotic periodicity introduced in the paper [5]. It is possible to extend in a similar way generalized asymptotic periodicity conditions introduced in the paper [6].

THEOREM 6.1. *Let G be a nonempty and unbounded subset of \mathbf{R}^m and let $\alpha \in \mathbf{R}^m \setminus \{0\}$ be given. Assume that π^x is G -asymptotically α -periodic, $y \in \Lambda_G(x)$, π_α is continuous at the point y .*

Then π^y is α -periodic.

PROOF. There is a sequence $\{t^n\}$ of elements of G such that

$$\pi(t^n, x) \longrightarrow y \text{ and } \|t^n\| \longrightarrow \infty \text{ as } n \longrightarrow \infty.$$

Since π_α is continuous at y , we have

$$\pi(\alpha, x) = \pi(\alpha, \lim \pi(t^n, x)) = \lim \pi(\alpha, \pi(t^n, x)) = \lim \pi(t^n + \alpha, x).$$

So

$$\rho(y, \pi(\alpha, x)) = \rho(\lim \pi(t^n, x), \lim \pi(t^n + \alpha, x))$$

and by virtue of Remark 6.1 we get $\rho(y, \pi(\alpha, y)) = 0$ □

Similarly we can prove the following

THEOREM 6.2. *Let G be a nonempty and unbounded subset of \mathbf{R}^m , K, P be subsets of $\{1, \dots, m\}$ such that $K \cap P = \emptyset$, $K \cup P \neq \emptyset$ and let $\alpha \in \mathbf{R}^m \setminus \{0\}$ be fixed. Assume that π^x is $(K\text{-positive})\text{-}(P\text{-negative})\text{-}G$ -asymptotically α -periodic. If $y \in \Lambda_G[(+; K), (-; P)](x)$ is such that π_α is continuous at y then π^y is α -periodic.*

The proof will be omitted. \square

THEOREM 6.3. *Let G be a nonempty and unbounded subset of \mathbf{R}^m , $\alpha \in \mathbf{R}^m \setminus \{0\}$ be fixed. If $x \in X$, $y \in \Lambda_G(x)$, π^y is G -semi-stable, π_α is continuous, π^y is α -periodic, then π^x is G -asymptotically α -periodic.*

PROOF. Assume that $x \in X$, $y \in \Lambda_G(x)$, π^y is G -semi-stable, π_α is continuous, π^y is α -periodic, but π^x is not G -asymptotically α -periodic. So there exist an $\varepsilon^0 > 0$ and a sequence $\{t^n\}$ of elements of the set G such that

$$(6.5) \quad \|t^n\| \rightarrow \infty \text{ as } n \rightarrow \infty$$

and

$$(6.6) \quad \rho(\pi(t^n + \alpha, x), \pi(t^n, x)) \geq \varepsilon^0 \text{ for every } n.$$

Since $y \in \Lambda_G(x)$ there is a sequence $\{s^n\}$ of elements of G such that

$$(6.7) \quad \|s^n\| \rightarrow \infty \text{ and } \pi(s^n, x) \rightarrow y \text{ as } n \rightarrow \infty.$$

Since π_α is continuous and $\pi(\alpha, y) = y$, we have

$$(6.8) \quad \pi(s^n + \alpha, y) = \pi(\alpha, \pi(s^n, x)) \rightarrow \pi(\alpha, y) = y \text{ as } n \rightarrow \infty.$$

We may assume (without loss of generality) that

$$(6.9) \quad \|t^n - s^n\| \rightarrow \infty \text{ as } n \rightarrow \infty$$

(if not, we can select suitable subsequences).

The condition (6.6) implies

$$(6.10) \quad \rho(\pi(t^n - s^n, \pi(s^n, x)), \pi(t^n - s^n, \pi(s^n + \alpha, x))) \geq \varepsilon^0.$$

On the other hand

$$(6.11) \quad \left\{ \begin{array}{l} \rho(\pi(t^n - s^n, \pi(s^n, x)), \pi(t^n - s^n, \pi(s^n + \alpha, x))) \\ \leq \rho(\pi(t^n - s^n, \pi(s^n, x)), \pi(t^n - s^n, y)) \\ + \rho(\pi(t^n - s^n, y), \pi(t^n - s^n, \pi(\alpha, y))) \\ + \rho(\pi(t^n - s^n, \pi(\alpha, y)), \pi(t^n - s^n, \pi(s^n + \alpha, x))). \end{array} \right.$$

The second addend of the sum on the right-hand side of (6.11) is equal to zero since π^y is α -periodic. The first one is as small as we wish, if n is sufficiently large (because of (6.9) and G -semistability of π^y). Because of the same reason the third addend is as small as wish (recall that $y = \pi(\alpha, y)$ and $y = \lim \pi(s^n + \alpha, x)$). So the left hand side must tend to zero. \square

