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EVOLUTION EQUATIONS OF SECOND ORDER WITH PARAMETER

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Abstract. The main object of this paper is the study of continuity and differentiability with respect to both h and t of a solution of a second order evolution problem with a parameter $h \in \Omega \subset \mathbf{R}^m$.

Introduction. Let X be a real Banach space. We consider the abstract initial value problem with a parameter $h \in \Omega \subset \mathbf{R}^m$

$$\frac{d^2u}{dt^2} = A(h,t)u + f(h,t), \quad t \in [0,T], \tag{1.1}$$

$$u(0) = u_h^0, \tag{1.2}$$

$$\frac{du}{dt}(0) = u_h^1, \tag{1.3}$$

where $(A(h,t))_{(h,t)\in\Omega\times[0,T]}$ is a family of linear operators from a real Banach space X into itself, $u\colon \mathbf{R} \to X$, $f\colon \Omega \times \mathbf{R} \to X$, Ω is an open subset of $\mathbf{R}^m$, and $u_h^0, u_h^1 \in X$ for $h \in \Omega$.

We make the following assumptions.

(Z_1) The domain $D(A(h,t)) = D$ is independent of t and h, D is dense in X. For each $h \in \Omega$ and $t \in [0,T]$, 0 belongs to the resolvent set of $A(h,t)$ and $A^{-1}(h,t)$ is a bounded operator.

(Z_2) For each $h \in \Omega$ and $t \in [0,T]$, $A(h,t)$ is the infinitesimal generator of a strongly continuous cosine family $\{C(h,t,\xi), \xi \in \mathbf{R}\}$ of bounded linear operators from X into itself.

(Z_3) For each $h \in \Omega$ and $t \in [0,T]$ there exists a linear operator $B(h,t) : X \to X$ such that $B^2(h,t) = A(h,t)$, the domain $D(B(h,t)) := D(B)$ is independent of h and t and 0 belongs to the resolvent set of $B(h,t)$.

(Z_4) For each $x \in D(B)$ and $h \in \Omega$ the mapping $[0,T] \ni t \to B(h,t)x$ is of class C^1.

If the assumptions Z_1–Z_4 are satisfied, then for each $\lambda > 0$ and $h \in \Omega$ there exists $\mathcal{R}(\lambda; A(h,t)) := (\lambda - A(h,t))^{-1}$ and

$$\| \mathcal{R}(\lambda; A(h,t) \| \leq \frac{M}{\lambda},$$

where $M \geq 1$ is a constant independent of λ, h, t ([3] p. 61).

DEFINITION 1. We say that a function $u(h,\cdot) : [0,T] \to X$, for $h \in \Omega$ is a solution of the problem (1.1)–(1.3) if

(i) $u(h,t) \in D$ for each $h \in \Omega$ and $t \in [0,T]$,
(ii) $u(h,\cdot)$ is of class C^2 on $(0,T]$ and of class C^1 on $[0,T]$,
(iii) $\frac{d^2u(h,t)}{dt^2} = A(h,t)u(h,t) + f(h,t)$, for $h \in \Omega$, $t \in [0,T]$, $u(h,0) = u_h^0$, $\frac{du(h,0)}{dt} = u_h^1$ for $h \in \Omega$.

DEFINITION 2. A family $S_h = S(h,t,s)$, for $t,s \in [0,T]$, $h \in \Omega$ is called a fundamental solution of the equation

$$\frac{d^2u}{dt^2} = A(h,t)u \quad \text{for} \quad t \in [0,T],\ h \in \Omega$$

if

D1) the mapping $[0,T] \times [0,T] \ni (t,s) \to S(h,t,s)x \in X$ is of class C^1 for $x \in X$, $h \in \Omega$, and

a) $S(h,t,t) = 0$ for $h \in \Omega$, $t \in [0,T]$,
b) $\frac{\partial}{\partial t}S(h,t,s) |_{t=s} x = x$, $\frac{\partial}{\partial s}S(h,t,s) |_{t=s} x = -x$ for $t,s \in [0,T]$ $h \in \Omega$, $x \in X$,

