UNIVERSITATIS IAGELLONICAE, ACTA MATHEMATICA FASCICULUS XXVI

1987

Another Generalization of Strong Unicity

Jan SUDOLSKI and Adam WÓJCIK

Abstract. In this note another concept of generalization of strong unicity is considered. Some properties and examples are given.

1. Introduction. The notion of strong unicity of best approximation was introduced by Newman and Shapiro, who proved the following

THEOREM 1.1, [11]. *Given a Haar subspace V of the Banach space $C\,[0, 1]$ of real- or complex-valued continuous functions on the interval $[0, 1]$ and an element $f \in C[0, 1]$, there exist $g \in V$ and a constant $r > 0$ such that for every $h \in V$*

$$(1.1)\qquad ||f-h|| \geqslant ||f-g|| + r||h-g||$$

in the real case or

$$(1.2)\qquad ||f-h||^2 \geqslant ||f-g||^2 + r||h-g||^2$$

in the complex one.

In the real case the element g satisfying (1.1) is called the strongly unique element of best approximation for f with respect to V. The strong unicity problem can be considered in an arbitrary normed linear space but, by Wulbert's remark [17], in a smooth space E there are no strongly unique elements of best approximation in the above sense with respect to any subspace of E.

The first concept of generalization of the strong unicity was given by McLaughlin and Somers [10]. If E is a real normed linear space and V is its subspace then, for a given $f \in E$, denote by $P_V(f)$ the set of all elements of best approximation to f in V. The strong unicity inequality is defined as follows: there exists $r > 0$ such that for every $h \in V$

$$(1.3)\qquad ||f-h|| \geqslant \operatorname{dist}(f, V) + r \operatorname{dist}(h, P_V(f)).$$

In the real space $C[0, 1]$ the only subspaces with the property that for every $f \in C[0,1]$ the inequality (1.3) holds are Haar subspaces. In general, (1.3) can be satisfied even in the case where V is not a Chebyshev subspace.

The aim of our paper is to generalize the notion of strong unicity in another way to include both real and complex cases of Theorem 1.1. Most of important properties of the strongly unique elements of best approximation are preserved and, as is shown by examples, this notion can be discussed in Hilbert spaces and L^p spaces. Notions which are not defined in the paper can be found in [13].

2. Definitions. Throughout the paper E will denote a normed linear space and G its closed subset.

DEFINITION 2.1. Let f be an element of the space E and $p \geq 1$. An element $g \in G$ is called *the p-strongly unique element of best approximation for f with respect to G* (briefly: p-SUBA) iff there exists a positive constant r such that for every $h \in G$ the following inequality holds

$$\|f-h\|^p \geq \|f-g\|^p + r\|g-h\|^p. \tag{2.1}$$

It is clear that if g is p-SUBA for f (with any p) then g is also the unique element of best approximation for f with respect to G.

Remark 2.2. If g is p-SUBA for f and $q > p$ then g is also q-SUBA for f.

DEFINITION 2.3. If for every $f \in E$ there exists p-SUBA with respect to G then G is called *the strongly Chebyshev set* (briefly: SCS). If p does not depend on f, then G is called *the p-strongly Chebyshev set* (briefly: p-SCS).

Observe that Theorem 1.1 can be reformulated as follows:

Every Haar subspace of $C[0, 1]$ is a p-SCS, where $p = 1$ in the real case and $p = 2$ in the complex case.

3. Characterizations. For $x \in E$ and $d > 0$, let $\bar{B}(x, d)$ denote the closed ball $\{y \in E: \|x-y\| \leq d\}$ and $B(x, d)$ the open ball $\{y \in E: \|x-y\| < d\}$. In the sequel we assume that $f \in E \setminus G$ and $g \in G$.

We start with some conditions equivalent to (2.1). We need two lemmas.

LEMMA 3.1. *Let $p, q \geq 1$ and $d > 0$. The following conditions are equivalent:*

there exists $r > 0$ such that for every $h \in G \cap B(g, d)$ the inequality

$$\|f-h\| \geq \|f-g\| + r\|g-h\|^p \tag{3.1}$$

holds,

there exists $r' > 0$ such that for every $h \in G \cap B(g, d)$ the inequality

$$\|f-h\|^q \geq \|f-g\|^q + r'\|g-h\|^p \tag{3.2}$$

holds.

