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MODIFIED MULTIVARIATE SECANT METHOD

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Abstract. In this paper an iterative method for solving the system of nonlinear equations is given.

I. Introduction.

The purpose of the present paper is to modify the multivariate secant method described by Janina Jankowska in [3] by imposing more conditions on the initial vectors so that the new iterative method will have a better convergence rate than the previous one. The system of nonlinear equations is of the form

$$f(x) = 0 \tag{1}$$

where $f : U \to \mathbf{R}^n$ is a vector function from the open subset U of the real Euclidean space $\mathbf{R}^n$ equipped with the norm,

$$\|x\| = |x_1| + |x_2| + \ldots + |x_n|,$$

to $\mathbf{R}^n$.

Assume that a set of $n+1$ initial vectors, $x^{(0)}, x^{(1)}, x^{(2)}, \ldots, x^{(n)}$, is given, the multivariate secant method for solving the system (1) is defined by the formula

$$x^{(k+n+1)} = x^{(k+n)} - X^{(k+n)} \left(F^{(k+n)}\right)^{-1} f^{(k+n)}, \quad k = 0, 1, 2, \ldots \tag{2}$$

where

$$f^{(m)} = f(x^{(m)}),$$
$$df^{(j)} = f^{(j+1)} - f^{(j)},$$
$$dx^{(j)} = x^{(j+1)} - x^{(j)}$$

and $X^{(m)}$ and $F^{(m)}$ are $n \times n$ matrices having as columns vectors, $dx^{(j)}$ and $df^{(j)}$ respectively, i. e.

$$X^{(m)} = [dx^{(m-n)}, \dots, dx^{(m-1)}] \text{ and}$$
$$F^{(m)} = [df^{(m-n)}, \dots, df^{(m-1)}], \quad (m \geq n).$$

It is assumed that $X^{(m)}$ and $F^{(m)}$ are nonsingular matrices for all $m \geq n$ (see [3] page 234).

II. Definitions.

DEFINITION 1. The value $[f(x)]$ of a vector function

$$f(x) = (f_1(x), f_2(x), \dots, f_n(x))$$

is defined by the formula

$$[f(x)] = f_1(x) + f_2(x) + \dots + f_n(x).$$

DEFINITION 2. The set of $n+1$ vectors, $\{x^{(i)}, x^{(i+1)}, \dots, x^{(i+n)}\}$, is said to be a mixture if $[f(x^{(i)})], [f(x^{(i+1)})], \dots, [f(x^{(i+n)})]$ have both nonnegative and negative values.

DEFINITION 3. The ordered set of $n+1$ vectors,

$$\{x^{(i)}, x^{(i+1)}, x^{(i+2)}, \dots, x^{(i+n)}\},$$

is said to be well-ordered provided that

$$\left\| f(x^{(i)}) \right\| \geq \left\| f(x^{(i+1)}) \right\| \geq \dots \geq \left\| f(x^{(i+n)}) \right\|.$$

DEFINITION 4. The ordered set of $n+1$ vectors is said to be primary if it is both mixture and well-ordered.

III. New Algorithm.

For any given $\varepsilon > 0$ and the primary set, $A = \{x^{(0)}, x^{(1)}, \dots, x^{(n)}\}$, of the initial $n+1$ vectors. We will modify the multivariate secant method described by Janina Jankowska in [3] by doing the iterations by the following steps.

Step 1. Compute $x^{(n+1)}$ according to (2). If $\|f(x^{(n+1)})\| < \varepsilon$, go to step 4, else go to step 2.

Step 2. Set $\bar{A} = \{x^{(0)}, x^{(1)}, \dots, x^{(n)}, x^{(n+1)}\}$. Choose the first i for which the set

$$A_i = \{x^{(0)}, \dots, x^{(i-1)}, \hat{x}^{(i)}, x^{(i+1)}, \dots, x^{(n+1)}\},$$

$i = 0, 1, 2, \dots, n$, ($\hat{x}^{(i)}$ means that $x^{(i)}$ is deleted), is a mixture. Go to step 3. (comment: there must be at least one value of i, $i \in \{0, 1, 2, \dots, n\}$, for which A_i is a mixture since the initial set A is primary.)

Step 3. Rearrange A_i to obtain the set A_i^* which is primary. Set $A = A_i^*$ and go to step 1.

Step 4. Stop, the approximated solution of the system (1) is $x^{(n+1)}$.

IV. Examples.

The following three examples use both the multivariate secant method from [3] and the proposed modified multivariate secant method. The methods were tested for two sets A, B of initial vectors. All examples use the value $\epsilon = 10^{-9}$.

