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EMBEDDING OF STRICTLY PSEUDOCONVEX DOMAINS INTO CONVEX DOMAINS AND A LOCAL RETRACTION

BY PIOTR JAKÓBCZAK

Abstract. We prove that Fornaess' construction of the embedding of the strictly pseudoconvex domain into a convex domain admits a local holomorphic retraction near a given boundary point.

In [1] Fornæss proved the following theorem on the embedding of strictly pseudoconvex domains in $\mathbf{C}^n$ into convex domains:

THEOREM ([1], Thm 9, p. 543). *Let D be a strictly pseudoconvex bounded domain in $\mathbf{C}^n$ with C^k boundary, $k \geq 2$. Then there exist an integer $m \geq n$, a strictly convex domain $C \subset \mathbf{C}^m$ with C^k boundary, a neighborhood D' of $\overline{D}$ in $\mathbf{C}^n$, and a holomorphic mapping $\psi\colon D' \longrightarrow \mathbf{C}^m$ such that ψ maps D' biholomorphically onto some closed complex submanifold $\psi(D')$ of $\mathbf{C}^m$, $\psi(D) \subset C$, $\psi(D'\backslash\overline{D}) \subset \mathbf{C}^m\backslash\overline{C}$, and $\psi(D')$ intersects ∂C transversally.*

(The notions of strictly pseudoconvex and strictly convex domains, which are now classical in complex analysis, will be explained below; we refer also the reader to standard monographs on the subject, e.g. to [2].)

This theorem has found many applications in complex analysis, e.g. to the problem of extension of holomorpic functions from submanifolds to complex domains with preservation of the smoothness on the boundary.

The aim of this note is to verify that Fornæss' construction of the embedding can be done in such a way that it has another useful property, namely that there exists the local holomorphic retraction from the convex domain C onto $\psi(D')$ near a point $\psi(z_0)$, where z_0 is a given point of ∂D. More precisely, we prove the following result:

THEOREM 1. *Let D be a strictly pseudoconvex bounded domain in $\mathbf{C}^n$ with C^k boundary, $k \geq 2$. Fix $z_0 \in \partial D$. Then there exist a strictly convex domain C with C^k boundary in some $\mathbf{C}^m$, a neighborhood D' of $\overline{D}$, a holomorphic mapping $\psi: D' \longrightarrow \mathbf{C}^m$, and a neighborhood W of $\psi(z_0)$ in $\mathbf{C}^m$ such that D, D', C, and ψ, satisfy all assertions of Fornæss' theorem, and moreover there exists a holomorphic mapping $\pi: W \longrightarrow \psi(D')$ such that for every $w \in W$, $\pi(w) \in W \cap \psi(D')$, for every $w \in W \cap \psi(D')$, $\pi(w) = w$, and for every $w \in (W \cap \overline{C})\backslash\psi(D')$, $\pi(w) \in \psi(D)$.*

Let D be a strictly pseudoconvex bounded domain in $\mathbf{C}^n$ with C^k boundary, $k \geq 2$. (This means that there exists a neighborhood V of ∂D and a strictly plurisubharmonic function $\rho \in C^k(V)$ such that for all $z \in V$, $\operatorname{grad}\rho(z) \neq 0$, and $D \cap V = \{z \in V \mid \rho(z) < 0\}$.) To give the proof of Theorem 1 we need the following proposition which is a special case of [1], Prop. 1, p. 530.