D2) for any $x \in D$, $t,s \in [0,T]$, and $h \in \Omega$ the following conditions are satisfied

a) $S(h,t,s)x \in D$,
b) the mapping $[0,T] \times [0,T] \ni (t,s) \to S(h,t,s)x$ is of class C^2,
c) $\frac{\partial^2}{\partial t^2}S(h.t,s)x = A(h,t)S(h,t,s)x$,
d) $\frac{\partial^2}{\partial s^2}S(h,t,s)x = S(h,t,s)A(h,s)x$,
e) $\frac{\partial}{\partial s}\frac{\partial}{\partial t}S(h,t,s) |_{t=s} x = 0$,

D3) for any $x \in D$, $t, s \in [0,T]$, $h \in \Omega$ the following conditions are satisfied

a) $\frac{\partial}{\partial s} S(h,t,s)x \in D$,

b) there exist the derivatives $\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s} S(h,t,s)x$ and $\frac{\partial^2}{\partial s^2}\frac{\partial}{\partial t} S(h,t,s)x$,

c) $\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial s} S(h,t,s)x = A(h,t)\frac{\partial}{\partial s} S(h,t,s)x$,

d) $\frac{\partial^2}{\partial s^2}\frac{\partial}{\partial t} S(h,t,s)x = \frac{\partial}{\partial t} S(h,t,s)A(h,s)x$,

e) the mapping $[0,T]\times[0,T] \ni (t,s) \to A(h,t)\frac{\partial}{\partial s} S(h,t,s)x$ is continuous,

D4) for $t, s, r \in [0,T]$, $h \in \Omega$,
$S(h,t,s) = S(h,t,r)\frac{\partial}{\partial r} S(h,r,s) - \frac{\partial}{\partial r} S(h,t,r)S(h,r,s)$.

Let, for a fixed $h_0 \in \Omega$, Y be the linear space $D(B)$ with the norm $\| \cdot \|_Y$ given by

$$\| y \|_Y := \| y \| + \| B(h_0,0)y \|, \quad \text{for} \quad y \in D(B).$$

It is obvious that the operator $\mathcal{A}(h,t) : Y \times X \to Y \times X$ defined by

$$\mathcal{A}(h,t) = \begin{bmatrix} 0 & I \\ A(h,t) & 0 \end{bmatrix} \quad \text{for} \quad h \in \Omega,\ t \in [0,T]$$

is linear with the domain $D(A) \times D(B)$. Let

$$F(h,t) := \begin{bmatrix} 0 \\ f(h,t) \end{bmatrix} \quad \text{for} \quad h \in \Omega,\ t \in [0,T].$$

Since the problem (1.1)–(1.3) can be reduced to the following problem

$$\frac{dU}{dt} = \mathcal{A}(h,t)U + F(h,t) \quad \text{for} \quad t \in [0,T] \tag{1.4}$$

$$U(0) = U_h^0 = \begin{bmatrix} u_h^0 \\ u_h^1 \end{bmatrix}, \tag{1.5}$$

we will only consider first order linear initial value problem in the space $Y \times X$.

Definition 3. A family $\{A(h,t)\}$, $(h,t) \in \Omega \times [0,T]$ is said to be uniformly stable approximated with respect to $h \in \Omega$, if there exists a sequence $\{A_n(h,t)\}$ of bounded linear operators $A_n(h,t) : X \to X$, $n = 1, 2, \ldots$ such that

1^0 the mapping $\Omega \times [0,T] \ni (h,t) \to A_n(h,t)x$ is continuous for $x \in X$, $n = 1, 2, \ldots$

2^0 $\lim_{n\to\infty}\{\sup \| [A_n(h,t) - A(h,t)]A^{-1}(h,t)x \| : (h,t) \in \Omega \times [0,T]\} = 0$ for $x \in X$ and the sequence $\{U_n(h,t,s\}$ of fundamental solutions of the problems

$$\begin{cases} \dfrac{du}{dt} = A_n(h,t) \\ u(s) = x \quad 0 \le s \le t \le T \end{cases}$$

is uniformly bounded i.e. there exists $M > 0$ such that

$$\| U_n(h,t,s) \| \leq M, \quad \text{for} \quad h \in \Omega,\ (t,s) \in \Delta_T = \{(t,s) : 0 \leq s \leq t \leq T\}.$$

The sequence $\{A_n(h,t)\}$ approximating the family $\{A(h,t)\}$, $h \in \Omega$, $t \in [0,T]$ has the form

$$A_n(h,t) := -nA(h,t)R(n; A(h,t)) \tag{1.6}$$

(cf [8] p. 204).