EXAMPLE 1. (from [1] page 319 or [6] page 138)

$$\begin{aligned} 0.5\sin(x_1 x_2) - 0.25 x_2/\pi - 0.5 x_1 &= 0 \\ (1 - 0.25/\pi)(e^{2x_1} - e) + e x_2/\pi - 2 e x_1 &= 0 \end{aligned} \tag{3}$$

$$A = \{(0,0), (1,1), (0.25, 3)\} \text{ and } B = \{(0,0), (1,1), (0.35, 3)\}.$$

EXAMPLE 2. (from [2] page 88)

$$\begin{aligned} e^{x_1} + x_1 x_2 - 1 &= 0 \\ \sin(x_1 x_2) + x_1 + x_2 - 1 &= 0 \end{aligned} \tag{4}$$

$$A = \{(1,1), (1,-1), (0,0)\} \text{ and } B = \{(2,0), (-1,0), (1,-1)\}.$$

EXAMPLE 3.

$$\begin{aligned} 2\log(x_1) + e^{x_2} - 2x_3 + 1 &= 0 \\ x_1 x_2 - \sin(x_2) + x_3 - 2 &= 0 \\ \log(x_1)\cos(x_2) - x_3 + 1 &= 0 \end{aligned} \tag{5}$$

$$A = \{(2,0,1),(2,-1,2),(2,1,2),(1,1,1)\} \text{ and}$$
$$B = \{(3,2,3),(1,-1,3),(2,1,0),(1,-1,0)\}.$$

The results are summarized in the following Table 1.

eq.	solutions	method	int. vectors	# of iter.
(3)	$x_1 = 0.29944869$	secant	A	16
	$x_2 = 2.83692777$		B	31
		mod. sec.	A	11
			B	12
(4)	$x_1 = 0.0$	secant	A	10
	$x_2 = 1.0$		B	39
		mod. sec.	A	5
			B	10
(5)	$x_1 = 2.71828183$	secant	A	32
	$x_2 = 0.0$		B	22
	$x_3 = 2.0$	mod. sec.	A	21
			B	17

Table 1.

Fig. 1, Fig. 2 and Fig. 3 give the comparison between the multivariate secant method and the modified multivariate secant method for example 1, example 2 and example 3 respectively.

V. Conclusion.

From the three examples in IV., we can see that the modified multivariate secant method does accelerate the convergence of the multivariate secant method by reducing an approximately thirty percent of the number of the iteration steps needed to achieve the required accuracy of solution.

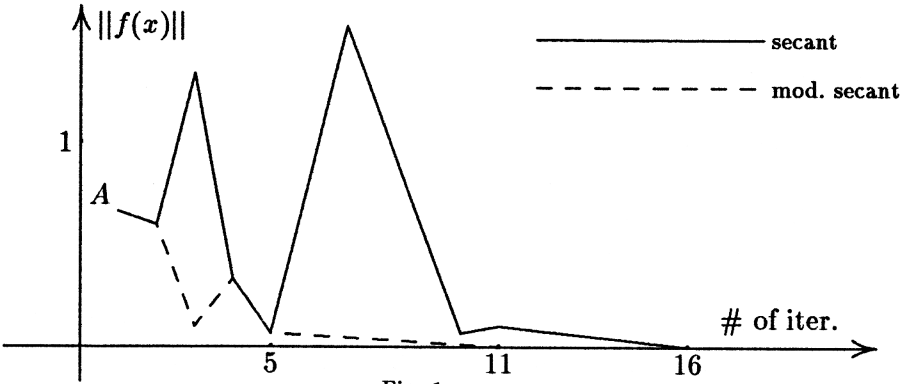

Fig. 1a.

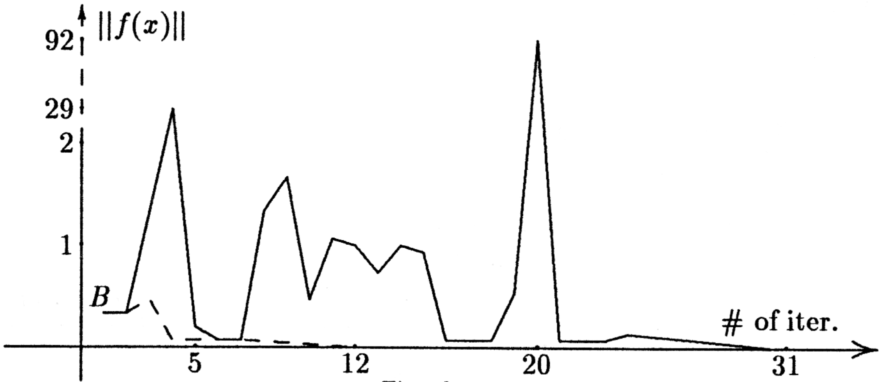

Fig. 1b.

