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THE DIFFERENCE METHOD FOR THE PARTIAL DIFFERENTIAL EQUATIONS OF THE THIRD ORDER

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1. In our preceding papers [1] and [2] we have given the difference method for partial differential equations of the even order. It turns out that the method can be applied also for the equations of odd order. In this paper we solve the equation of the third order

$$\frac{\partial u}{\partial t} = f\left(t, \mathbf{x}, u, \frac{du}{d\mathbf{x}}, \frac{d^2u}{d\mathbf{x}^2}, \frac{d^3u}{d\mathbf{x}^3}, \right) \tag{1.1}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_p)$.

2. Difference quotients.

We shall use the concise notation of the papers [1] and [3], cf. Fig. 1. The difference quotients have the following form:

First order:

$$v^{M_j} = h^{-1} \cdot \left(v^{j(M)} - v^M\right) \quad (j = 1, \ldots, p). \tag{2.1}$$

Second order:

$$v^{M_{jj}} = h^{-1} \cdot \left[h^{-1} \cdot \left(v^{j(M)} - v^M\right) - h^{-1} \cdot \left(v^M - v^{-j(M)}\right)\right], \tag{2.2}$$

i.e.

$$v^{M_{jj}} = h^{-2} \cdot \left(v^{j(M)} - 2v^{M} + v^{-j(M)}\right) \quad (j = 1, \dots, p). \tag{2.3}$$

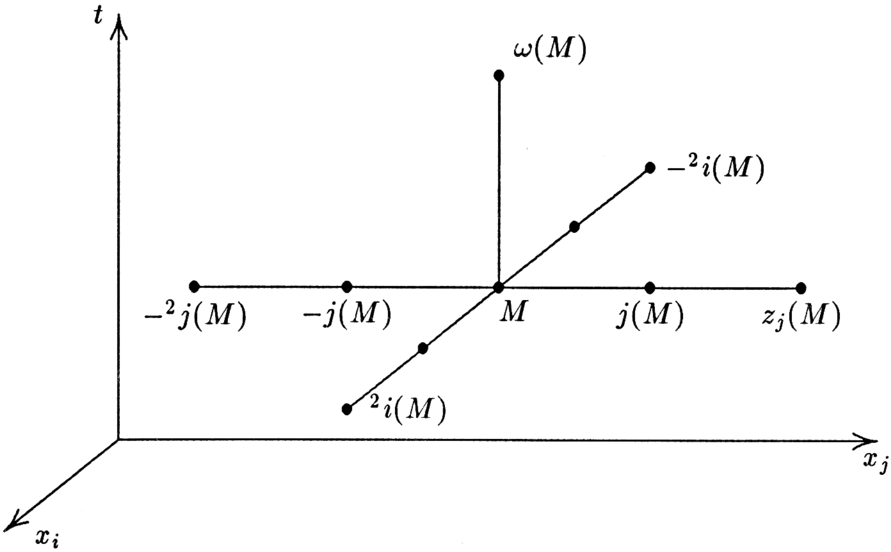

Fig. 1. The nodal points with indices M, $j(M)$, ${}^2j(M)$, $-j(M)$, $-{}^2j(M)$ and $\omega(M)$.

Third order:

$$\begin{aligned} v^{M_{jjj}} =& h^{-1} \cdot \Big[h^{-2} \cdot \left(v^{{}^2j(M)} - 2v^{j(M)} + v^{M}\right) \\ & - h^{-2} \cdot \left(v^{j(M)} - 2v^{M} + v^{-j(M)}\right)\Big], \end{aligned} \tag{2.4}$$

i.e.

$$v^{M_{jjj}} = h^{-3} \cdot \left(v^{{}^2j(M)} - 3v^{j(M)} + 3v^{M} - v^{-j(M)}\right), \quad (j = 1, \dots, p). \tag{2.5}$$

The coefficients of the difference quotients are taken from the Pascal triangle, the signs are as in Fig. 2.