DEFINITION 4. A family $\{A(h,t)\}$, $h \in \Omega$, $t \in [0,T]$ is called uniformly stable in Ω if there are constants $M \geq 1$ and ω such that

$$(\omega, \infty) \subset P(A(h,t)) \quad \text{for} \quad t \in [0,T],\ h \in \Omega$$

and

$$\left\| \prod_{j=1}^{k} \mathcal{R}(\lambda; A(h,t_j)) \right\| \leq M(\lambda - \omega)^{-k} \quad \text{for} \quad \lambda > \omega,\ h \in \Omega.$$

Let the assumption Z_1 be fulfilled.

DEFINITION 5. ([8] p. 193). The Cauchy problem

$$\begin{cases} \dfrac{du}{dt} = A(h,t)u \\ u(s) = x_h \end{cases} \tag{1.7}$$

is said to be uniformly correct if:

1^0 for each $s \in [0,T]$, $x_h \in D$, there exists a unique solution $u_h = u(h,t,s)$ of (1.7) on the segment $[s,T]$ for $h \in \Omega$,

2^0 the function u_h and $(u_h)'_t$ are continuous for $t, s \in \Delta_T$ and $h \in \Omega$,

3^0 for $h \in \Omega$ the solution depends continuously on the initial data.

It is well known ([8] Chapter II §2) that the problem (1.7) is uniformly correct if $A(h,t)$ is bounded for $h \in \Omega$, $t \in [0,T]$ and the mapping $[0,T] \ni t \to A(h,t)$ is strongly continuous. In this case there exists a fundamental solution of (1.7).

LEMMA 1. *If the mapping $\Omega \times [0,T] \ni (h,t) \to A(h,t)x$ is continuous for $x \in D$, then the mapping $\Omega \times [0,T] \ni (h,t) \to \mathcal{A}(h,t) \begin{bmatrix} x \\ y \end{bmatrix}$ is continuous for $x \in D$, $y \in D(B)$.*

PROOF. Let $(h_0, t_0) \in \Omega \times [0,T]$.

$$\begin{aligned}
&\| (\mathcal{A}(h,t) - \mathcal{A}(h_0,t_0)) \begin{bmatrix} x \\ y \end{bmatrix} \| \\
&= \| \begin{bmatrix} 0 & I \\ A(h,t) & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 & I \\ A(h_0,t_0) & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \| \\
&= \| \begin{bmatrix} 0 \\ (A(h,t) - A(h_0,t_0))x \end{bmatrix} \| = \| (A(h,t) - A(h_0,t_0))x \| \xrightarrow[(h,t)\to(h_0,t_0)]{} 0.
\end{aligned}$$

LEMMA 2. *If the mapping $[0,T] \ni t \to A(h,t)x$ is of class C^1 for $h \in \Omega$, $x \in D$, then the mapping $[0,T] \ni t \to \mathcal{A}(h,t) \begin{bmatrix} x \\ y \end{bmatrix}$ is of class C^1 for $h \in \Omega$, $\begin{bmatrix} x \\ y \end{bmatrix} \in D \times D(B)$.*

PROOF. It is the same as the proof of Lemma 1.

LEMMA 3. *If the assumption Z_1 is satisfied then the operator $\mathcal{A}^{-1}(h,t)$ is bounded.*

PROOF. Let $x \in D(B)$ and $y \in X$. By the assumption Z_1 we have

$$\begin{aligned}
&\| \mathcal{A}^{-1}(h,t) \begin{bmatrix} x \\ y \end{bmatrix} \| = \| \begin{bmatrix} 0 & A^{-1}(h,t) \\ I & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \| \\
&\leq \| A^{-1}(h,t)y \| + \| x \| \leq \| A^{-1}(h,t) \| \| y \| + \| x \| \leq C \| \begin{bmatrix} x \\ y \end{bmatrix} \| .
\end{aligned}$$

Let $\mathcal{A}_n(h,t) : Y \times X \to Y \times X$, $n = 1, 2, \ldots$, be defined by

$$\mathcal{A}_n(h,t) = \begin{bmatrix} 0 & I \\ A_n(h,t) & 0 \end{bmatrix} \quad \text{for} \quad h \in \Omega,\ t \in [0,T],$$

where $A_n(h,t)$ is defined by (1.6).

