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A NOTE ON UNIQUENESS OF STOCHASTIC NAVIER—STOKES EQUATIONS

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Abstract. We give two versions of uniqueness theorems for general stochastic Navier–Stokes equations with feedback in the forces and in the noise term.

1. Introduction

We consider the following system of stochastic Navier–Stokes equations

$$(1) \qquad \begin{cases} du = [\nu\Delta u + <u, \nabla> u + f(t,u)]dt + g(t,u)dw_t\,, \\ \operatorname{div} u = 0. \end{cases}$$

This system was first considered in [1] with $g(t,u) = 1$, f independent of u, and one-dimensional Wiener process. It has been investigated since then by many authors (see, for example, [12], [13]) but first existence results covering the case of g depending on u appeared in 1991: [3] for dimension $n = 1$, [2] for $n = 2$, and [5] for $n \leq 4$. The paper [5] uses the methods of nonstandard analysis but a standard proof of a similar result is also available [7]. Note that (1) is not covered by the existing general theory of stochastic partial differential equations because of quadratic type of nonlinearity.

The existence results of [2] and [5] concern weak solution. For dimension $n = 2$ stronger solutions are found in [7] and for the special case of periodic boundary conditions further regularity is shown in [5], see Theorems 2.3, 2.4 below.

Uniqueness was proved in [5] for $n = 2$ in a narrow class od solutions with general form of the coefficients f and g, under some Lipschitz conditions. In [9] we find a uniqueness theorem ($n = 2$) in a broader class of solutions (weak solutions) but in a more particular situation (f independent of t and u,

g independent of t) and under more restrictive conditions. (in particular, the Lipschitz condition involves a different norm).

Here we generalize both results cited above. We prove uniqueness of weak solutions in the general situation and under the Lipschitz condition weaker than in [9] and slightly weaker than in [5], and in the class broader than in [5]. We also show that for $n = 3$ we have at most one strong solution; this result is possibly with empty content because we do not if there exist any (an open question even in the deterministic case $g = 0$).

2. Preliminaries

Notations: Let $D \subset \mathbb{R}^n$, $n \leq 4$, be a bounded domain with the boundary of class C^2.

H is the closure of the set $\{u \in C_0^\infty(D, \mathbb{R}^n)\colon \operatorname{div} u = 0\}$ in the L^2 norm $|u| = (u, u)^{1/2}$,

$$(u, v) = \sum_{j=1}^{n} \int_D u^j(x) v^j(x) dx.$$

V is the closure of $\{u \in C_0^\infty(D, \mathbb{R}^n)\colon \operatorname{div} u = 0\}$ in the norm $|u| + \|u\|$ where $\|u\| = ((u, u))^{1/2}$,

$$((u, v)) = \sum_{j=1}^{n} \left(\frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_j}\right).$$

H and **V** are Hilbert spaces with scalar products $(\cdot, \cdot)$ and $((\cdot, \cdot))$ respectively.

By A we denote the self-adjoint extension of the operator $-\Delta$ in **H** and by $\{e_k\}$ the orthonormal basis of its eigenfunctions with the corresponding eigenvalues $\lambda_k \nearrow \infty$. Note that $((u, v)) = \sum_{k=1}^{\infty} \lambda_k u_k v_k$, where $u_k = (u, e_k)$.

By **V**$'$ we denote the space dual to **V**, with the duality extending the scalar product in **H**.

We put

$$b(u, v, z) = \sum_{i,j=1}^{n} \int_D u^j(x) \frac{\partial v^i}{\partial x_j}(x) z^i(x) dx = (< u, \nabla > v, z)$$

whenever the integrals make sense. Note that for $u, v, z \in \mathbf{V}$ we have $b(u, v, z) = -b(u, z, v)$, hence $b(u, v, v) = 0$.