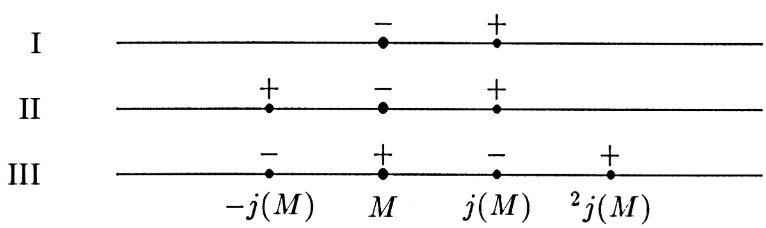

Fig. 2. Difference quotients of the first, second and third order and the signs of corresponding coefficients. All coefficients are taken from Pascal's triangle.

3. The boundary value problems.

We consider the equation

$$\frac{\partial u}{\partial t} = f\left(t, \mathbf{x}, u, \frac{\partial u}{\partial \mathbf{x}}, \frac{\partial^2 u}{\partial \mathbf{x}^2}, \frac{\partial^3 u}{\partial \mathbf{x}^3}\right), \tag{3.1}$$

where $u = u(t,\mathbf{x})$, $\mathbf{x} = (x_1, \ldots, x_p)$ and

$$\frac{\partial^i u}{\partial x^i} = \left(\frac{\partial^i u}{\partial {x_1}^i}, \frac{\partial^i u}{\partial {x_2}^i}, \ldots, \frac{\partial^i u}{\partial {x_p}^i}\right) \quad (i = 1,2,3). \tag{3.2}$$

We assume that the function $f\left(t, \mathbf{x}, u, \overset{1}{q}, \overset{2}{q}, \overset{3}{q}\right)$, $\overset{i}{\mathbf{q}} = \left(\overset{i}{q}_1, \overset{i}{q}_2, \ldots, \overset{i}{q}_p\right)$ $(i = 1,2,3)$ is of the class C^1 in the set

$$\mathcal{D}_1 = \Big\{0 \leq t \leq T, 0 \leq x_j \leq a, -\infty < u < +\infty, -\infty < \overset{i}{q}_j < +\infty$$
$$(j = 1, \ldots, p; i = 1,2,3)\Big\}$$

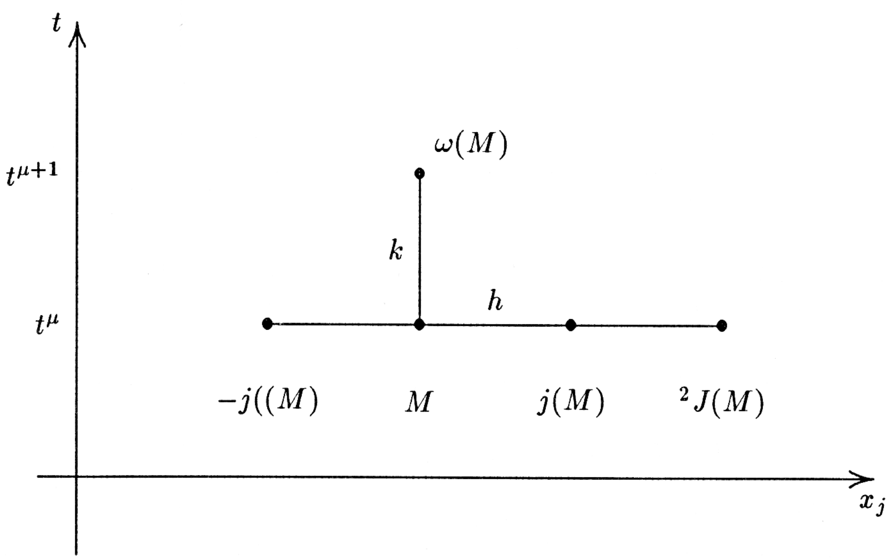

Fig. 3. Basic configuration of the nodal points.