LEMMA 4. *If the assumptions $(Z_1) - (Z_3)$ are fulfilled and the mapping $[0,T] \ni t \longrightarrow A(h,t)x$ is of class C^1 for $h \in \Omega$, $x \in D$, then the sequence $\{\mathcal{A}_n(h,t)\}$ satisfies the following conditions:*

1^0 *$\mathcal{A}_n(h,t)$ is a bounded operator for $n \in \mathbf{N}$, $h \in \Omega$, $t \in [0,T]$,*

2^0 *the mapping $[0,T] \ni t \to \mathcal{A}_n(h,t)$, $h \in \Omega$, is strongly continuously differentiable,*

3^0 *$\| \mathcal{A}_n^{-1}(h,t) \| \leq C$ (C does not depend on h, t, n),*

4^0 *the sequence* $\{\mathcal{A}'_n(h,t)\mathcal{A}_n^{-1}(h,t)\}$ *is strongly and uniformly convergent to a bounded operator,*

5^0 $\lim_{n\to\infty}\{\sup_{0\le t\le T} \| [\mathcal{A}_n(h,t) - \mathcal{A}(h,t)]\mathcal{A}^{-1}(h,t) \begin{bmatrix} y \\ x \end{bmatrix} \|\} = 0$ *for* $h \in \Omega$, $(y,x) \in Y \times X$.

PROOF. It is an immediate consequence of Theorem 2 in [1].

LEMMA 5. *If the assumptions* (Z_1)–(Z_4) *are fulfilled, the mapping* $(0,T] \ni t \to A(h,t)x$, *is of dass* C^1 *for* $h \in \Omega$, *and the family* $\{A(h,t)\}$, *for* $h \in \Omega$, $t \in [0,T]$ *is uniformly stable with stability constans* $M \ge 1$ *and* $\omega = 0$, *then*

$$\| \mathcal{V}_n(h,t,s) \| \le M \quad (M \text{ does not depend on } h,t,s,n), \tag{1.8}$$

where $\mathcal{V}_n(h,t,s)$ *is the fundamental solution corresponding to the operator* $\mathcal{A}_n(h)$, $\lim_{n\to\infty} \mathcal{V}_n(h,t,s) = \mathcal{V}(h,t,s)$ *strongly in* $Y \times X$ *and uniformly in* $\Omega \times [0,T] \times [0,T]$, *and* $\mathcal{V}(h,t,s)$ *is the fundamental solution corresponding to the operator* $\mathcal{A}(h,t)$.

PROOF. It is an immediate consequence of Theorems 3 and 4 in [1].

COROLLARY. *Let the assumptions of Lemma 5 be satisfied. Then the family* $\{\mathcal{A}(h,t)\}$, $h \in \Omega$, $t \in [0,T]$ *is uniformly stable approximated with respect to* $h \in \Omega$.

THEOREM 1. *Suppose that*

(*a*) *the assumptions* (Z_1)–(Z_4) *are fulfilled,*

(*b*) *the mapping* $\Omega \times [0,T] \ni (h,t) \to A(h,t)x$ *is continuous for* $x \in D$,

(*c*) *the mapping* $[0,T] \ni t \to A(h,t)x$ *is of class* C^1 *for* $h \in \Omega$, $x \in D$,