PROPOSITION 2. *Let D be a strictly pseudoconvex bounded domain in $\mathbf{C}^n$ with C^k boundary. Let $z_0 \in \partial D$ be fixed. Then there exist a strictly convex set $C \subset \mathbf{C}^n$ with C^k boundary, a neighborhood $\widehat{D}$ of $\overline{D}$, a holomorphic mapping $\phi: \widehat{D} \longrightarrow \mathbf{C}^n$, and a neighborhood U of z_0 in $\mathbf{C}^n$ such that $z_0 \in U \subset\subset \widehat{D} \cap V$, and:*

(i) $\phi(D) \subset C$,
(ii) $\phi(\overline{D}) \subset \overline{C}$,
(iii) $\phi(\{z \in U \mid \rho(z) < 0\}) \subset C$,
(iv) $\phi(\{z \in U \mid \rho(z) > 0\}) \subset \mathbf{C}^n\backslash\overline{C}$,
(v) $\phi\big|_U: U \longrightarrow \phi(U)$ *is a biholomorphism.*

(We say that a domain C is *strictly convex with C^k boundary* if C is bounded and there exists a strictly convex function $\tau \in C^k(\mathbf{C}^n)$ such that $C = \{w \in \mathbf{C}^n \mid \tau(w) < 0\}$).

From now on we fix a point $z_0 \in \partial D$.

For every point $z \in \partial D$, choose a strictly convex domain $C_z \subset \mathbf{C}^n$ with C^k boundary, a neighborhood $\widehat{D}_z$ of $\overline{D}$, a holomorphic mapping $\phi_z: \widehat{D}_z \longrightarrow \mathbf{C}^n$, and a neighborhood U_z of z in $\mathbf{C}^n$ with properties listed in the assertion of Proposition 1. Since ∂D is compact, we can choose a finite number of points $z_1, \ldots, z_k \in \partial D$ such that $\partial D \subset U_{z_1} \cup \ldots \cup U_{z_k}$. We may assume that $z_1 = z_0$. We can also shrink the neighborhoods $U_{z_2}, \ldots, U_{z_k}$ so that they still cover ∂D but

$$z_0 \notin U_{z_2} \cup \ldots \cup U_{z_k}. \tag{1}$$

Choose a neighborhood $\widehat{D}$ of $\overline{D}$ such that $\widehat{D} \subset\subset D \cup (U_{z_1} \cup \ldots \cup U_{z_k})$.

Define

$$\psi_1 = (\phi_{z_1}, \dots, \phi_{z_k}) \colon \widehat{D} \longrightarrow \mathbf{C}^{kn}.$$

Then $\psi_1(D) \subset C_{z_1} \times \dots \times C_{z_k}$, $\psi_1(\overline{D}) \subset \overline{C_{z_1}} \times \dots \times \overline{C_{z_k}}$ (by Proposition 1 (i) and (ii)), and $\psi_1(\widehat{D} \setminus \overline{D}) \subset \mathbf{C}^{kn} \setminus (\overline{C_{z_1}} \times \dots \times \overline{C_{z_k}})$ (since every point $z' \in \widehat{D} \setminus \overline{D}$ belongs to a certain set of the form $\{z \in U_{z_j} \mid \rho(z) > 0\}$ for some $j = 1, \dots, k$, and so $\phi_{z_j}(z') \notin \overline{C_{z_j}}$ by Proposition 1 (iv)). Denote $C_{z_j} = C_j$, $j = 1, \dots, k$.

We may assume that the neighborhood $\widehat{D}$ of $\overline{D}$, chosen above, is a domain of holomorphy. Then there exist a positive integer s and a holomorphic mapping $\psi_2 \colon \widehat{D} \longrightarrow \mathbf{C}^s$ which maps $\widehat{D}$ biholomorphically onto some closed submanifold $\psi_2(\widehat{D})$ of $\mathbf{C}^s$. Since $\overline{D}$ is compact, there exists a ball $C_{k+1} \subset \mathbf{C}^s$ such that $\psi_2(D) \subset C_{k+1}$. Then the mapping $\psi = (\psi_1, \psi_2) \colon \widehat{D} \longrightarrow \mathbf{C}^{kn+s}$ has following properties:

PROPOSITION 3 ([1], Theorem 4, p.537).