The admissible boundary problem for the differential equation (3.1) requires some comments. We have chosen the basic configuration of the nodal points as in Fig. 3 and as a consequence the values v^M at the hyperplaine $x_j = 0$

$(j = 1,\dots,p)$ and two hyperplanes $x_j = a$, $x_j = a - h$ $(j = 1,\dots,p)$ are needed, cf. Fig. 4. The simplest admissible boundary problem has the form:

$$(3.3)\qquad \begin{cases} u(0,\mathbf{x}) = \varphi_0(\mathbf{x}), \\ u(t,\mathbf{x}) = \varphi_j(t,\mathbf{x}), & \text{for } x_j = 0, \\ u(t,\mathbf{x}) = \psi_j(t,\mathbf{x}), & \text{for } x_j = a, \\ \frac{\partial u(t,\mathbf{x})}{\partial x_j} = \gamma_j(t,\mathbf{x}), & \text{for } x_j = a. \end{cases}$$

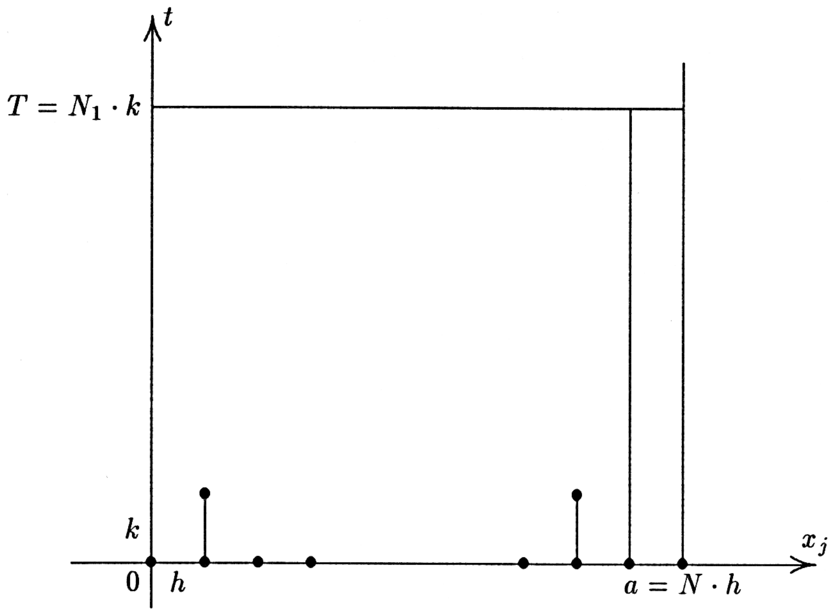

Fig. 4. To begin the calculations we have to know the values of the discrete function v^M on the hyperplanes $t = 0$, $x_j = 0$, $x_j = a - h$, $x_j = a$, $j = 1,\dots,p$.

We assume that the solution $u(t,\mathbf{x})$ of the problem (3.1) (3.3) exists in the set

$$\mathcal{D} = \{0 \le t \le T,\ 0 \le x_j \le a \ \ (j = 1,\dots,p)\},$$

The difference equation also requires comments. We have noted in our previous papers [1] and [2] that the coefficients with negative signs in the difference quotient of the highest order are decisive for the proof of convergence of the difference method. In our case the difference quotient (2.5) of the third order contains terms with negative signs:

$$(3.4)\qquad v^{j(M)} \quad \text{and} \quad v^{-j(M)}.$$

We form the arithmetic mean of these coefficients:

$$v^M_{mean} = \frac{1}{p} \sum_{j=1}^{p} \frac{1}{2} \left(v^{j(M)} + v^{-j(M)} \right), \tag{3.5}$$

we introduce the time difference quotient with the mean value (3.5):

$$k^{-1} \cdot \left(v^{\omega(M)} - v^M_{mean} \right), \tag{3.6}$$

and write the difference equation in the form:

$$k^{-1} \cdot \left(v^{\omega(M)} - v^M_{mean} \right) = f\left(t^\mu, x^m, v^M, v^{M1}, v^{M2}, v^{M3} \right). \tag{3.7}$$

Here, as in the previous papers [1] and [2], we denote

$$v^{M1} = \left(v^{M_1}, v^{M_2}, \ldots, v^{M_p} \right), \tag{3.8}$$

$$v^{M2} = \left(v^{M_{11}}, v^{M_{22}}, \ldots, v^{M_{pp}} \right), \tag{3.9}$$

$$v^{M3} = \left(v^{M_{111}}, v^{M_{222}}, \ldots, v^{M_{ppp}} \right), \tag{3.10}$$

the difference quotients of the first, second and third order v^{M_j}, $v^{M_{jj}}$ and $v^{M_{jjj}}$ being given by (2.1), (2.3) and (2.5), respectively.