We shall need some well-known inequalities for b (see [11] for example) and we list them here for reference:

$$|b(u,v,z)| \leq c\,\|u\|\,\|v\|\,\|z\|, \tag{2}$$
$$|b(u,v,z)| \leq c\,\|u\|\,|Av|\,|z|, \tag{3}$$
$$|b(u,v,z)| \leq c\,|Au|\,\|v\|\,|z|, \tag{4}$$
$$|b(u,v,z)| \leq c\,|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}|v|^{\frac{1}{2}}\|v\|^{\frac{1}{2}}\|z\| \text{ if } n=2, \tag{5}$$

for suitable u, v, and z. The inequality (2) allows us to define a $\mathbf{V}'$-valued quadratic form $B(u)$ by

$$B(u)(z) = b(u,u,z).$$

Wiener processes with values in $\mathbf{H}$ are defined as in [8] for example:

DEFINITION 2.1. Let $Q : \mathbf{H} \to \mathbf{H}$ be a linear, non-negative, trace class operator. An $\mathbf{H}$-valued square integrable stochastic process $w(t)$, $t \geq 0$, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ is a *Wiener process* with the covariance operator Q if
1) $w(0) = 0$,
2) $Ew(t) = 0$, $\mathrm{Cov}[w(t) - w(s)] = (t-s)Q$, for all s, $t \geq 0$,
3) w has independent increments,
4) w has continuous paths,
5) w is adapted to $(\mathcal{F}_t)$.

Such a Wiener process has the following structure: let $\{d_i\}$ be an orthonormal set of eigenvectors of Q with eigenvalues γ_i, so that $\mathrm{tr}Q = \sum_{i=1}^{\infty} \gamma_i$; then $w(t) = \sum_{i=1}^{\infty} \beta_i(t) d_i$, where β_i are mutually independent real Wiener processes with $E(\beta_i^2(t)) = \gamma_i t$.

The stochastic integral $\int g\,dw$ is defined in [8] as follows. First we introduce the space of integrands. For any Hilbert space $\mathbf{Y}$ we denote by $\mathcal{M}(\mathbf{Y})$ the space of all stochastic processes

$$g : [0,T] \times \Omega \longrightarrow \mathcal{L}(\mathbf{H},\mathbf{Y})$$

such that

$$E\left(\int_0^T |g(t)|^2_{\mathbf{H},\mathbf{Y}}\,dt\right) < \infty$$

and for all $h \in \mathbf{H}$, $g(t)h$ is a $\mathbf{Y}$-valued stochastic process measurable with respect to the filtration $\mathcal{F}_t$.

The stochastic integral $\int_0^t g(s)dw(s) \in \mathbf{Y}$ is defined for all $g \in \mathcal{M}(\mathbf{Y})$ by

$$\int_0^t g(s)dw(s) = L^2 - \lim_{m\to\infty} \sum_{i=1}^{m} \int_0^t g(s)d_i\, d\beta_i(s).$$

Note that if we write $w_k(t) = (w(t), e_k)$ and $w^{(m)}(t) = \sum_{k=1}^m w_k(t)e_k$, then $w^{(m)}$ is a Wiener process with covariance $Q_m = \mathrm{Pr}_m Q \mathrm{Pr}_m$ and

$$\int_0^t g(s)dw^{(m)}(s) \xrightarrow{L^2} \int_0^t g(s)dw(s) \quad \text{in } \mathbf{H}.$$

If $g \in \mathcal{L}(\mathbf{H},\mathbf{H})$ and $v \in \mathbf{H}$ we write (v,g) for the element of $\mathbf{H}$ given by $((v,g),u) = (v,gu)$ for $u \in \mathbf{H}$ (i.e. $(v,g) = g'v$), so that $\int_0^t (v, g(s))dw(s)$ makes sense.

We can now explain what we mean by a solution of (1).