(i) $\psi \colon \widehat{D} \longrightarrow \psi(\widehat{D}) \subset \mathbf{C}^{kn+s}$ *is a biholomorphism,*
(ii) $\psi(D) \subset C_1 \times \dots \times C_k \times C_{k+1}$,
(iii) $\psi(\widehat{D} \setminus \overline{D}) \subset \mathbf{C}^{kn+s} \setminus (\overline{C_1} \times \dots \times \overline{C_k} \times \overline{C_{k+1}})$.

We take a neighborhood $U =: U_{z_1}$ of $z_0 = z_1$ chosen in accordance with Proposition 2 and (1). Consider the set

$$W = \phi_{z_1}(U) \times \mathbf{C}^{(k-1)n+s}.$$

This is a neighborhood of $\psi(z_0)$ in $\mathbf{C}^{kn+s}$. By Proposition 2 (v) the mapping $\chi =: \left(\phi_{z_1}\big|_U\right)^{-1} \colon \phi_{z_1}(U) \longrightarrow U$ is a biholomorphism. Since for $w = (w_1, w_2, \dots, w_{k+1}) \in W$ we have $w_1 \in \phi_{z_1}(U)$, the mapping χ is well defined at w_1 and $\chi(w_1) \in U$. Let

$$\pi_1 \colon W \ni w = (w_1, w_2, \dots, w_{k+1}) \longrightarrow w_1 \in \phi_{z_1}(U).$$

Define

$$\pi = \psi \circ \chi \circ \pi_1 \colon W \longrightarrow \psi(U). \tag{2}$$

Then π is holomorphic. Suppose that

$$w = (w_1, w_2, \dots, w_{k+1}) \in W \cap \psi(\widehat{D}).$$

Then

$$w = (w_1, w_2, \dots, w_{k+1}) = (\phi_{z_1}(z), \dots, \phi_{z_k}(z), \psi_2(z)) = \psi(z)$$

for a unique $z \in \widehat{D}$ since ψ_2 is one-to-one. Moreover, by definition,

$$\pi(w) = \psi \circ \chi(\phi_{z_1}(z)) = \psi((\chi \circ \phi_{z_1})(z)) = \psi(z),$$

since $\chi \circ \phi_{z_1}$ is the identity. Hence $\pi(w) = w$, and so $\pi \colon W \longrightarrow W \cap \psi(U)$ is a holomorphic retraction of W onto $W \cap \psi(\widehat{D})$. Moreover, if

$$w = (w_1, \ldots, w_{k+1}) \in W \cap (C_1 \times \ldots \times C_{k+1}),$$

then $w_1 \in \phi_{z_1}(U) \cap C_1 = \phi_{z_1}(U \cap D)$, and so

$$\pi(w) \ \in \ \psi(U \cap D) \ \subset \ C_1 \times \ldots \times C_{k+1}.$$

Therefore π maps $W \cap (C_1 \times \ldots \times C_{k+1})$ onto $W \cap \psi(\widehat{D}) \cap (C_1 \times \ldots \times C_{k+1})$.

PROPOSITION 4 ([1], Proposition 6, p.541). *Let D, V, ρ, z_0, C, $\widehat{D}$, ϕ, U be the same as in Proposition 2. Let m be some positive integer, and let $\Gamma \colon \widehat{D} \longrightarrow \mathbf{C}^m$ be a (arbitrary) holomorphic mapping. Set $\eta = (\phi, \Gamma) \colon \widehat{D} \longrightarrow \mathbf{C}^{n+m}$. Then there exist an open set U' in $\mathbf{C}^n$ with $z_0 \in U' \subset\subset U$ and a strictly convex bounded domain $C' \subset \mathbf{C}^{n+m}$ with C^k boundary such that:*

(i) $\eta(D) \subset C'$,
(ii) $\eta(\overline{D}) \subset \overline{C'}$,
(iii) $\eta(\{z \in U' \mid \rho(z) < 0\}) \subset C'_p$,
(iv) $\eta(\{z \in U' \mid \rho(z) > 0\}) \subset \mathbf{C}^{n+m} \backslash \overline{C'}$.