(*d*) *the family* $\{A(h,t)\}$, $h \in \Omega$, $t \in [0,T]$ *is uniformly stable in* Ω, *then the mapping*

$$\Omega \times [0,T] \times [0,T] \ni (h,t,s) \to \mathcal{V}(h,t,s) \begin{bmatrix} y \\ x \end{bmatrix}$$

is continuous for $y \in Y$, $x \in X$, *where* $\mathcal{V}(h,t,s)$ *is the fundamental solution to the problem*

$$\frac{dU}{dt} = \mathcal{A}(h,t)U, \tag{1.8}$$

$$U(0) = U_h^0 = \begin{bmatrix} u_h^0 \\ u_h^1 \end{bmatrix}. \tag{1.9}$$

PROOF. From ([1], Theorem 1]) the problem (1.8)–(1.9) is uniformly correct. Using Lemma 5 and Theorem 3 in [2] we can see that the mapping

$$\Omega \times [0,T] \times [0,T] \ni (h,t,s) \to \mathcal{V}_n(h,t,s) \begin{bmatrix} y \\ x \end{bmatrix}$$

is continuous for $y \in Y$, $x \in X$, $n \in \mathbf{N}$ and

$$\lim_{n\to\infty} \mathcal{V}_n(h,t,s) = \mathcal{V}(h,t,s)$$

strongly in $Y \times X$ and uniformly in $\Omega \times [0,T] \times [0,T]$. This ends the proof.

DEFINITION 6. For $h \in \Omega$, $t,s \in [0,T]$ we define the family of operators $\mathcal{S}(h,t,s) : X \to X$ by the formula

$$\mathcal{S}(h,t,s)x := \Pi_1 \mathcal{V}(t,s) \begin{bmatrix} 0 \\ x \end{bmatrix} \quad \text{for} \quad x \in X,$$

where $\Pi_1 \begin{bmatrix} x \\ y \end{bmatrix} := x$ for $x, y \in X$. It folows from [7] (Theorem 4.1) that the family $\{\mathcal{S}(h,t,s)\}$, $t,s \in [0,T]$, $h \in \Omega$ is the fundamental solution of the equation

$$\frac{d^2u}{dt^2} = A(h,t)u \quad \text{for} \quad t \in [0,T],\ h \in \Omega.$$

LEMMA 6. *If the assumptions of Theorem 1 are fulfilled then*

$$\lim_{h\to h_0} \frac{\partial}{\partial s}\mathcal{S}(h,t,s)x = \frac{\partial}{\partial s}\mathcal{S}(h_0,t,s)x \quad \textit{and} \quad \lim_{h\to h_0} \mathcal{S}(h,t,s)x = \mathcal{S}(h_0,t,s)x$$

uniformly in $[0,T] \times [0,T] \times K$, *where* K *is a compact subset of* X.

PROOF. It folows from [7] (Theorem 4.1) that

$$\mathcal{V}(h,t,s) = \begin{bmatrix} -\frac{\partial}{\partial s}\mathcal{S}(h,t,s) & \mathcal{S}(h,t,s) \\ -\frac{\partial}{\partial t}\frac{\partial}{\partial s}\mathcal{S}(h,t,s) & \frac{\partial}{\partial t}\mathcal{S}(h,t,s) \end{bmatrix}.$$

By Theorem 1

$$\lim_{h\to h_0} \mathcal{S}(h,t,s)x = \mathcal{S}(h_0,t,s)x \quad \text{for} \quad x \in X$$

and

$$\lim_{h\to h_0} \frac{\partial}{\partial s}\mathcal{S}(h,t,s) = \frac{\partial}{\partial s}\mathcal{S}(h_0,t,s)y \quad \text{for} \quad y \in Y$$

uniformly in $[0,T] \times [0,T]$.

Similarly as in [7] (Theorem 4.1) we can prove that (1.9) holds for $y \in X$, not only for $y \in Y$.

Repeating now the reasoning of Proposition 1 in [10] we get Lemma 6.

In the sequel we assume that X is a reflexive Banach space.