DEFINITION 2.2. Let $u_0 \in \mathbf{H}$, $f : [0,\infty) \times \mathbf{V} \to \mathbf{V}'$, and $g : [0,\infty) \times \mathbf{V} \to \mathcal{L}(\mathbf{H},\mathbf{V}')$. A stochastic process $u(t,\omega)$ is a ***weak solution of the stochastic Navier-Stokes equations*** (1) if

$$E\left(\sup_{t\le T} |u(t)|^2 + \int_0^T \|u(t)\|^2 dt\right) \le c(T) \quad \text{for all } T \tag{6}$$

and

$$u(t) = u_0 + \int_0^t [-\nu A u(s) - B(u(s)) + f(s,u(s))]ds + \int_0^t g(s,u(s))dw(s) \tag{7}$$

holds as an identity in $\mathbf{V}'$ (the first integral is understood in the sense of Bochner). We say that u is a *strong solution* if (6) is replaced by

$$P\left(\sup_{t\in[0,T]} \|u(t)\|^2 + \int_0^T |Au(t)|^2 dt < \infty\right) = 1 \text{ for all } T. \tag{8}$$

Note that ∇p as an element of $\mathbf{V}'$ is equal to 0: $\nabla p[v] = (p, \mathrm{div} v) = 0$.

For completeness of exposition we give two existence theorems, one from [5] and one from [7]. The first is on existence of solutions for $n \le 4$ and the second shows additional regularity of solution in dimension 2.

We consider the set $K_m = \{v : \|v\| \le m\} \subseteq \mathbf{V}$ with the strong topology of $\mathbf{H}$. In the theorem below, continuity on each K_m turns out to be the appropriate condition for the coefficients f, g. Note that this is weaker than continuity on $\mathbf{V}$ in either the $\mathbf{H}$-norm or the weak topology of $\mathbf{V}$.

THEOREM 2.3. [5] *Suppose that $u_0 \in \mathbf{H}$ and*

$$f : [0,\infty) \times \mathbf{V} \to \mathbf{V}', \quad g : [0,\infty) \times \mathbf{V} \to \mathcal{L}(\mathbf{H},\mathbf{H})$$

are jointly measurable functions with the following properties

(i) $f(t,\cdot) \in C(K_m, \mathbf{V}'_{\mathbf{weak}})$ *for all* m,
(ii) $g(t,\cdot) \in C(K_m, \mathcal{L}(\mathbf{H},\mathbf{H})_{\mathbf{weak}})$ *for all* m,
(iii) $|f(t,u)|_{\mathbf{V}'} + |g(t,u)|_{\mathbf{H},\mathbf{H}} \le a(t)(1+|u|)$ *where* $a \in L^2(0,T)$ *for all* T.

Then there exists a probability space Ω and a Wiener process w such that equation (7) has a weak solution u on Ω.

THEOREM 2.4. [7] *Suppose that $u_0 \in \mathbf{V}$ and $f(t,u) \in \mathbf{H}$, $g(t,u) \in \mathcal{L}(\mathbf{H},\mathbf{V})$ with*

$$|f(t,u)| + |g(t,u)|_{\mathbf{H},\mathbf{V}} \le a(t)(1+\|u\|)$$

where $\int_0^T a^2(t)dt < \infty$, all T.
Then (1) has a strong solution.

3. The case $n = 2$ – weak solutions

The theorem proved in this section: requires that the coefficients f and g be Lipschitz continuous in u.
A1. $|f(t,u) - f(t,v)|_{\mathbf{V}'} \le c_1\|u-v\| + c_2|u-v|$,
A2. $|g(t,u) - g(t,v)|^2_{\mathcal{L}(\mathbf{H},\mathbf{H})} \le c_3\|u-v\|^2 + c_4|u-v|^2$,
A3. $c_1 + c_3 \operatorname{tr} Q \le 2\nu$.

