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ON COMPLEXIFIED NORM

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Abstract. Let E be a real unitary space with a scalar product $\langle x,y\rangle$, let $||x||_E := \sqrt{<x,x>}$ and let $\tilde{E} = E \oplus \mathbf{i}\, E$ denote the complexification of E. We give a short and elementary proof of the following effective formula for the complexified norm in $\tilde{E}$, namely

If $||x+\mathbf{i}\,y||_c := \inf\{ \sum_{finite} |c_j|\,||v_j||_E : x+\mathbf{i}\,y = \sum_{finite} c_j v_j,\ c_j \in \mathbf{C},\ v_j \in E\}$,

then

$$||x+\mathbf{i}\,y||_c = \sqrt{<x,x> + <y,y> + 2\sqrt{<x,x><y,y> - <x,y>^2}}.$$

We also show that if $\dim E \geq 2$ then the norm $||\ \ ||_c$ is totally noneuclidean and mention some applications of the above formula.

1. Introduction. Let E denote a real unitary space with a scalar product denoted by $<x,y>$ and a euclidean norm $||x||_E := \sqrt{<x,x>}$, when $x,y \in E$. The complexification $\tilde{E}$ of the space E is a space $E \oplus \mathbf{i}\, E$, where $\mathbf{i}^2 = -1$. $\tilde{E}$ endowed with addition and multiplication by complex numbers defined in an obvious way (cf. [D, GL, L]) is a complex vector space. Define the so called *crossnorm* or *complexified norm* by the formula

$$||x+\mathbf{i}\,y||_c := \inf\{\sum_{j=1}^{p} |c_j|\,||v_j||_E : \quad x+\mathbf{i}\,y = \sum_{j=1}^{p} \mathrm{Re}\, c_j\, v_j + \mathbf{i} \sum_{j=1}^{p} \mathrm{Im}\, c_j\, v_j,$$
$$c_j \in \mathbf{C},\ v_j \in E,\ j=1,\dots,p,\ p \in \mathbf{N}\}, \qquad x,y \in E.$$

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As we remember the necessary and sufficient condition for a norm $|| \ \ ||$ on $\tilde{E}$ to be euclidean (i. e. generated by some scalar product) is to fulfil the so called parallelogram condition

$$||z+w||^2 + ||z-w||^2 - 2||z||^2 - 2||w||^2 = 0 \quad \text{for any vectors } z, w \in \tilde{E}.$$

The scalar product in the complex space $\tilde{E}$ is then given by the formula

$$<z,w> := \frac{1}{4}\left(||z+w||^2 - ||z-w||^2 + \mathbf{i}\,||z+\mathbf{i}\,w||^2 - \mathbf{i}\,||z-\mathbf{i}\,w||^2\right)$$

for any $z, w \in \tilde{E}$.

Define the so called measure of noneuclidity of the norm as the number (cf. [GL])

$$\gamma(|| \ \ ||) := \sup\left\{\frac{||z+w||^2 + ||z-w||^2 - 2||z||^2 - 2||w||^2}{2(||z||^2 + ||w||^2)} : z, w \in \tilde{E}\setminus\{0\}\right\}.$$

Obviously $\gamma(|| \ \ ||)$ belongs to $[0,1]$, cf. [GL]. In the case where $\gamma(|| \ \ ||) = 1$ we call the norm $|| \ \ ||$ totally noneuclidean. The aim of this note is to prove the following

THEOREM 1.1. *(i) (cf. [D]). Keeping the above notation we have the effective formula for the complexified norm*

$$(*) \qquad ||x+\mathbf{i}\,y||_c = \sqrt{<x,x> + <y,y> + 2\sqrt{<x,x><y,y> - <x,y>^2}}, \quad x+\mathbf{i}\,y \in \tilde{E}.$$

(ii) The norm $|| \ \ ||_c$ is totally noneuclidean, i. e. $\gamma(|| \ \ ||_c) = 1$.

The effective formula for the complexified norm was proved earlier in [D], but the proof given there is technical (with tedious calculations) and needs some facts from functional analysis. Therefore, we find it useful to present a short direct proof of the mentioned formula together with the interesting and presumable property of the complexified norm to be totally noneuclidean.