THEOREM 2. *If*

1^0 *the family* $\{A(h,t)\}$ *satisfies the assumptions of Theorem 1,*

2^0 $u_h^0,\ u_h^1 \in D$ *for* $h \in \Omega$,

3^0 *the mappings* $\Omega \ni h \to u_h^0 \in X$, $\Omega \ni h \to u_h^1 \in X$ *and* $f\colon \Omega \times [0,T] \to X$ *are continuous,*

4^0 *there exists* $M > 0$ *such that* $\| f(h,t_1) - f(h,t_2) \| \leq M \mid t_1 - t_2 \mid$ *for* $t_1, t_2 \in [0,T]$ *then for any* $h \in \Omega$ *there exists exactly one solution of the problem (1.1)–(1.3) and it is given by the formula*

$$(1.10)\quad u(h,t) = -\frac{\partial}{\partial s}\mathcal{S}(h,t,s)\mid_{s=0} u_h^0 + \mathcal{S}(h,t,s)u_h^1 + \int_0^t \mathcal{S}(h,t,s)f(h,s)ds$$

and

$$\lim_{h\to h_0} u(h,t) = u(h_0,t)$$

uniformly in $[0,T]$.

PROOF. It follows from [5] that there exists exactly one solution of the problem (1.1)–(1.3) and it is given by the formula

$$u(h,t) = -\frac{\partial}{\partial s}\mathcal{S}(h,t,s)\mid_{s=0} u_h^0 + \mathcal{S}(h,t,0)u_h^1 + \int_0^t \mathcal{S}(h,t,s)f(h,s)ds.$$

Thus

$$u(h,t) - u(h_0,t) = \left[-\frac{\partial}{\partial s}\mathcal{S}(h,t,s)\mid_{s=0} + \frac{\partial}{\partial s}\mathcal{S}(h_0,t,s)\mid_{s=0}\right] u_h^0$$

$$-\frac{\partial}{\partial s}\mathcal{S}(h_0,t,s)\mid_{s=0} (u_h^0 - u_{h_0}^0) + [\mathcal{S}(h,t,0) - \mathcal{S}(h_0,t,0)]u_h^1$$

$$+\mathcal{S}(h_0,t,0)(u_h^1 - u_{h_0}^1) + \int_0^t [\mathcal{S}(h,t,s) - \mathcal{S}(h_0,t,s)]\, f(h,s)ds$$

$$+\int_0^t \mathcal{S}(h_0,t,s)[f(h,s) - f(h_0,s)]ds.$$

Let K be a compact neighborhood of h_0. Since the mappings 3^0 are continuous, the sets

$$K_1 = \{u_h^0 :\ h \in K\}, \qquad K_2 = \{u_h^1 :\ h \in K\},$$

$$K_3 = \{f(h,s) :\ h \in K,\ s \in [0,T]\}$$

are compact in X. By Lemma 6

$$\left[-\frac{\partial}{\partial s}\mathcal{S}(h,t,s)\mid_{s=0}+\frac{\partial}{\partial s}\mathcal{S}(h_0,t,s)\mid_{s=0}\right]u_h^0 \xrightarrow[h\to h_0]{} 0$$

and

$$[\mathcal{S}(h,t,0)-\mathcal{S}(h_0,t,0)]u_h^1 \xrightarrow[h\to h_0]{} 0$$

uniformly in $[0,T]$.

The family $\{\frac{\partial}{\partial s}\mathcal{S}(h,t,s)\}$, $h\in\Omega$, $t,s\in[0,T]$ is a strongly continuous family of bounded operators. Then by the Banach–Steinhaus Theorem it is uniformly bounded. We have the same for the family $\{\mathcal{S}(h,t,s)\}$. Thus

$$\lim_{h\to h_0}\frac{\partial}{\partial s}\mathcal{S}(h_0,t,0)(u_h^0-u_{h_0}^0)=0,$$

$$\lim_{h\to h_0}\mathcal{S}(h_0,t,0)(u_h^1-u_{h_0}^1)=0,$$

$$\lim_{h\to h_0}\int_0^t\mathcal{S}(h_0,t,0)[f(h,s)-f(h_0,s)]ds=0$$

uniformly in $[0,T]$. This ends the proof.

COROLLARY. *If u is the solution of the problem (1.1)–(1.3) then the mapping*

$$u:\Omega\times[0,T]\ni(h,t)\longrightarrow u(h,t)\in X$$

is continuous.

Differentation with respect to a parametr.