The proof of this proposition is similar to the proof of Proposition 6, p. 541 from [1].

Take $\widehat{D}$, $U_{z_1}, \ldots, U_{z_k}$, $C_1, \ldots, C_k, C_{k+1}$, and ψ as in Proposition 3. Consider the mapping $\psi = (\psi_1, \psi_2) = (\phi_{z_1}, \ldots, \phi_{z_k}, \psi_2)$. Choose any $p \in \partial$D. Then there exists $j = 1, \ldots, k$ such that $p \in U_{z_j}$, and moreover D, V, ρ, p, C_j, $\widehat{D}$, ϕ_{z_j}, U_{z_j} satisfy the assertions of Proposition 2. Fix this j. We may consider ψ in the form

$$\psi = \eta = \left(\phi_{z_1}, \ldots, \phi_{z_j}, \ldots, \phi_{z_k}, \psi_2\right) = \left(\phi_{z_j}, \Gamma\right),$$

and then apply Proposition 4 with D, V, ρ, p, C_j, $\widehat{D}$, ϕ_{z_j}, U_{z_j}. Then there exists an open set U'_p in $\mathbf{C}^n$ with $p \in U'_p \subset\subset U_{z_j}$ and a strictly convex bounded open set $C'_p \subset \mathbf{C}^{kn+s}$ with C^k boundary such that

$$\begin{aligned} &(i) \ \ \psi(D) \subset C'_p, \\ &(ii) \ \ \psi(\overline{D}) \subset \overline{C'_p}, \\ &(iii) \ \ \psi(\{z \in U'_p \mid \rho(z) < 0\}) \subset C'_p, \\ &(iv) \ \ \psi(\{z \in U'_p \mid \rho(z) > 0\}) \subset \mathbf{C}^{kn+s} \backslash \overline{C'_p}. \end{aligned} \tag{3}$$

By compactness we may take a finite number of points $p_1, \dots, p_t \in \partial D$ such that $\partial D \subset U_{p_1} \cup \dots \cup U_{p_t}$. Then

$$\psi(D) \subset C'_{p_1} \cap \dots \cap C'_{p_t}. \tag{4}$$

Moreover, we may still asume that $z_0 = p_1$ and that $z_0 \notin U_{p_2} \cup \dots \cup U_{p_t}$. Denote $C'_{p_j} = C'_j$, $j = 1, \dots, t$. Then $\psi(D) \subset C'_1 \cap \dots \cap C'_t$, and moreover from the above considerations it follows that we can choose a neighborhood $\widehat{D}'$ of $\overline{D}$ such that $\widehat{D}' \subset (D \cup U_{p_1} \cup \dots \cup U_{p_t}) \cap \widehat{D}$, and

$$\psi(\widehat{D}' \backslash \overline{D}) \subset \mathbf{C}^{nt+s} \backslash \left(\overline{C'_1} \cap \dots \cap \overline{C'_t}\right). \tag{5}$$

We also note that from Proposition 3 (iii) it follows that

$$\overline{\psi(\widehat{D} \backslash \widehat{D}')} \cap (\overline{C_1} \times \dots \times \overline{C_k} \times \overline{C_{k+1}}) = \emptyset.$$

Since the set $\overline{C_1} \times \dots \times \overline{C_k} \times \overline{C_{k+1}}$ is convex in $\mathbf{C}^{kn+s}$, there exists a function σ of class C^k which is strictly convex in $\mathbf{C}^{kn+s}$, and such that $\sigma < 0$ on $\overline{C_1} \times \dots \times \overline{C_k} \times \overline{C_{k+1}}$ and $\sigma > 0$ on $\overline{\psi(\widehat{D} \backslash \widehat{D}')}$.
Set $C'_{t+1} = \{z \in \mathbf{C}^{kn+s} \mid \sigma(z) < 0\}$. Then $\psi(D) \subset \overline{C_1} \times \dots \times \overline{C_k} \times \overline{C_{k+1}} \subset C'_{t+1}$ and

$$\overline{\psi(\widehat{D} \backslash \widehat{D}')} \cap \overline{C'_{t+1}} = \emptyset \tag{6}$$

