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EXTREMAL VALUES OF REAL FUNCTIONS ON SUBMANIFOLDS IN $\mathbf{R}^m$

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Abstract. In the paper we give a generalization of the Lagrange method of finding the extremal values on submanifolds of real functions.

In this short note we want to give a method for finding extremal on a submanifold in $\mathbf{R}^m$ values of a real function defined on some neighbourhood of the submanifold. Before we state the main result of the paper we shall give some basic notions and summarize the main results concerning the topic. In the sequel N will denote a d-dimensional submanifold in $\mathbf{R}^m$ $(d, m \geq 1)$ of class C^1 and Ω some open set in $\mathbf{R}^m$ such that $N \subset \Omega$. We denote the tangent space to N at some point $a \in N$ by T_aN. We shall impose some additional conditions on N, as a need arrise. We also assume that $f : \Omega \longrightarrow \mathbf{R}$.

For our purposes we shall need the following classical characterization of a tangent space:

LEMMA 1. *A vector $v \in \mathbf{R}^m$ belongs to T_aN for some $a \in N$ if and only if:*

$$\text{There are: a sequence } \{h_\nu\} \text{ in } \mathbf{R}^m, \text{ a sequence } \{\lambda_\nu\},\ \lambda_\nu \geq 0,$$
$$\text{such that } a + h_\nu \in N \text{ and } h_\nu \to 0,\ \lambda_\nu h_\nu \to v,\ (\nu \to +\infty).$$

We know the following necessary condition for a mapping f to have an extremum at a point $a \in N$.

PROPOSITION 2. *If a function $f|_N : N \longrightarrow \mathbf{R}$ has an extremum at a and f is differentiable at a, then*

$$f'(a)|_{T_aN} = 0.$$

From now on we assume that our function f is n-differentiable at $a \in N$, $n \geq 2$. Assume that $f'(a) = \ldots = f^{(n-1)}(a) = 0$. Define Φ as the following form of degree n:

$$\Phi\colon T_aN \ni h \longrightarrow f^{(n)}(a)(h)^{(n)} := f^{(n)}(a)(h, \ldots, h) \in \mathbf{R}.$$

We say that the form Φ of degree n is *positive (negative) definite* if $\Phi(v) > 0$ ($\Phi(v) < 0$) for any $v \in T_aN \setminus \{0\}$. The form Φ is said to be *positive (negative) semidefinite* if $\Phi(v) \geq 0$ ($\Phi(v) \leq 0$) for any $v \in T_aN$. The form Φ is said to be *indefinite* if there are two vectors $v, w \in T_aN$ such that $\Phi(v) > 0$ and $\Phi(w) < 0$.

An easy computation based on Taylor's formula gives the following proposition.

PROPOSITION 3.
(1) If Φ is positive (negative) definite, then $f|_N$ has a strict minimum (maximum) at a.
(2) If Φ is indefinite, then $f|_N$ has no extremum at a.

Remark that if in Proposition 3 we assume only that Φ is semidefinite, then we do not get the result as in Proposition 3.

On the other hand, if Φ is not zero and n is an odd number, then in view of Proposition 3 $f|_N$ has no extremum at a.

If $f'(a) = \ldots = f^{(n-1)}(a) = 0$, then to some (relatively large) extent the properties of $f|_N$ are hidden in the properties of Φ. Let us remark here that this depends only on the restriction of $f|_N$. This piece of information we also get from the following theorem.

PROPOSITION 4. *If in our situation $f|_N = 0$, then $\Phi = 0$.*

PROOF. One needs to show that $f^{(n)}(a)(v)^{(n)} = 0$, if only $v \in T_aN$. Fix $v \in T_aN$. In view of Lemma 1 there are sequences

$$\{h_\nu\} \text{ in } \mathbf{R}^m, \quad \{\lambda_\nu\},\ \lambda_\nu \geq 0$$

such that

$$a + h_\nu \in N \text{ and } h_\nu \to 0, \quad \lambda_\nu h_\nu \to \nu, \quad (\nu \to +\infty).$$

In view of Taylor's formula we get

$$0 = f(a + h_\nu) - f(a) = \frac{1}{n!} f^{(n)}(a)(h_\nu)^{(n)} + \eta(h_\nu)\|h_\nu\|^n,$$

where

$$\eta\colon \Omega - a \longrightarrow \mathbf{R}, \quad \eta(0) = 0, \quad \eta - \text{ continuous at } 0.$$

We then have

$$\frac{1}{n!} f^{(n)}(a)(\lambda_\nu h_\nu)^{(n)} + \eta(h_\nu)\|\lambda_\nu h_\nu\|^n = 0.$$

And consequently

$$f^{(n)}(a)(v)^{(n)} = 0,$$

which completes the proof.