REMARK 3.1. We can replace A2 and A3 by slightly less restrictive
A2'. $\operatorname{tr}\Big((g(t,u)-g(t,v))Q(g(t,u)-g(t,v))^T\Big) \le c_5\|u-v\|^2 + c_7|u-v|^2$,
A3'. $c_1 + c_5 \le 2\nu$.
and the proof remains virtually the same, see Remark 3.3. The conditions in [9] were A1 with $c_1 = c_2 = 0$ and A2' with $c_5 = 0$. In [5] we find A1–3 with $c_1 = 0$.

THEOREM 3.2. *Suppose that A1–A3 (or A1, A2', and A3') hold. Then there is at most one weak solution to the stochastic Navier–Stokes equation.*

PROOF. Let u_1 and u_2 be two solutions of (1) with the same initial value. The Itô formula (see [8], for example) applied to $|u_1(t) - u_2(t)|^2$ yields

$$
\begin{aligned}
|u_1(t)-u_2(t)|^2 &= -2\nu \int_0^t \|u_1(s) - u_2(s)\|^2 ds \\
&+ 2\int_0^t (B(u_1(s)) - B(u_2(s)), u_1(s) - u_2(s))ds \\
&+ 2\int_0^t (f(s, u_1(s)) - f(s, u_2(s)), u_1(s) - u_2(s))ds \\
&+ 2\int_0^t (u_1(s) - u_2(s), g(s, u_1(s)) - g(s, u_2(s)))dw_s \\
&+ \int_0^t \mathrm{tr}\Big[\big(g(s, u_1(s)) - g(s, u_2(s))\big)Q\big(g(s, u_1(s)) - g(s, u_2(s))\big)^T\Big] ds.
\end{aligned}
$$

We denote $\beta = 2\nu - c_1 - c_3\mathrm{tr}Q$ which is positive by A3. Taking account of A1 and A2 (or A1 and A2') we have

$$
\begin{aligned}
&\Big|\int_0^t (f(s, u_1(s)) - f(s, u_2(s)), u_1(s) - u_2(s))ds\Big| \\
&\le \int_0^t |f(s, u_1(s)) - f(s, u_2(s))|_{V'} \|u_1(s) - u_2(s)\|ds \\
&\le c_1 \int_0^t \|u_1(s) - u_2(s)\|^2 ds \\
&\quad + c_2 \int_0^t |u_1(s) - u_2(s)|\, \|u_1(s) - u_2(s)\|ds \\
&\le c_1 \int_0^t \|u_1(s) - u_2(s)\|^2 ds \\
&\quad + \frac{\beta}{2}\int_0^t \|u_1(s) - u_2(s)\|^2 ds + \frac{2c_2^2}{\beta}\int_0^t |u_1(s) - u_2(s)|^2 ds
\end{aligned}
$$

and

$$
\int_0^t \mathrm{tr}\Big[\big(g(s, u_1(s)) - g(s, u_2(s))\big)Q\big(g(s, u_1(s)) - g(s, u_2(s))\big)^T\Big] ds
$$

$$
\le \int_0^t \mathrm{tr}Q|g(s, u_1(s)) - g(s, u_2(s))|^2_{\mathcal{L}(\mathbf{H},\mathbf{H})} ds \tag{9}
$$

$$
\le c_3\mathrm{tr}Q \int_0^t \|u_1(s) - u_2(s)\|^2 ds + c_4 \int_0^t |u_1(s) - u_2(s)|^2 ds. \tag{10}
$$

We use (5) and $(B(u), u) = 0$ to deduce that

$$\begin{aligned}
\int_0^t &(B(u_1(s)) - B(u_2(s)), u_1(s) - u_2(s))ds \\
&\leq 2^{3/2} \int_0^t \|u_1(s) - u_2(s)\| \, \|u_2\| \, |u_1(s) - u_2(s)| ds \\
&\leq \frac{\beta}{2} \int_0^t \|u_1(s) - u_2(s)\|^2 ds + \frac{16}{\beta} \int_0^t \|u_2(s)\|^2 |u_1(s) - u_2(s)|^2 ds.
\end{aligned}$$