By (4), (5), and (6), we therefore obtain

PROPOSITION 5. *Let ψ be as in Proposition 3. Then:*

(i) $\psi \colon \widehat{D} \longrightarrow \psi(\widehat{D}) \subset \mathbf{C}^{kn+s}$ *is a biholomorphism,*
(ii) $\psi(D) \subset C'_1 \cap \dots \cap C'_{t+1}$,
(iii) $\psi\left(\widehat{D} \backslash \overline{D}\right) \subset \mathbf{C}^{kn+s} \backslash \left(\overline{C'_1} \cap \dots \cap \overline{C'_{t+1}}\right)$.

Set now $U' := U'_{p_1} \subset\subset U_{z_1} = U_{z_0} = U$, where U'_{p_1} is chosen with respect to the point $p_1 = z_0$ according to Proposition 4 and (3). (The inclusion follows from Proposition 4.) Consider the set

$$W' = \phi_{z_1}(U') \times \mathbf{C}^{(n-1)k} \times \mathbf{C}^s.$$

This is a neighborhood of $\psi(z_0)$ in $\mathbf{C}^{kn+s}$ and the mapping

$$\chi' = (\phi_{z_1}|_{U'})^{-1} \colon \phi_{z_1}(U') \longrightarrow U'$$

is a biholomorphism. Similarly to (2) we define

$$\pi' = \psi \circ \chi' \circ \pi_1 : W' \longrightarrow \psi(U').$$

(This is a restriction of π from (2) to W'.) Then π' is holomorphic and it is a holomorphic retraction from W' onto $W' \cap \psi(U')$. (The argument is the same as that concerning π from (2).) Moreover, if

$$w = (w_1, w_2, \dots, w_{k+1}) \in W' \cap (C'_1 \cap \dots \cap C'_{t+1}),$$

then in particular

$$w = \psi(z) = (\phi_{z_1}(z), \phi_{z_2}(z), \dots, \phi_{z_k}(z), \psi_2(z)) = (\phi_{z_1}(z), \Gamma(z)) \in C'_1$$

for some $z \in U' = U'_{p_1}$, and by Proposition 4, we have $\rho(z) < 0$, i.e. $z \in D$. Hence $z \in D \cap U'$, and so

$$\pi'(w) \in \psi(U' \cap D) \subset C'_1 \cap \dots \cap C'_{t+1}.$$

Therefore, π' maps $W' \cap (C'_1 \cap \dots \cap C'_{t+1})$ onto $\psi(\widehat{D}) \cap (C'_1 \cap \dots \cap C'_{t+1})$.

Hence we have obtained the following result:

PROPOSITION 6. *Let D be a strictly pseudoconvex domain in $\mathbf{C}^n$ with C^k boundary. Let z_0 be fixed in ∂D. Then there exist a positive integer N, a neighborhood $\widehat{D}$ of $\overline{D}$, a holomorphic mapping $\psi : \widehat{D} \longrightarrow \mathbf{C}^N$ which maps $\widehat{D}$ biholomorphically onto $\psi(\widehat{D})$, and a finite number of strictly convex domains $C_1, \dots, C_r$ in $\mathbf{C}^N$ with C^k boundaries such that*