Proof. It is clear that if $B < A$ then

$$qB^{q-1}(A-B) \leq A^q - B^q \leq qA^{q-1}(A-B).$$

Hence by (3.1) we obtain

$$\|f-h\|^q-\|f-g\|^q \geq q\|f-g\|^{q-1}(\|f-h\|-\|f-g\|) \geq rq\|f-g\|^{q-1}\|g-h\|^p.$$

Conversely, by (3.2)

$$\|f-h\|-\|f-g\| \geq \frac{r'}{q\|f-h\|^{q-1}}\|g-h\|^p \geq \frac{r'}{q(\|f-g\|+d)^{q-1}}\|g-h\|^p.$$

LEMMA 3.2. *If G is boundedly compact and g is the unique element of best approximation for f in G then for every $q \geq 1$ and every $d > 0$ there exists a positive constant r such that for every $h \in G \setminus B(g, d)$*

$$\|f-h\|^q \geq \|f-g\|^q + r\|h-g\|^q. \tag{3.3}$$

Proof. Fix $q \geq 1$ and $d > 0$. Let us consider the continuous positive function

$$F(h) := (\|f-h\|^q - \|f-g\|^q)\cdot\|g-h\|^{-q}$$

defined on $G \setminus B(g, d)$. If G is bounded then the set $K := G \setminus B(g, d)$ is compact and F attains its minimum m on K, whence (3.3) is satisfied with $r = m$. If G is unbounded, it is evident that

$$\lim_{\|g-h\|\to\infty} F(h) = 1 .$$

Hence there exists $M > 0$ such that $F(h) \geq \frac{1}{2}$ for $h \in G \setminus B(g, M)$. Then (3.3) holds for $r = \min(n, 1/2)$ where n is the minimum value of F on the compact set

$$\{h \in G\colon d \leq \|h-g\| \leq M\}.$$

For a fixed number $p \geq 1$, put

$$R_p(t) := \begin{cases} t^p, & \text{if } 0 \leq t \leq 1 \\ t, & \text{if } t > 1. \end{cases}$$

PROPOSITION 3.3. *Let $f \in E \setminus G$ and $g \in G$. The following statements are equivalent:*

(i) *there exists $r_1 > 0$ such that for every $h \in G$*

$$\|f-h\|^p \geq \|f-g\|^p + r_1\|g-h\|^p,$$

(ii) *for every $d > 0$ there exists $r_2 > 0$ such that for every $h \in G \cap \bar{B}(g, d)$*

$$\|f-h\| \geq \|f-g\| + r_2\|g-h\|^p,$$

(iii) *there exists $r_3 > 0$ such that for every $h \in G$*

$$\|f-h\| \geq \|f-g\| + r_3 R_p\left(\frac{\|g-h\|}{\|f-g\|}\right),$$

(iv) *there exists $r_4 > 0$ such that for every $h \in G$*

$$\|f-h\|^p \geq \|f-g\|^p + r_4 R_p\left(\frac{\|g-h\|}{\|f-g\|}\right).$$

If moreover

(a) *G is starlike with respect to g*

or

(b) *G is boundedly compact and g is the unique element of best approximation for f in G then* (i)—(iv) *are equivalent to each of the following conditions:*

(v) *there exist* $D>0$ *and* $r_5>0$ *such that for every* $h\in G\cap \bar{B}(g, D)$

$$\|f-h\|\geqslant \|f-g\|+r_5\|g-h\|^p,$$

(vi) *there exist* $D>0$ *and* $r_6>0$ *such that for every* $h\in G\cap \bar{B}(g, D)$

$$\|f-h\|^p\geqslant \|f-g\|^p+r_6\|g-h\|^p.$$

Proof. (i) $\Rightarrow$ (ii) follows from Lemma 3.1.

(ii) $\Rightarrow$ (iii). By the argument of the proof of Lemma 3.2 there exists $d>\|f-g\|$ such that for every $h\in G\setminus\bar{B}(g, d)$

$$\|f-h\|-\|f-g\|\geqslant \tfrac{1}{2}\|g-h\| = \tfrac{1}{2}\|f-g\|R_p\left(\frac{\|g-h\|}{\|f-g\|}\right).$$

By (ii), for every $h\in G\cap\bar{B}(g, d)$

$$\|f-h\|-\|f-g\|\geqslant r_2\|f-g\|^pR_p\left(\frac{\|g-h\|}{\|f-g\|}\right).$$

Hence (iii) holds with $r_3 = \min(\tfrac{1}{2}\|f-g\|,\ r_2\|f-g\|^p)$.

(iii) $\Rightarrow$ (iv) follows from the first part of the proof of Lemma 3.1.

(iv) $\Rightarrow$ (i). Again by the argument of the proof of Lemma 3.2 there exists $M>\|f-g\|$ such that for every $h\in G\setminus\bar{B}(g, M)$

$$\|f-h\|^p-\|f-g\|^p\geqslant \tfrac{1}{2}\|g-h\|^p.$$

Let $h\in G\cap\bar{B}(g, M)$. If $\|g-h\|\leqslant\|g-f\|$ then

$$\|f-h\|^p-\|f-g\|^p\geqslant \frac{r_4}{\|f-g\|^p}\|g-h\|^p.$$

If $\|g-h\|\geqslant\|f-g\|$ then

$$\|f-h\|^p-\|f-g\|^p\geqslant \frac{r_4}{\|f-g\|\,\|g-h\|^{p-1}}\|g-h\|^p\geqslant \frac{r_4}{\|f-g\|M^{p-1}}\|g-h\|^p.$$

Thus we can take $r_1 = \min(1/2,\ r_4/M^p)$.