2. Proof of Theorem 1.1.

PROOF OF (I). Denote

$$(0)\qquad D(x,y) := \sqrt{<x,x> + <y,y> + 2\sqrt{<x,x><y,y> - <x,y>^2}}, \qquad x,y \in E.$$

Now we will prove two lemmas.

LEMMA 2.1 (cf. [D, p. 50]). *For any $x, y \in E$ there exist a couple of orthogonal vectors $u, v \in E$ and an orthogonal matrix $A = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{bmatrix}$ such that*

$$x = \alpha_1 u + \alpha_2 v, \qquad y = \beta_1 u + \beta_2 v.$$

PROOF OF LEMMA 2.1. Assume $<x,y> \neq 0$. If $\varphi := \frac{1}{2} \operatorname{arcctg} \frac{<x,x> - <y,y>}{2<x,y>}$, then

$$A := \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \quad \text{and} \quad u := x\cos\varphi + y\sin\varphi, \; v := -x\sin\varphi + y\cos\varphi$$

fulfil the conditions required in Lemma 2.1. □

LEMMA 2.2. *Let $(v_j \in E : j \in S)$ be a finite sequence of vectors. Then*

$$(\text{i1})\qquad D\Big(\sum_{j\in S} \beta_j v_j, \sum_{j\in S} \gamma_j v_j\Big) \le \sum_{j\in S} \sqrt{\beta_j^2 + \gamma_j^2}\,||v_j||_E \qquad \text{for any } \beta_j, \gamma_j \in \mathbf{R}.$$

PROOF OF LEMMA 2.2. Denote $x = \sum_{j\in S} \beta_j v_j$, $y = \sum_{j \in S} \gamma_j v_j$ and preserve the notation used in Lemma 2.1. It is easy to see that

$$(1)\qquad D(x,y) = D(\alpha_1 u + \alpha_2 v, \beta_1 u + \beta_2 v) = D(u,v) = ||u||_E + ||v||_E.$$

Denote

$$u^\circ := \begin{cases} \frac{u}{||u||}, & \text{when } u \neq \mathbf{0} \\ 0, & \text{when } u = \mathbf{0} \end{cases} \quad \text{and} \quad v^\circ := \begin{cases} \frac{v}{||v||}, & \text{when } v \neq \mathbf{0} \\ 0, & \text{when } v = \mathbf{0}. \end{cases}$$

Due to (1) by the Schwarz inequality the following holds

$$\begin{aligned} D(x,y) &= <u,u^\circ> + <v,v^\circ> = <\alpha_1 x + \beta_1 y,\, u^\circ> + <\alpha_2 x + \beta_2 y,\, v^\circ> \\ &= <\sum_{j\in S}(\alpha_1\beta_j + \beta_1\gamma_j)\, v_j, u^\circ> + <\sum_{j\in S}(\alpha_2\beta_j + \beta_2\gamma_j)\, v_j, v^\circ> \\ &= \sum_{j\in S}(\alpha_1\beta_j + \beta_1\gamma_j) \; <v_j,u^\circ> + \sum_{j\in S}(\alpha_2\beta_j + \beta_2\gamma_j) \; <v_j,v^\circ> \\ &\le \sum_{j\in S}\sqrt{(\alpha_1\beta_j + \beta_1\gamma_j)^2 + (\alpha_2\beta_j + \beta_2\gamma_j)^2}\sqrt{<v_j,u^\circ>^2 + <v_j,v^\circ>^2} \\ &= \sum_{j\in S}\sqrt{\beta_j^2 + \gamma_j^2}\sqrt{<v_j,u^\circ>^2 + <v_j,v^\circ>^2} \\ &\le \sum_{j\in S}\sqrt{\beta_j^2 + \gamma_j^2}\,||v_j||_E. \end{aligned}$$

□

Now we are able to finish the proof of the formula (*). Note that we can write the definition of the complexified norm in the following way

$$(1) \qquad ||x + \mathbf{i}\, y||_c = \inf\{\sum_{j\in S}\sqrt{\beta_j^2 + \gamma_j^2}\,||v_j|| : \; x = \sum_{j\in S}\beta_j v_j,\; y = \sum_{j\in S}\gamma_j\, v_j\}.$$

