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On the Theorem on Difference Inequalities

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In [1] A. Fitzke presented a proof of convergence of the difference method for the non-linear parabolic equation with mixed derivatives

$$u_t = f(t, x, u, u_x, u_{xx}) \tag{1}$$

where $x = (x_1, \ldots, x_n)$, $u_x = (u_{x_1}, \ldots, u_{x_n})$

$$u_{xx} = (u_{x_1x_1}, \ldots, u_{x_1x_n}, u_{x_2x_2}, \ldots, u_{x_2x_n}, u_{x_3x_3}, \ldots, u_{x_nx_n})$$

with initial and boundary conditions

$$\begin{cases} u(0, x) = \varphi(x) & \\ u(t, x) = \varphi_j(x) & \text{for } x_j = 0, \quad j = 1, \ldots, n \\ u(t, x) = \psi_j(x) & \text{for } x_j = a, \quad j = 1, .., n. \end{cases} \tag{2}$$

This proof was obtained with the aid of the theorem on difference inequalities (cf. theorem 1 in [1]), assuming $0 \leqslant f_u \leqslant L$. In this note we observe that by virtue of this theorem we can obtain the similar result for $f_u \leqslant -L < 0$ and that the theorem 2 in [1] also holds if we replace the assumption $0 \leqslant f_u \leqslant L$ by $|f_u| \leqslant L$.

1. We denote

$$D_T = \{(t, x)\colon 0 \leqslant t \leqslant T,\ 0 \leqslant x_j \leqslant a,\ j = 1, \ldots, n\}.$$

We define a set of nodal points $(t^\mu, x^m) = (\mu k, m_1 h, \ldots, m_n h)$ of D_T, $\mu = 0, 1, \ldots, N_1$, $m_j = 0, 1, \ldots, N$, $j = 1, \ldots, n$, $k = T/N_1$, $h = a/N$. We shall write shortly M for $(\mu, m) = (\mu, m_1, \ldots, m_n)$. We shall also use the notations $+M$ for $(\mu+1, m)$ and $M \pm i$ or $M \pm i \pm j$ for the multiindices $(\mu, m_1, \ldots, m_i \pm 1, \ldots, m_n)$ and $(\mu, m_1, \ldots, m_i \pm 1, \ldots, m_j \pm \pm 1, \ldots, m_n)$ resp. $(i \neq j)$. Let

$$D_h = \{M = (\mu, m);\ 1 \leqslant \mu \leqslant N_1 - 1,\ 1 \leqslant m_j \leqslant N-1,\ j = 1, \ldots, n\}$$

$$D'_h = \{M = (\mu, m);\ \mu = 0 \text{ or it exists such } j\text{, that either } m_j = 0 \text{ or } m_j = N\}.$$

The values of the function v defined in D_T at the nodal point (t^μ, x^m) for $M \in D_h \cup D'_h$ will be denoted by v^M.

Following [1] we define the terms

$$\delta_i v^M = (2h)^{-1}(v^{M+i} - v^{M-i}),$$
$$\delta_{ii} v^M = h^{-2}(v^{M+i} - 2v^M + v^{M-i}),$$
$$\sigma_{ii} v^M = \frac{1}{n-1}\,\delta_{ii} \sum_{\substack{j=1\\ j\neq i}}^{n} (\alpha v^{M+j} + \beta v^M + \alpha v^{M-j}),$$

for $i = 1, \ldots, n$; α, β are positive constants

$$\sigma_{ij} v^M = (4h^2)^{-1}(v^{M+i+j} - v^{M+i-j} - v^{M-i+j} + v^{M-i-j}),$$

for $i = 1, \ldots, n-1$, $j = 2, \ldots, n$, $i<j$.
For abbreviation we shall write δv^M for $(\delta_2 v^M, \ldots, \delta_n v^M)$ and σv^M for $(\sigma_{11} v^M, \ldots, \sigma_{1n} v^M, \sigma_{22} v^M, \ldots, \sigma_{2n} v^M, \sigma_{33} v^M, \ldots, \sigma_{nn} v^M)$.

We claim that the function $f(t, x, u, p, q)$ and the positive constants α, β, k, h satisfy the following

ASSUMPTIONS A.