Inserting the last three estimates into the formula for $|u_1(s) - u_2(s)|^2$ above we have

$$\begin{aligned}
|u_1(t) - u_2(t)|^2 \leq & c_5 \int_0^t |u_1(s) - u_2(s)|^2 ds \\
& + c_6 \int_0^t \|u_2(s)\|^2 |u_1(s) - u_2(s)|^2 ds \\
& + \int_0^t (u_1(s) - u_2(s), g(s, u_1(s)) - g(s, u_2(s))) dw_s
\end{aligned}$$

for some constants c_5, c_6. Now, following the idea of the uniqueness proof of [9] (see also [10], p.264) we consider the process

$$\eta(t) = \exp\left(-c_6 \int_0^t \|u_2(s)\|^2 ds \right).$$

Computing the differential $d(\eta(t)|u_1(t) - u_2(t)|^2)$ we obtain

$$\begin{aligned}
\eta(t)|u_1(t) - u_2(t)|^2 \leq & c_5 \int_0^t \eta(s) |u_1(s) - u_2(s)|^2 ds \\
& + \int_0^t \eta(s)(u_1(s) - u_2(s), g(s, u_1(s)) - g(s, u_2(s))) dw_s
\end{aligned}$$

and taking mathematical expectation we arrive at

$$E\left(\eta(t)|u_1(t) - u_2(t)|^2\right) \leq E c_5 \int_0^t \eta(t) |u_1(s) - u_2(s)|^2 ds.$$

Using the Gronwall lemma we get $E\eta(t)|u_1(t) - u_2(t)|^2 = 0$ hence since $\int_0^t \|u_2(s)\|^2 ds < \infty$ almost everywhere by (6), we find that $|u_1(t) - u_2(t)|^2 = 0$ which finishes the proof. □

REMARK 3.3. If we assume A2', A3' instead of A2, A3, then the only difference in the proof is that (9) is redundant and A2' yields (10) directly with $c_5 = c_3 \mathrm{tr} Q$ (from that on we employ A3' instead of A3).

4. The case $n = 3$ – strong solutions

Now, to prove uniqueness of strong solutions we need the following conditions.

B1. $|f(t,u) - f(t,v)| \le c_7 \|u - v\|$,

B2. $|g(t,u) - g(t,v)|_{\mathcal{L}(\mathbf{H},\mathbf{V})} \le c_8 \|u - v\|$,

REMARK 4.1. In a similar manner as in Remark 3.1 we can also relax B2:

B2'. $\operatorname{tr}\Big(A^{1/2}(g(t,u) - g(t,v))Q(g(t,u) - g(t,v))^T A^{1/2}\Big) \le c_7' \|u - v\|^2$,

with the same proof.

THEOREM. *Suppose that B1, B2 (or B1, B2') hold. If u_1 and u_2 are solutions satisfying* (8) *with the same initial value, then they coincide.*

PROOF. We apply the Itô formula to $\|u_1(t) - u_2(t)\|^2$ obtaining

$$
\begin{aligned}
&\|u_1(t) - u_2(t)\|^2\\
&= -2\nu \int_0^t |Au_1(s) - Au_2(s)|^2 ds\\
&\quad + 2\int_0^t (B(u_1(s)) - B(u_2(s)), Au_1(s) - Au_2(s))ds\\
&\quad + 2\int_0^t (f(s,u_1(s)) - f(s,u_2(s)), Au_1(s) - Au_2(s))ds\\
&\quad + 2\int_0^t (Au_1(s) - Au_2(s), g(s,u_1(s)) - g(s,u_2(s)))dw_s\\
&\quad + \int_0^t \operatorname{tr}\Big[A^{1/2}(g(s,u_1(s)) - g(s,u_2(s)))Q\big(g(s,u_1(s)) - g(s,u_2(s))\big)^T A^{1/2}\Big]ds.
\end{aligned}
$$

