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TIME DEPENDENT CIRCULAR OPERATORS

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Abstract. We consider some circular commutation relations, where the unitary circulating groups are replaced by suitable one–parameter unitary valued functions. Moreover, the circular operators depend also on the time variable.

1. Let $\{e_n\}_{n\geq 0}$ be an energetic basis of the separable complex Hilbert space H. Let t be the real parameter varying over the totality $\mathbf{R}^1$ of all reals.

In all what follows we assume, that we are given the diagonal operators $K(t)$, $R(t)$, namely:

$$K(t)e_n = k_n(t)e_n \qquad n = 0,1,2,\dots, t \in \mathbf{R}^1 \tag{1.0}$$

$$R(t)e_n = r_n(t)e_n \qquad n = 0,1,2,\dots, t \in \mathbf{R}^1 \ , \tag{1.1}$$

and

(1.2) The real functions $k_n(t)$, $r_n(t)$ are continuous in t all over the real line $\mathbf{R}^1$, for $n = 0,1,2,\dots$.

Let S be the unilateral isometric shift i.e. S is a linear bounded operator in H such that $Se_n = e_{n+1}$ for $n = 0,1,2,\dots$. Suppose now that S satisfies for all real t the circular relation

$$e^{-iK(t)}Se^{iK(t)} = e^{iR(t)}S \ . \tag{1.3}$$

It follows then that for $n = 0,1,2,\dots$ and all t

$$e^{i(k_n(t)-k_{n+1}(t))}e_{n+1} = e^{ir_{n+1}(t)}e_{n+1} \ ; \tag{1.4}$$

Consequently, for all t, n,

$$k_n(t) - k_{n+1}(t) = r_{n+1}(t) + 2p(n,t)\pi \tag{1.5}$$

where $p(n,t)$ is an integer valued function.

Notice, that if $K(t) = tN$, where N is the quantum number operator – $Ne_n = ne_n$ $(n = 0,1,2,\dots)$ and $R(t) \equiv -tI$, then (1.3) holds true and we can put $p(n,t) \equiv 0$.

Let us define the linear manifold M of finite vectors i.e.

$$M = \left\{ f \colon f = \sum_{m=0}^{n(f)} a_m(f) e_m ,\ n(f) < +\infty \right\} .$$

We say that the linear operator V belongs to the M–class, if $D(V) = M$.

We are interested in the solutions $V(t)$ of M–class, to equation

$$e^{-iK(t)} V(t) e^{iK(t)} f = e^{iR(t)} V(t) f \ , \tag{1.6}$$

for $t \in \mathbf{R}^1$ and $f \in M$.

We assume once for all, that (1.3) holds true; this implies the consistency relations (1.5). Next, defining $Z(t) = S^* V(t)$ we get that

$$\begin{aligned} e^{-iK(t)} Z(t) e^{iK(t)} &= e^{-iK(t)} S^* V(t) e^{iK(t)} \\ &= e^{-iK(t)} S^* e^{iK(t)} e^{-iK(t)} V(t) e^{iK(t)} \ . \end{aligned}$$

By (1.3) we derive the commutation relation $e^{-iK(t)} S^* e^{iK(t)}\ S^* e^{-iR(t)}$. It follows then by (1.6) that for $f \in M$, and all t,

$$Z(t) e^{iK(t)} f = e^{iK(t)} Z(t) f \ . \tag{1.7}$$

When taking $f = e_n$, we derive therefore, that for all t and all n

$$\left(e^{i(k_n(t) - K(t))} - 1 \right) Z(t) e_n = 0 \ . \tag{1.8}$$

Let $V(t)e_n = \sum_{m=0}^{\infty} a_m^{(n)}(t) e_m$ be the Fourier expansion of $V(t)e_n$ with respect to the energetic basis $\{e_n\}_{n \geq 0}$. It follows then that

$$Z(t) e_n = S^* V(t) e_n = \sum_{m=1}^{\infty} a_m^{(n)}(t) e_{m-1} \tag{1.9}$$

and consequently by (1.8)

$$\sum_{m=1}^{\infty} a_m^{(n)}(t) \left(e^{ik_n(t) - ik_{m-1}(t)} - 1 \right) e_{m-1} = 0 \ . \tag{1.10}$$

It follows that

$$a_m^{(n)}(t)\left(e^{ik_n(t)-ik_{m-1}(t)}-1\right)=0 \tag{1.11}$$

if $n \neq m-1$. Let us now define the set

$$A_{n,m,p}=\left\{t\in\mathbf{R}^1: k_n(t)-k_{m-1}(t)=2p\pi\right\}\ ,$$

where p is an integer and $n \neq m-1$.
We define $A_{n,m}=\bigcup_p A_{n,m,p}$ where $n \neq m-1$. If t does not belong to $A_{n,m}$, then $k_n(t)-k_{m-1}(t)\neq 2p\pi$ for every integer p. We define $B_{n,m}=\mathbf{R}^1-A_{n,m}$. It follows that if $t\in B_{n,m}$, then $e^{i(k_n(t)-k_{m-1}(t))}\neq 1$. It follows then that if $t\in B_{n,m}$ then $a_m^{(n)}(t)=0$ – notice that $n\neq m-1$. We conclude that:

(1.12) If $n\neq m-1$ and $B_{n,m}$ is dense in $\mathbf{R}^1$, then $a_m^{(n)}(t)\equiv 0$, provided $V(t)e_n$ is t–continuous for each n.

It follows that:

(1.13) If sets $B_{n,m}$ for $n\neq m-1$ $(n,m\in\mathbf{Z}^+)$ are dense, then $a_m^{(n)}(t)\equiv 0$ for $n\neq m-1$, provided $V(t)e_n$ is t–continuous for each n.

Consequently,

(1.14) If for all $n\neq m-1$ the sets $B_{n,m}$ are dense, then $V(t)e_n=v_n(t)e_{n+1}$ provided that the functions $\mathbf{R}^1\ni t\to V(t)e_n$ are continuous for each $n=0,1,2,\dots$

Indeed, it is sufficient to define $v_n(t)=a_{n+1}^{(n)}(t)$.

COROLLARY 1.0. *Let N be the quantum number operator and $V(t)$ of class M. Then, if $V(t)f$ is t–continuous for $f\in M$, $e^{-itN}V(t)e^{itN}=e^{-it}V(t)$ for all t, then $V(t)e_n=v_n(t)e_{n+1}$ for scalar t–continuous functions $v_n(t)$, for $n=0,1,2,3,\dots$.*

COROLLARY 1.1. *Let $k_n(t)$ be polynomials in t with real coefficients such that $k_n\neq k_m$ for $n\neq m$. It is plain that the union of all roots of all equations*

$k_n(t) - k_{m-1}(t) = 2p\pi$ (p–integers $n \neq m-1$) *is a countable set. Hence, its complement is dense in* $\mathbf{R}^1$ and (1.14) *applies.*

References

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