Now we are coming to the main point of the paper.

THEOREM 5. *Let us assume that $f\colon \Omega \longrightarrow \mathbf{R}$, where Ω is open in $\mathbf{R}^m$ and N is a d-dimensional submanifold of class C^k in $\mathbf{R}^m$, such that $N \subset \Omega$, $k \in \{1, 2, \dots, +\infty\}$. Assume additionally that $N = h^{-1}(0)$, where $h\colon \Omega \longrightarrow \mathbf{R}^{m-d}$ is a submersion of the class C^k and f is k-differentiable at $a \in N$. Let us assume that for some n, $1 \le n < k$, we have*

$$f'(a) = \dots = f^{(n-1)}(a) = 0$$

and

$$f^{(n)}(a)|_{T_a N} = 0.$$

Then there is a function $\tilde{f}\colon \Omega \longrightarrow \mathbf{R}$, k-differentiable at $a \in N$ such that

(1) $$\tilde{f}|_N = f|_N,$$

(2) $$\tilde{f}'(a) = \dots = \tilde{f}^{(n)}(a) = 0.$$

PROOF. In the first part of our proof we show that there are homogeneous polynomials w_j of degree $n-1$ such that

(I) $$f^{(n)}(a)(x)^{(n)} = \sum_{j=1}^{m-d} w_j(x) h_j'(a)(x).$$

Then we shall prove that the function $\tilde{f}$ given by the formula

(II) $$\tilde{f}(x) = f(x) - \frac{1}{n!} \sum_{j=1}^{m-d} w_j(x-a) h_j(x)$$

is a function we are looking for.

Let us begin with the proof of (I). Remark that $f^{(n)}(a)(x)^{(n)}$ is a homogeneous polynomial of degree n, vanishing on T_aN. The forms $h_j'(a)(x)$ generate the ideal of polynomials vanishing on T_aN. Consequently, there are homogeneous polynomials $w_j(x)$ of degree $n-1$, as desired in (I).

PROOF OF (II).

Remark that $\tilde{f}$ is k-differentiable at $a \in N$ $(k \geq n)$. The functions f and $\tilde{f}$ are equal on N, because $h|_N = 0$. This gives (1). For the proof of (2) let us compare both sides of the formula

$$\tilde{f}(x) = f(x) - \frac{1}{n!}\sum_{j=1}^{m-d} w_j(x-a)h_j(x).$$

From Taylor's formula and the assumptions about f and h_j we get

$$\begin{aligned} f(x) &= f(a) + \frac{1}{n!}f^{(n)}(a)(x-a)^{(n)} + R_f(x-a), \\ &\quad \text{where } R_f(x-a) = o(\|x-a\|^n), \\ h_j(x) &= h_j'(a)(x-a) + R_{h_j}(x-a), \\ &\quad \text{where } R_{h_j}(x-a) = o(\|x-a\|), \end{aligned}$$

for $j = 1, \dots, m-d$.

Consequently

$$\begin{aligned} \tilde{f}(x) = & f(a) + \frac{1}{n!}f^{(n)}(a)(x-a)^{(n)} + R_f(x-a) \\ & -\frac{1}{n!}\sum_{j=1}^{m-d} w_j(x-a)\left[h_j'(a)(x-a) + R_{h_j}(x-a)\right]. \end{aligned}$$

Remark that

$$R_f(x-a) - \frac{1}{n!}\sum_{j=1}^{m-d} w_j(x-a)R_{h_j}(x-a) = o(\|x-a\|^n).$$

So

$$\tilde{f}(x) = u_0(x-a) + u_1(x-a) + \ldots + u_{n-1}(x-a) + u_n(x-a) + R(x-a),$$

where $R(x-a)=o(\|x-a\|^n)$, and u_i is a homogeneous polynomial of degree i , $i=0,\dots,n$. Let us see what the polynomials u_i look like.