The boundary values for the difference problem correspond to the boundary values (3.3) for the differential problem. They have the following form:

$$\begin{cases} v^M = \varphi_0(\mathbf{x}^m), & \text{for } M = (0, m), \\ v^M = \varphi_j(t^\mu, \mathbf{x}^m), & \text{for } m_j = 0, \\ v^M = \psi_j(t^\mu, \mathbf{x}^m), & \text{for } m_j = N, \\ v^{M_j} = \delta_j(t^\mu, \mathbf{x}^m), & \text{for } m_j = N, \end{cases} \tag{3.11}$$

for $j = 1, 2, \ldots, p$, cf. Fig. 5.

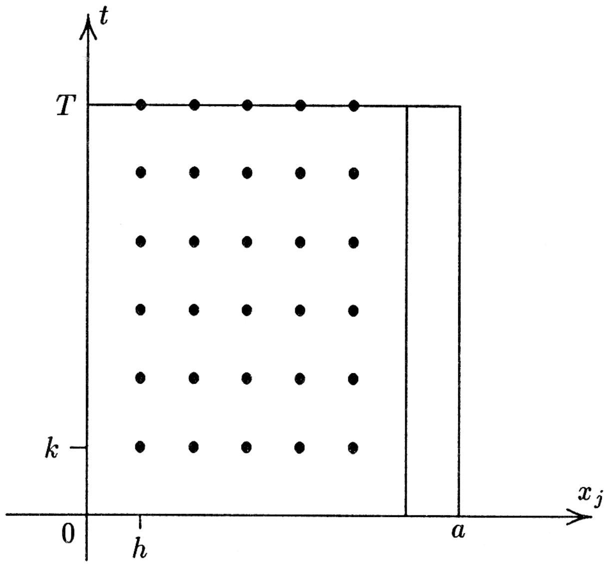

Fig. 5. With the aid of the discrete boundary conditions (3.11) we can calculate the values v^M on the hyperplanes $t = 0$, $x_j = 0$, $x_j = a - h$, $x_j = a$, $j = 1, \ldots, p$.

We assume that

(3.12) $$\left|\frac{\partial f}{\partial u}\right| \leq \mathcal{L}, \text{ in the set } \mathcal{D}_1$$

(3.13) $$\left|\frac{\partial f}{\partial \overset{s}{q}_j}\right| \leq \Gamma_s,\ (s = 1, 2), \text{ in the set } \mathcal{D}_1$$

(3.14) $$0 < g \leq \frac{\partial f}{\partial \overset{3}{q}_j} \leq \mathcal{G}, \text{ in the set } \mathcal{D}_1$$

$(j = 1, 2, \ldots, p)$

We assume also that the mesh size h for the space coordinates x_j $(j = 1, \ldots, p)$ and k for the time coordinate t satisfy the conditions:

(3.15) $$g \cdot \frac{3}{h^3} - \Gamma_2 \cdot \frac{2}{h^2} - \Gamma_1 \cdot \frac{1}{h} \geq 0,$$

(3.16) $$-\mathcal{G} \cdot \frac{3}{h^3} - \Gamma_2 \cdot \frac{2}{h^2} - \Gamma_1 \cdot \frac{1}{h} + \frac{1}{k} \cdot \frac{1}{2p} \geq 0,$$

(3.17) $$-\mathcal{G} \cdot \frac{3}{h^3} - \Gamma_2 \cdot \frac{2}{h^2} + \frac{1}{k} \cdot \frac{1}{2p} \geq 0,$$

We definee the error η^M by

$$(3.18)\qquad k^{-1}\left(u^{\omega(M)} - u^M_{mean}\right) = f\left(t^\mu, \mathbf{x}^m, u^M, u^{M1}, u^{M2}, u^{M3}\right) + \eta^M,$$

and we have

$$(3.19)\qquad \varepsilon(h,k) \longrightarrow 0, \quad \text{as } h,k \to 0,$$

where

$$(3.20)\qquad \varepsilon(h,k) = \max_M |\eta^M|.$$

(3.19) means that the difference equation is consistent with the differential equation.