Due to (0), Lemma 2.2 and (1) we obtain

$$(2) \qquad D(x,y) \le ||x + \mathbf{i}\, y||_c.$$

In order to get the inequality opposite to (2) we fix $x, y \in E$ and put $x = \alpha_1 u + \alpha_2 v$, $y = \beta_1 u + \beta_2 v$ as in Lemma 2.1. Then

$$\begin{aligned} D(x,y) &= D(\alpha_1 u + \alpha_2 v, \beta_1 u + \beta_2 v) \\ &= ||u|| + ||v|| = \sqrt{\alpha_1^2 + \beta_1^2}\,||u|| + \sqrt{\alpha_2^2 + \beta_2^2}\,||v|| \ge ||x + \mathbf{i}\, y||_c. \end{aligned}$$

and the proof of part (i) of the theorem is completed.

Proof of (ii). Take any vectors $x, y \in E$ such that $<x,y>=0$, $<x,x>=<y,y>=1$ and define $z := x + \mathbf{i}\, x \in \tilde{E}$, $w := y + \mathbf{i}(-y) \in \tilde{E}$. Then it is not difficult to calculate that $||z||_c^2 = ||w||_c^2 = 2$ and $||z+w||_c^2 = ||(x+y) + \mathbf{i}\,(x-y)||_c^2 = 8$, $||z+w||_c^2 = ||(x-y) + \mathbf{i}\,(x+y)||_c^2 = 8$, so

$$\frac{||z+w||_c^2 + ||z-w||_c^2 - 2||z||_c^2 - 2||w||_c^2}{2(||z||_c^2 + ||w||_c^2)} = 1,$$

hence $\gamma(||\;\;||_c) = 1$. The proof of Theorem 1.1 is completed.

3. Applications.

REMARK 1. It is worth recalling that if the series of homogeneous polynomials $\sum_{k=0}^{\infty} f_k(x)$ is convergent for every $x \in B := \{x \in E : <x, x> < 1\}$, then the complexified series $\sum_{k=0}^{\infty} \tilde{f}_k(x + \mathbf{i}\, y)$ is convergent for $x + \mathbf{i}\, y \in \tilde{B} := \{x + \mathbf{i}\, y \in \tilde{E} : \ ||x+\mathbf{i}\, y||_c < 1\}$ (here the complexification $\tilde{f}_k$ of a homogeneous polynomial f_k is defined by the formula $\tilde{f}(x+\mathbf{i}\, y) = \sum_{j=0}^{k} \binom{k}{j} \mathbf{i}^j \widehat{f}(x, ..., \overset{j}{x}, y, ..., y)$, where $\widehat{f}$ is a unique symmetrical k-linear map such that $\widehat{f}(x, ..., x) = f(x)$ for $x \in E$), cf. [D].

In particular we get the following

THEOREM. *Any harmonic function defined on a euclidean ball* $B = \{x \in \mathbf{R}^n : ||x|| < R\}$ *in* $\mathbf{R}^n$ *can be uniquelly holomorphically extended to the holomorphic function* F *defined on the complexification* $\tilde{B} = \{x \in \mathbf{C}^n : ||x+\mathbf{i}\, y||_c < R\}$ *of the ball* B.

The ball $\tilde{B}$ is the so called Lie ball, cf. [H]. For other useful and nontrivial applications of Theorem 1.1, cf. e. g. [D, M, MW, S].

REMARK 2. Preserving the notation of Section 2 one can check that

$$D(x, y) = \sqrt{\lambda_1(x, y)} + \sqrt{\lambda_2(x, y)} \quad \text{for any} \quad x, y \in E,$$

where $\lambda_1(x, y)$, $\lambda_2(x, y)$ are the eigenvalues of Gramm's matrix

$$\begin{bmatrix} <x, x> & <x, y> \\ <y, x> & <y, y> \end{bmatrix}.$$

Using the above formula and properties of eigenvalues one can give another proof of Lemma 2.2.

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