$$u_0(x-a)=f(a),$$
$$u_1(x-a)=\ldots=u_{n-1}(x-a)=0,$$
$$u_n(x-a)=\frac{1}{n!}f^{(n)}(a)(x-a)^{(n)}-\frac{1}{n!}\sum_{j=1}^{m-d}w_j(x-a)h_j'(a)(x-a)$$
$$=\frac{1}{n!}\sum_{j=1}^{m-d}w_j(x-a)h_j'(a)(x-a)-\frac{1}{n!}\sum_{j=1}^{m-d}w_j(x-a)h_j'(a)(x-a)=0$$

On the other hand

$$\tilde{f}(x)=\tilde{f}(a)+\frac{1}{1!}\tilde{f}'(a)(x-a)+\ldots+\frac{1}{n!}\tilde{f}^{(n)}(a)(x-a)^{(n)}+R_{\tilde{f}}(x-a),$$
$$\text{where } R_{\tilde{f}}(x-a)=o(\|x-a\|^n).$$

Consequently

$$\tilde{f}(x)=v_0(x-a)+v_1(x-a)+\ldots+v_n(x-a)+R_f(x-a),$$

where v_i is a homogeneous polynomial of degree i, $i=0,\dots,n$. In view of Theorem 7.1.2. in [C], we have the following equality

$$u_0+u_1+\ldots+u_n=v_0+v_1+\ldots+v_n.$$

The identity principle for polynomials implies that

$$\frac{1}{i!}\tilde{f}^{(i)}(a)=v_i=u_i=0,\quad i=1,2,\dots,n,$$

which completes the proof.

REMARK 6. If f is $(n+1)$-differentiable at $a\in N$ then in view of Theorem 5, the defiteness of form Ψ given by the formula

$$\Psi\colon T_aN\ni h\longrightarrow \tilde{f}^{(n+1)}(a)(h)^{(n+1)}\in\mathbf{R}$$

may imply that f has an extremum at a.

Remark that in view of Theorem 4 the form Ψ depends only on $f|_N$, and does not depend on h and the extension of f.

REMARK 7. Let us remark that for $n = 1$ Theorem 5 is the well-known the Lagrange Theorem. In this case we may easily find the desired modification of a function f. We need only to solve the following system of equations

$$f'(a) = \sum_{j=1}^{m-d} \lambda_j h_j'(a), \quad \lambda_j \in \mathbf{R}, \quad j = 1, \dots, m-d.$$

The desired modification is given by the formula

$$\tilde{f}(x) = f(x) - \sum_{j=1}^{m-d} \lambda_j h_j(x),$$

and we may operate it using the second derivative if only $k \geq 2$.

EXAMPLE 8. Let $N = \{(x,y) \in \mathbf{R}^2 : x^3 - \sin y = 0\}$, $a = (0,0)$. Then $T_aN = \mathbf{R} \times \{0\}$. Put $f : \mathbf{R}^2 \ni (x,y) \longrightarrow x^4 + y^3 \in \mathbf{R}$. Then $f'(a) = f''(a) = 0$ and $f^{(3)}(a)|T_aN = 0$ but $f^{(3)}(a) \neq 0$ because $\frac{\partial^3 f}{\partial y^3}(a) = 6$. From the proof of Theorem 5 we get $\tilde{f}(x,y) = f(x,y) + y^2(x^3 - \sin y)$, so $\tilde{f}'(a) = \tilde{f}''(a) = \tilde{f}^{(3)}(a) = 0$ and $\frac{\partial^4 \tilde{f}}{\partial x^4}(a) = 24$, which proves that $\tilde{f}$ and hence f has a minimum on N at a (see Proposition 3).

REMARK 9. It is worth underlining that the method of getting a new function $\tilde{f}$ in Theorem 5 may be continued inductively in the following sense: Assume that $f'(a) = \ldots = f^{(n-1)}(a) = 0$ and that $f^{(n)}(a)|T_aN = 0$ for some $1 \leq n < k$, then in view of Theorem 5 we get some new function $\tilde{f}$ with $\tilde{f}|N = f|N$ but such that $\tilde{f}'(a) = \ldots = \tilde{f}^{(n)}(a) = 0$. If the form $\tilde{f}^{(n+1)}(a)$ is definite or indefinite, then we know whether at the point a the function f has an extremum but if $\tilde{f}^{(n+1)}(a)|T_aN = 0$, then we may apply Theorem 5 to $\tilde{f}$ instead of f and $n+1$ instead of n (under the assumption $k > n+1$). Consequently, we see that this procedure leads us to solving the problem of extremality of f as long as the suitable forms are not semidefinite on the tangent space.

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