7. Examples.

I. Let $f : \mathbf{R}^5 \rightarrow \mathbf{R}$ be continuous and such that for every (x^0, y^0) in \mathbf{R}^2 and every C^1 -functions $\alpha, \beta : \mathbf{R} \rightarrow \mathbf{R}$ there is exactly one solution

$$(7.0) \quad u((\cdot, \cdot); (x^0, y^0), \alpha, \beta)$$

of the Darboux problem

$$(7.1) \quad \begin{cases} u_{xy} = f(x, y, u, u_x, u_y) \text{ in } \mathbf{R}^2 \\ u(x, 0) = \alpha(x), u(0, y) = \beta(y) \text{ for } x, y \in \mathbf{R} \end{cases}$$

and the mapping

$$(7.2) \quad (x^0, y^0, \alpha, \beta) \mapsto u(\cdot, \cdot; (x^0, y^0), \alpha, \beta)$$

is continuous (in the topology induced by the Euclidean norm on \mathbf{R}^2 and the uniform convergence on compact sets in suitable function spaces).

Consider now

$$Z := \mathbf{R}^2 \times \{\text{pairs } (\alpha, \beta) \text{ of } C^1\text{-functions from } \mathbf{R} \text{ into } \mathbf{R}\}$$

provided with the metric ρ defined as follows

$$\begin{aligned} \rho(((x, y), (\alpha, \beta)), ((w, z), (\gamma, \delta))) &:= ((x - w)^2 + (y - z)^2)^{\frac{1}{2}} \\ &+ \min\{1, \sup\{|\alpha(\xi) - \gamma(\xi)| : \xi \in \mathbf{R}\} + \sup\{|\beta(\xi) - \delta(\xi)| : \xi \in \mathbf{R}\}\}. \end{aligned}$$

Put now

$$(7.3) \quad X := \{((x, y), (\alpha, \beta)) \in Z : \alpha(x) = \beta(y)\}$$

and define a mapping $\pi : \mathbf{R}^2 \times X \rightarrow X$ by the formula

$$(7.4) \quad \begin{aligned} \pi((x, y), ((r, s), (\alpha, \beta))) &:= \\ ((x + r, y + s), (u(\cdot, y + s; (r, s), \alpha, \beta), u(x + r, \cdot; (r, s), (\alpha, \beta)))) \end{aligned}$$

We have the following

PROPOSITION 7.1. *The triplet $(X, \mathbf{R}^2; \pi)$ with X and π defined by (7.3) and (7.4) respectively is a dynamical system.*

An elementary proof will be omitted. \square

If we assume that the existence of the unique solution (7.0) is assured only for $(x^0, y^0) \in [0, \infty) \times [0, \infty)$, $\alpha : [x^0, \infty) \rightarrow \mathbf{R}$, $\beta : [y^0, \infty) \rightarrow \mathbf{R}$, then the above construction will induce a dynamical semi-system.

A natural simplification of these examples are obtainable in the case of an “autonomous” equation $u_{xy} = f(u)$.

Existence and uniqueness questions for solutions of equations mentioned above will be discussed separately in another paper.

II. Assume that $(X, \mathbf{R}; \lambda)$ and $(X, \mathbf{R}; \sigma)$ are two dynamical systems (with (X, ρ) being a metric space) such that

$$\sigma(t, \lambda(s, x)) = \lambda(s, \sigma(t, x))$$

for all $x \in X$, $t, s \in \mathbf{R}$. It is easy to see that $\pi: \mathbf{R}^2 \times X \rightarrow X$ defined by the formula

$$\pi((t, s), x) := \sigma(t, \lambda(s, x))$$

gives a dynamical system $(X, \mathbf{R}^2; \pi)$.

References

1. Bhatia N.P., Szegő G.P., *Stability Theory of Dynamical Systems*, Springer Verlag, Berlin–Hiedleberg–New York, 1970.
2. Pelczar A., *Stability of sets in pseudo-dynamical systems (I), II–IV*, Bull. Acad. Polon. Sci., Sér. Sci. Math., Astronom. et Phys. **19** (1971), 13–17; **19** (1971), 951–957; **20** (1972), 673–677; **21** (1973), 911–916.
3. Pelczar A., *Stability of motions in pseudo-dynamical systems*, Bull. Acad. Polon. Sci., Sér. Sci. Math., Astronom. et Phys., **21** (1973), 393–398.
4. Pelczar A., *Semi-stability of motions and regular dependence of limit sets on points*, Ann. Polon. Math. **42** (1983), 263–282.
5. Pelczar A., *Asymptotically Periodic Motions*, Bull. Acad. Polon. Sci., Math. **33** (1985), 314–319.
6. Pelczar A., *Remarks on some generalizations of asymptotic periodicity*, Ann. Polon. Math. **51** (1990), 259–268.
7. Pelczar A., *Remarks on some “multidimensional” dynamical systems*, Abstracts of Short Communications,, International Congress of Mathematicians, ICM–94, Zürich, 3–11 August 1994, 201.

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