We define also the error

$$(3.21)\qquad r^M = u^M - v^M.$$

4. The difference inequalities

$$(4.1)\qquad s^{\mu\sim} \le \mathcal{L}.|r^A| + \varepsilon(k,k),$$

$$(4.2)\qquad z^{\mu\sim} \le \mathcal{L}.|r^C| - \varepsilon(k,k),$$

for the equation of the third order.

Let us denote

$$(4.3)\qquad s^\mu = \max_m r^{\mu,m} = r^{\mu,b} = r^B,$$

$$(4.4)\qquad s^{\mu+1} = \max_m r^{\mu+1,m} = r^{\mu+1,a} = r^{\omega(A)},$$

and

$$(4.5)\qquad s^{\mu\sim} = k^{-1}\cdot\left(s^{\mu+1} - s^\mu\right),$$

where $\mathbf{a} = (a_1,\dots,a_p)$, $\mathbf{b} = (b_1,\dots,b_p)$, $\omega(A) = (\mu+1, a)$.

The difference quotient (4.5) can be written in the form:

$$(4.6)\qquad s^{\mu\sim} = k^{-1}\cdot\left(r^{\omega(A)} - r^A_{mean}\right) + k^{-1}\cdot\left(r^A_{mean} - r^B\right).$$

But from the definition (3.21) of the error r^M it follows that

$$r^{\omega(A)} - r^A_{mean} = \left(u^{\omega(A)} - u^A_{mean}\right) - \left(v^{\omega(A)} - v^A_{mean}\right), \tag{4.7}$$

therefore the equations (3.18) and (3.7) yield

$$\begin{aligned} k^{-1}\cdot\left(r^{\omega(A)} - r^A_{mean}\right) =& \eta^A + f\left(t^\mu, \mathbf{x}^a, u^A, u^{A1}, u^{A2}, u^{A3}\right) \\ & - f\left(t^\mu, \mathbf{x}^a, v^A, v^{A1}, v^{A2}, v^{A3}\right). \end{aligned} \tag{4.8}$$

The mean value theorem gives the folowing formula for $s^{\mu\sim}$:

$$\begin{aligned} s^{\mu\sim} =& \eta^A + \frac{\partial f}{\partial u}(\sim)\cdot r^A + \sum_{j=1}^{p} {}^1\mathcal{D}_j \cdot \frac{1}{h}\left[\left(r^{j(A)} - r^B\right) - \left(r^A - r^B\right)\right] \\ & + \sum_{j=1}^{p} {}^2\mathcal{D}_j \cdot \frac{1}{h^2}\left[\left(r^{j(A)} - r^B\right) - 2\left(r^A - r^B\right) + \left(r^{-j(A)} - r^B\right)\right] \\ & + \sum_{j=1}^{p} {}^3\mathcal{D}_j \cdot \frac{1}{h^3}\left[\left(r^{2j(A)} - r^B\right) - 3\left(r^A - r^B\right) + 3\left(r^A - r^B\right)\right. \\ & \left. - \left(r^{-j(A)} - r^B\right)\right] + \frac{1}{k}\cdot\left(r^A_{mean} - r^B\right), \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} r^A_{mean} - r^B &= \frac{1}{p}\sum_{j=1}^{p}\left[\frac{1}{2}\left(r^{j(A)} + r^{-j(A)}\right)\right] - r^B \\ &= \frac{1}{p}\sum_{j=1}^{p}\left[\frac{1}{2}\left(r^{j(A)} - r^B\right) + \frac{1}{2}\left(r^{-j(A)} - r^B\right)\right], \end{aligned} \tag{4.10}$$

because of the definition (3.5) of the mean value v^M_{mean}. In the formula (4.9) we have introduced the value r^B at appropriate places and the derivatives at suitable point $(\sim)$ have been denoted shortly as

$$\left\{\begin{aligned} {}^3\mathcal{D}_j &= \frac{\partial f}{\partial \overset{3}{q_j}}(\sim), \\ {}^2\mathcal{D}_j &= \frac{\partial f}{\partial \overset{2}{q_j}}(\sim), \\ {}^1\mathcal{D}_j &= \frac{\partial f}{\partial \overset{1}{q_j}}(\sim). \end{aligned}\right. \tag{4.10a}$$