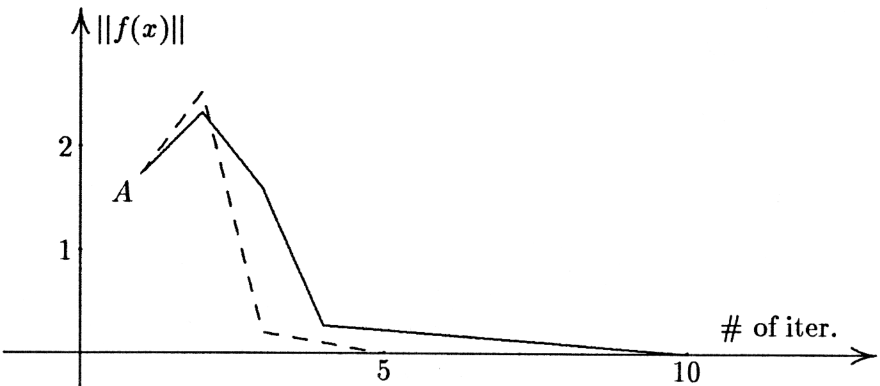

Fig. 2a.

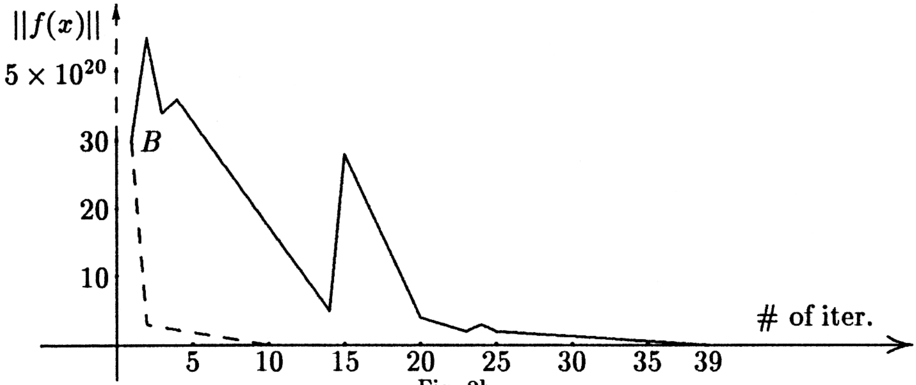

Fig. 2b.

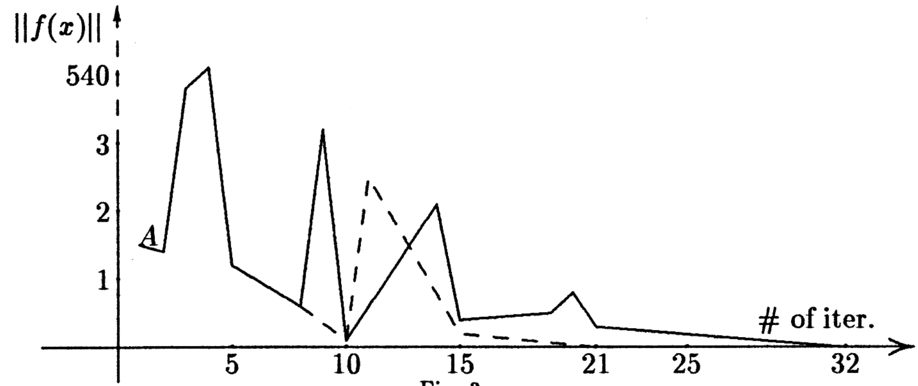

Fig. 3a.

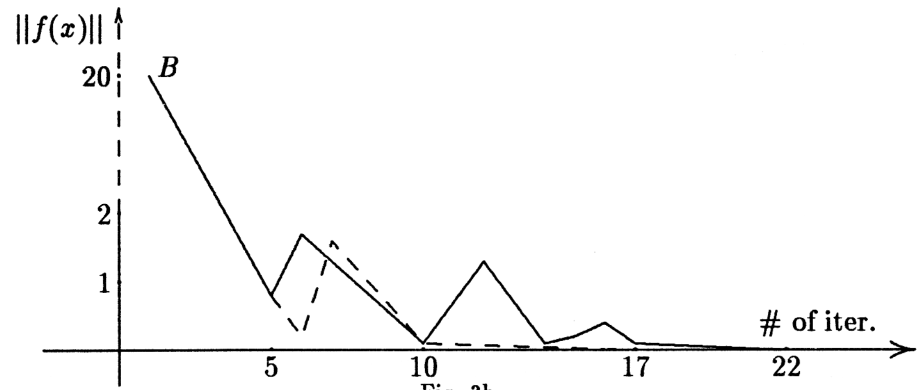

Fig. 3b.

References

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