THEOREM 3. *If*

1^0 *the assumptions of Theorem 2 are fulfilled,*

2^0 *$u(h,\cdot)$ is the solution of the problem (1.1)–(1.3),*

3^0 *the mappings $\Omega\ni h\to A(h,\cdot)x$ for $x\in D$, $\Omega\ni h\to f(h,\cdot)$, $\Omega\ni h\to u_h^0$, $\Omega\ni h\to u_h^1$ are differentiable at h_0, then the mapping $\Omega\ni h\to u(h,\cdot)$ is differentiable at h_0 and*

$$u'(h_0,t)=-\frac{\partial}{\partial s}\mathcal{S}(h_0,t,s)\mid_{s=0}(u_{h_0}^0)'+\mathcal{S}(h_0,t,0)(u_{h_0}^1)'$$
$$+\int_0^t\mathcal{S}(h_0,t,s)[A'(h_0,s)u(h_0,s)+f'(h_0,s)]ds,$$

where "$'$" denotes the differentation with respect to the parameter h.

PROOF. Let $u(h,\cdot)$ be the solution of the problem (1.1) - (1.3). The function

$$v(h,t) = \frac{u(h,t) - u(h_0,t)}{h - h_0}$$

for $h \neq h_0$, is the solution of the problem

$$\begin{cases} \dfrac{d^2 v}{dt^2} = A(h,t)v + F(h,t) \\ v(0) = v_h^0 \\ \dfrac{dv}{dt}(0) = v_h^1 \end{cases}$$

where

$$F(h,t) = \begin{cases} \dfrac{A(h,t) - A(h_0,t)}{h - h_0} u(h_0,t) + \dfrac{f(h,t) - f(h_0,t)}{h - h_0} & \text{for} \quad h \neq h_0 \\ A'(h_0,t)u(h_0,t) + f'(h_0,t) \quad \text{for} \quad h = h_0 \end{cases}$$

$$v_h^0 = \begin{cases} \dfrac{u_h^0 - u_{h_0}^0}{h - h_0} & \text{for} \quad h \neq h_0 \\ (u_{h_0}^0)' & \text{for} \quad h = h_0 \end{cases}$$

$$v_h^1 = \begin{cases} \dfrac{u_h^1 - u_{h_0}^1}{h - h_0} & \text{for} \quad h \neq h_0 \\ (u_{h_0}^1)' & \text{for} \quad h = h_0. \end{cases}$$

It follows from the assumptions of Theorem 3 that the mapping

$$(h,t) \longrightarrow \begin{cases} \frac{f(h,t) - f(h_0,t)}{h - h_0} & \text{for} \quad h \neq h_0 \\ f'(h_0,t) & \text{for} \quad h = h_0 \end{cases}$$

is continuous. We have

$$\frac{A(h,t) - A(h_0,t)}{h - h_0} u(h_0,t)$$
$$= \frac{A(h,t) - A(h_0,t)}{h - h_0} A^{-1}(h_0,0)A(h_0,0)A^{-1}(h_0,t)A(h_0,t)u(h_0,t).$$

Since the mapping $[0,T] \ni t \longrightarrow A(h_0,t)u(h_0,t)$ is continuous, the mapping

$$(h,t) \longrightarrow \begin{cases} \frac{A(h,t) - A(h_0,t)}{h - h_0} u(h_0,t) & \text{for} \quad h \neq h_0 \\ A'(h_0,t)u(h_0,t) & \text{for} \quad h = h_0 \end{cases}$$

is continuous. By Theorem 2 the function

$$\tilde{v}(h,t) = \frac{\partial}{\partial s}\mathcal{S}(h,t,s)\mid_{s=0} v_h^0 + \mathcal{S}(h,t,s)v_h^1 + \int_0^t \mathcal{S}(h,t,s)F(h,s)ds$$

is continuous and

$$\tilde{v}(h,t) = \begin{cases} v(h,t) & \text{for} \quad h \neq h_0 \\ u'(h_0,t) & \text{for} \quad h = h_0. \end{cases}$$

Hence

$$u'(h_0,t) = -\frac{\partial}{\partial s}\mathcal{S}(h_0,t,s)\mid_{s=0} (u_{h_0}^0)' + \mathcal{S}(h_0,t,s)(u_{h_0}^1)' + \int_0^t \mathcal{S}(h_0,t,s)[A'(h_0,s)u(h_0,s) + f'(h_0,s)]ds.$$

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