A word concerning the difficulties inherent in the formula (4.9) is now in order. We have already remarked that in the difference quotient of the highest order, i.e. in the line with h^{-3} in (4.9), there are terms

$$r^{j(A)} - r^B \quad \text{and} \quad r^{-j(A)} - r^B$$

with negative coefficients decisive for the proof of convergence of the difference method.

These terms appear also in the formula (4.10) and can be rewritten jointly in the following way:

(4.11)

$$\begin{aligned} s^{\mu\sim} =& \eta^A + \frac{\partial f}{\partial u}(\sim)\cdot r^A + \sum_{j=1}^{p}\left(r^{2j(A)} - r^B\right)\cdot {}^3\mathcal{D}_j\cdot\frac{1}{h^3} \\ &+ \sum_{j=1}^{p}\left(r^A - r^B\right)\cdot\left[{}^3\mathcal{D}_j\cdot\frac{+3}{h^3} + {}^2\mathcal{D}_j\cdot\frac{-2}{h^2} + {}^1\mathcal{D}_j\cdot\frac{-1}{h}\right] \\ &+ \sum_{j=1}^{p}\left(r^{j(A)} - r^B\right)\cdot\left[{}^3\mathcal{D}_j\cdot\frac{-3}{h^3} + {}^2\mathcal{D}_j\cdot\frac{1}{h^2} + {}^1\mathcal{D}_j\cdot\frac{1}{h} + \frac{1}{k}\cdot\frac{1}{2p}\right] \\ &+ \sum_{j=1}^{p}\left(r^{j(A)} - r^B\right)\cdot\left[{}^3\mathcal{D}_j\cdot\frac{-1}{h^3} + {}^2\mathcal{D}_j\cdot\frac{1}{h^2} + \frac{1}{k}\cdot\frac{1}{2p}\right]. \end{aligned}$$

There is no difficulty with the first and the second sum $\sum_{j=1}^{p}$ on the right-hand side of the formula (4.11).

In the first sum we have

$$(4.12) \qquad r^{2j(A)} - r^B \le 0,$$

because of the definition (4.3) of the maximal value r^B, and

$$(4.13) \qquad {}^3D_j > 0,$$

because of the assumption (3.14), hence the term in the first sum is non-positive:

$$(4.14) \qquad \left(r^{2j(A)} - r^B\right)\cdot{}^3\mathcal{D}_j\cdot\frac{1}{h^3} \le 0,$$

and the first sum $\sum_{j=1}^{p}$ can be dropped.

In the second sum in the formula (4.11) we have

$$(4.15)\quad {}^3\mathcal{D}_j\cdot\frac{+3}{h^3}+{}^2\mathcal{D}_j\cdot\frac{-2}{h^2}+{}^1\mathcal{D}_j\cdot\frac{-1}{h}\geq g\cdot\frac{+3}{h^3}+\Gamma_2\cdot\frac{-2}{h^2}+\Gamma_1\cdot\frac{-1}{h}\geq 0,$$

for sufficiently small h, because of the assumption (3.15), and $r^A - r^B \leq 0$, because of the definition (4.3) of the maximal value r^B. Hence the second sum in the formula (4.11) is non–positive and can be dropped.

The third and fourth sums $\sum_{j=1}^{p}$ in the formula (4.11) should be treater care.

In the third sum we have

$$(4.16)\quad {}^3\mathcal{D}_j\cdot\frac{-3}{h^3}+{}^2\mathcal{D}_j\cdot\frac{1}{h^2}+{}^1\mathcal{D}_j\cdot\frac{1}{h}+\frac{1}{k}\cdot\frac{1}{2p}\geq \mathcal{G}\cdot\frac{-3}{h^3}-\Gamma_2\cdot\frac{1}{h^2}-\Gamma_1\cdot\frac{1}{h}+\frac{1}{k}\cdot\frac{1}{2p}\geq 0,$$

because of the assumption (3.16) (for a fixed h satisfying (3.15) there exists a sufficiently small value k such that (3.16) holds).