(i) f is of class C^1 for $(t, x) \in D_T$, $u \in R^1$, $p \in R^n$, $q \in R^{n(n+1)/2}$

(ii) $f_u \leqslant -L < 0$

(iii) $1 + kf_u - \dfrac{2\beta k}{h^2} \sum_{i=1}^{n} f_{q_{ii}} > 0$

(iv) $f_{q_{ii}} > 0$ for $i = 1, \ldots, n$

(v) $|f_{p_i}| < \dfrac{2}{h}\left(\beta f_{q_{ii}} - \dfrac{2\alpha}{n-1} \sum_{\substack{j=1\\ j\neq i}}^{n} f_{q_{jj}}\right)$ for $i = 1, \ldots, n$

(vi) $|f_{q_{ij}}| < \dfrac{4\alpha}{n-1}(f_{q_{ii}} + f_{q_{jj}})$ for $i = 1, \ldots, n-1$, $j = 1, \ldots, n$,

(vii) $2\alpha + \beta = 1$

2. Theorem. *Let u^M and v^M satisfy*

(2.1) $$u^{+M} = u^M + kf(t^\mu, x^m, u^M, \delta u^M, \sigma u^M) + k\varepsilon^M$$

(2.2) $$v^{+M} = v^M + kf(t^\mu, x^m, v^M, \delta v^M, \sigma v^M) \text{ for } M \in D_h$$

(2.3) $$u^M = v^M \textit{ for } M \in D'_h$$

and f, α, β, k, h satisfy the assumptions A. Then $r^M = u^M - v^M$ satisfies the inequality

$$|r^M| \leqslant \frac{\varepsilon}{L}(1-(1-kL)^\mu), \text{ where } \varepsilon = \max_{D_h}|\varepsilon^M|. \tag{2.4}$$

Proof. Using the mean value theorem we obtain, because of the definition of ε

$$\frac{r^{+M}-r^M}{k} \leqslant f_u(\sim)r^M + \sum_{i=1}^{n} f_{p_i}(\sim)\delta_i r^M + \sum_{i=1}^{n}\sum_{i\leqslant j\leqslant n} f_{q_{ij}}(\sim)\sigma_{ij}r^M + \varepsilon,$$

It is obvious (cf. [2]) that the function

$$R^\mu = \frac{\varepsilon}{L}(1-(1-kL)^\mu)$$

satisfies the inequality

$$\frac{R^{\mu+1}-R^\mu}{k} \geqslant -LR^\mu + \varepsilon$$

and because of $\delta_i R^\mu = 0$, $\sigma_{ij}R^\mu = 0$ for $i = 1, \ldots, n$, $i \leqslant j \leqslant n$, it satisfies also

$$\frac{R^{\mu+1}-R^\mu}{k} \geqslant f_u(\sim)R^\mu + \sum_{i=1}^{n} f_{p_i}(\sim)\delta_i R^\mu + \sum_{i=1}^{n}\sum_{i\leqslant j\leqslant n} f_{q_{ij}}(\sim)\sigma_{ij}R^\mu + \varepsilon. \tag{2.5}$$

Then in virtue of theorem 1 of [1] we obtain

$$r^M \leqslant R^\mu$$

In a similar way as in [1] we can obtain the inequality $r^M \geqslant -R^\mu$, then $|r^M| \leqslant R^\mu$. That finishes the proof of (2.4).

The inequality (2.5) holds, because $f_u \leqslant -L < 0$ implies $f_u R^\mu \leqslant -LR^\mu$. But if we assume in the theorem 2 in [1] that $|f_u| \leqslant L$, we have also $f_u R^\mu \leqslant LR^\mu$ and we can obtain the inequality similar to (2.5). Then theorem 2 in [1] is true provided that $|f_u| \leqslant L$.

References

[1] A. Fitzke, *Method of difference inequalities for parabolic equations with mixed derivatives*, Ann. Polon. Math. XXXI (1975), 121—129.

[2] Z. Węglowski, *O stabilności metod różnicowych dla równań cząstkowych*, Roczniki PTM, Matematyka Stosowana IV (1975), 77—85.

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