(i) $\psi(D) \subset C_1 \cap \dots \cap C_r$,
(ii) $\psi(\widehat{D} \setminus \overline{D}) \subset \mathbf{C}^N \setminus (\overline{C_1} \cap \dots \cap \overline{C_r})$,
(iii) $\psi(z_0) \in \partial C_1 \setminus (\partial C_2 \cup \dots \cup \partial C_r)$,
(iv) *there exist a neighborhood W of $\psi(z_0)$ in $\mathbf{C}^N$ and a holomorphic retraction $\pi : W \longrightarrow W \cap \psi(\widehat{D})$ such that $\pi(W \cap (C_1 \cap \dots \cap C_r)) = W \cap (C_1 \cap \dots \cap C_r) \cap \psi(\widehat{D})$.*

(Since $\psi(z_0) \in \partial(C_1 \cap \dots \cap C_r)$, condition (iii) means that $\psi(z_0) \in \partial C_1$ and $\psi(z_0) \in \operatorname{int} C_j$ for $j = 2, \dots, r$.)

We may also assume that $W \cap \partial C_j = \emptyset$ for $j = 2, 3, \dots, r$.

We conclude the proof of Theorem 1 by the following observation: Fornæss' construction of the passage from $\psi(D)$ embedded into $C_1 \cap \dots \cap C_r$ to $\psi(D)$ embedded into one strictly convex domain C can be done in such a way that if we choose an open neighborhood T of these points of $\partial(\overline{C_1} \cap \dots \cap \overline{C_r})$ which belong to more than one ∂C_j, then the set $\overline{C_1} \cap \dots \cap \overline{C_r}$ may only change those points which belong to T. Therefore we may assume that this step of Fornæss' construction does not change W and $W \cap (C_1 \cap \ldots \cap C_r)$.

NOTE. The existence of the embedding of strictly pseudoconvex domain into a convex one, which admits a local retraction near a given point (as described in Theorem 1), was found application to the study of the Carathéodory metric in strictly pseudoconvex domains by using the properties of that metric for strictly convex domains; this was communicated to me by Jarnicki and Pflug.

PROBLEM. A closer examination of the proof of Fornaess' embedding theorem shows that in fact a stronger result than that described in Theorem 1 is true:

THEOREM 2. *Let D be a strictly pseudoconvex bounded domain in $\mathbf{C}^n$ with C^k boundary, $k \geq 2$. Let $\epsilon > 0$ be given. Then there exist a compact subset $K \subseteq \partial D$ such that $\mu(\partial D \backslash K) < \epsilon$ (where μ denotes the surface measure on ∂D), a strictly convex domain C with C^k boundary in some $\mathbf{C}^m$, a neighborhood D' of $\overline{D}$, a biholomorphic mapping $\psi\colon D' \longrightarrow \mathbf{C}^m$, and a neighborhood W of $\psi(\partial D \backslash K)$ in $\mathbf{C}^m$, such that D, D', C, and ψ satisfy assertions of Fornaess' theorem, and moreover there exists a holomorphic mapping $\pi\colon W \longrightarrow \psi(D')$ such that for every $w \in W$, $\pi(w) \in W \cap \psi(D')$, for every $w \in W \cap \psi(D')$, $\pi(w) = w$, and for every $w \in (W \cap \overline{C}) \backslash \psi(D')$, $\pi(w) \in \psi(D)$. Moreover, given any finite number $z_1, \ldots, z_s$ of points of ∂D, one can choose K and W in such a way that $z_1, \ldots, z_s \in \partial D \backslash K$ (i.e. the retraction π is defined in some neighborhood of $\psi(z_1), \ldots, \psi(z_s)$).*

However, it is not known whether one can choose an embedding ψ such that for *every* point $\psi(z) \in \psi(\partial D)$ there exist some neighborhood W_z of the point $\psi(z)$ in $\mathbf{C}^m$ and a holomorphic retraction from W_z onto $\psi(D')$.

References

1. Fornæss J-E., *Embedding strictly pseudoconvex domains in convex domains*, Amer. J. Math. **98, No.1** (1976), 529–569.
2. Krantz S.G., *Function Theory of Several Complex Variables*, John Wiley & Sons, 1982.

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e-mail: jakobcza@im.uj.edu.pl