In addition we have $r^{j(A)} - r^B \leq 0$, because of the definition (4.3) of the maximal value r^B so that the third sum in (4.11) is non–positive and can be dropped.

The fourth sum in (4.11) can be dropped also since

$$(4.17)\qquad {}^3\mathcal{D}_j\cdot\frac{-1}{h^3}+{}^2\mathcal{D}_j\cdot\frac{1}{h^2}+\frac{1}{k}\cdot\frac{1}{2p}\geq \mathcal{G}\cdot\frac{-1}{h^3}-\Gamma_2\cdot\frac{1}{h^2}+\frac{1}{k}\cdot\frac{1}{2p}\geq 0,$$

because of the assumption (3.12) and the inequality $r^{-j(A)} - r^B l0$ for the maximal value r^B, cf. the definition (4.3).

Thus (4.11) reduces to the form

$$(4.18)\qquad s^{\mu\sim}\leq \eta^A+\frac{\partial f}{\partial u}(\sim)\cdot r^A.$$

From the assumption (3.12) and the definition (3.20) of $\varepsilon(h,k)$ it follows that

$$(4.19)\qquad s^{\mu\sim}\leq \mathcal{L}\cdot|r^A|+\varepsilon(h,k).$$

This ends the proof of the first difference inequality for the equation of the third order.

In a similar way we can introduce the minimum values

$$(4.20)\qquad z^\mu=\min_m r^{\mu,m}=r^{\mu,d}=r^D,$$

$$(4.21)\qquad z^{\mu+1}=\min_m r^{\mu+1,m}=r^{\mu+1,c}=r^C,$$

where $C = (\mu, c)$, $D = (\mu, d)$, $c = (c_1, \dots, c_p)$, $d = (d_1, \dots, d_p)$, and starting from the definition

$$z^{\mu\sim} = k^{-1} \cdot \left(z^{\mu+1} - z^{\mu}\right),$$

we can repeat the argument connected with formulas (4.3) up to (4.19), the sense of the corresponding inequalities being reserved. In particular, from equality (4.9) and (4.10) for $s^{\mu\sim}$, we obtain the corresponding equality for $z^{\mu\sim}$, if s^{μ}, A, B in (4.9) and (4.10) are replaced by z^{μ}, C, D, respectively.

It turns out that we can drop the sum $\sum_{j=1}^{p}$ in the equality for $z^{\mu\sim}$ just obtained, since they are non-negative, and this leads us to the second inequality

$$z^{\mu\sim} \geq -\mathcal{L} \cdot |r^{C}| - \varepsilon(h,k). \tag{4.23}$$

This ends the proof of the second difference inequality for the equation of the third order.

5. Main theorem.

THEOREM 1. *Under the assumptions of section* **3**, *the difference method is convergent.*

PROOF. We define first

$$R^{\mu} = \max_{m} |r^{M}|, \quad \text{for} \quad M = (\mu, m) \tag{5.1}$$

and obtain

$$R^{\mu\sim} \leq \max\left(s^{\mu\sim}, -z^{\mu\sim}\right), \tag{5.2}$$

because of Lemma 3.

On the right-hand side of the formula (5.2) we can use the difference inequalities (4.1) and (4.2) and this leads us to the difference inequality

$$R^{\mu\sim} \leq \mathcal{L}.R^{\mu} + \varepsilon(h,k), \quad R^{0} = 0, \tag{5.3}$$

because of Lemma 4.

This yields the error estimate

$$|r^{M}| \leq \frac{\varepsilon(h,k)}{\mathcal{L}} \cdot \left(e^{\mathcal{L}k\mu} - 1\right), \tag{5.4}$$

for $M = (\mu, m)$ $(\mu = 0, 1, \dots, N_1)$, $kN_1 = T$.
The convergence of the difference method follows from the error estimate (5.4) and the condition (3.19).

This completes the proof of the Theorem 1.

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