To estimate the term involving the operator B we use (3) and (4).

$$
\begin{aligned}
&\int_0^t (B(u_1(s)) - B(u_2(s)), Au_1(s) - Au_2(s))ds\\
&\quad = \int_0^t \Big(b(u_1(s), u_1 - u_2(s), Au_1(s) - Au_2(s))\\
&\qquad - b(u_1(s) - u_2(s), u_2(s), Au_1(s) - Au_2(s))\Big)ds\\
&\quad \le \int_0^t \Big(|Au_1(s)|\, \|u_1 - u_2(s)\|\, |Au_1(s) - Au_2(s)|\\
&\qquad + \|u_1(s) - u_2(s)\|\, |Au_2(s)|\, |Au_1(s) - Au_2(s)|\Big)ds
\end{aligned}
$$

$$\leq \nu \int_0^t |Au_1(s) - Au_2(s)|^2 ds$$

$$+ \int_0^t c_9 \|u_1 - u_2(s)\|^2 \left(|Au_1(s)|^2 + |Au_2(s)|^2\right) ds$$

The estimates of the remaining non-stochastic integrals are similar as in the proof of Theorem 3.2. Using B1 and B2, respectively, we obtain

$$\int_0^t (f(s, u_1(s)) - f(s, u_2(s)), Au_1(s) - Au_2(s)) ds$$

$$\leq \int_0^t |f(s, u_1(s)) - f(s, u_2(s)| \cdot |Au_1(s) - Au_2(s)| ds$$

$$\leq \nu \int_0^t |Au_1(s) - Au_2(s)|^2 ds + \int_0^t c_{10} \|u_1(s) - u_2(s)\|^2 ds,$$

$$\int_0^t \operatorname{tr}\left[A^{1/2} (g(s, u_1(s)) - g(s, u_2(s))) Q (g(s, u_1(s)) - g(s, u_2(s)))^T A^{1/2}\right] ds$$

$$\leq \int_0^t \operatorname{tr} Q |g(s, u_1(s)) - g(s, u_2(s))|^2_{\mathbf{H}, \mathbf{V}} ds$$

$$\leq \int_0^t c_{11} \|u_1(s) - u_2(s)\|^2 ds.$$

Going back to the formula for $\|u_1(t) - u_2(t)\|^2$ we have

$$\|u_1(t) - u_2(t)\|^2 \leq \int_0^t c_9 \|u_1 - u_2(s)\|^2 \left(|Au_1(s)|^2 + |Au_2(s)|^2\right) ds$$

$$+ \int_0^t (c_{10} + c_{11}) \|u_1(s) - u_2(s)\|^2 ds$$

$$+ \int_0^t (Au_1(s) - Au_2(s), g(s, u_1(s)) - g(s, u_2(s))) dw_s.$$

We now introduce the auxiliary process

$$\xi(t) = \exp\left(-c_9 \int_0^t \left(|Au_1(s)|^2 + |Au_2(s)|^2\right) ds\right).$$

Computing the differential $d(\xi(t)\|u_1(t) - u_2(t)\|^2)$ and taking mathematical expectation we get

$$E\left(\xi(t)\|u_1(t) - u_2(t)\|^2\right) \leq E c_{12} \int_0^t \xi(s) \|u_1(s) - u_2(s)\|^2 ds.$$

Using the Gronwall lemma we get $E\left(\xi(t)\|u_1(t)-u_2(t)\|^2\right)=0$. Since

$$\int_0^t \left(|Au_1(s)|^2+|Au_1(s)|^2\right)ds<\infty$$

almost everywhere by (8), we find that $\|u_1(t)-u_2(t)\|^2=0$ which finishes the proof. □

REMARK 4.3. Similarly as in the deterministic case, various versions of the notion of strong solution lead to the same result.

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