(v) $\Leftrightarrow$ (vi) follows from Lemma 3.1.

(i) $\Rightarrow$ (vi) is obvious.

(a) and (v) $\Rightarrow$ (ii). Fix $d>D$ and take $h\in G\cap\bar{B}(g, d)$. Since G is starlike,

$$h' := g+\frac{D}{d}(h-g)\in G \quad\text{and}\quad \|h'-g\|\leqslant D.$$

Hence

$$r_5\|g-h'\|^p \leqslant \|f-h'\|-\|f-g\| = \left\|\left(1-\frac{D}{d}\right)(f-g)+\frac{D}{d}(f-h)\right\| - \|f-g\| \leqslant$$

$$\leqslant \frac{D}{d}(\|f-h\|-\|f-g\|).$$

So we obtain

$$r_5\left(\frac{D}{d}\right)^{p-1}\|h-g\|^p \leqslant \|f-h\|-\|f-g\|$$

what gives (ii).

(b) and (vi) $\Rightarrow$ (i) follows from Lemma 3.2.

Remark 3.4. By a slight modification of the well known Kolmogorov criterion [6] we obtain the p-strong Kolmogorov criterion: if for $f \in E\setminus G$ and $g \in G$ there exists a positive constant r such that for every $h \in G$

$$\sup\{\mathrm{Re}\, L(g-h):\ L \in M(f-g)\} \geqslant r R_p\left(\frac{\|g-h\|}{\|f-g\|}\right), \tag{3.4}$$

where $M(f-g) := \{L \in E^*: \|L\| = 1,\ L(f-g) = \|f-g\|\}$, then g is p-SUBA for f with respect to G.

By [16], if $p = 1$, and G is starlike with respect to g then (3.4) is also a necessary condition for the strong uniqueness. On the other hand, for $p > 1$ the necessity of (3.4) fails to hold in a large class of spaces (e.g. if E is smooth and G is its subspace then by Theorem I.1.1, [13], the left-hand side of (3.4) is less than or equal to zero while, as is shown in Examples 5.3 and 5.4, p-strongly unique elements of best approximation can occur).

We define (set-valued) metric projection onto G as the mapping

$$P_G:\ E \ni x \to P_G(x) = \{z \in G:\ \|x-z\| = \mathrm{dist}(x, G)\}.$$

If G is a Chebyshev set then P_G is a vector-valued function.

Problem 3.5. *Given $f \in E\setminus G$ and $g \in P_G(f)$, find a necessary and sufficient condition on g to be the p-strongly unique element of best approximation to f in G (for some p).*

4. Properties. In Theorem 4.2 some basic properties of strongly Chebyshev sets are gathered. It is seen that such sets must be "very good" and that the property "to be a strongly Chebyshev set" is rather rare, e.g. the only strongly Chebyshev subspaces of $C[a, b]$ are Haar subspaces (see [9]) and the only strongly Chebyshev sets in a Hilbert space are closed convex subsets (see Corollary 5.5). The following lemma is a generalization of the known theorem of Freud (see, e.g. [12], Corollary 2.4.7).

LEMMA 4.1. *Let G be a closed subset of E and $f \in E\setminus G$. If g is p-SUBA for f with respect to G then there exist a neighbourhood U of f and a constant M such that for every $k \in U$ and $h \in P_G(k)$*

$$\|g-h\| \leqslant M\|f-k\|^{1/p}.$$

Proof. For a fixed $k \in E$ and $h \in P_G(k)$, by Proposition 3.3 we get

$$r R_p\left(\frac{||g-h||}{||f-g||}\right) \leqslant ||f-h|| - ||f-g|| \leqslant ||f-k|| + ||k-h|| - ||f-g|| \leqslant$$
$$\leqslant ||f-k|| + ||k-g|| - ||f-g|| \leqslant 2||f-k|| .$$

Hence for every $k \in B(f, r/2)$

$$||g-h||^p \leqslant \frac{2}{r} ||f-g||^p ||f-k|| .$$

Observe that h need not be the unique element of best approximation.

THEOREM 4.2. *Let G be a strongly Chebyshev set in the space E. Then the following conditions are satisfied.*

(i) *G is approximatively compact (i.e. for every $f \in E$ and for every sequence $(g_n) \subset G$ with $\lim_{n\to\infty} ||g_n - f|| = \mathrm{dist}(f, G)$ there exist $g \in P_G(f)$ and $(g_{n_k}) \subset (g_n)$ such that $g_{n_k} \to g$).*

(ii) *The metric projection P_G is continuous.*

(iii) *G is B-connected (i.e. the intersection of G with any open ball is connected).*

(iv) *G is $\bar{B}$-connected (i.e. the intersection of G with any closed ball is connected).*

(v) *If E is complete and G is locally compact then G is a sun (i.e. if $g = P_G(f)$ for any $f \in E$ then $g = P_G(g + t(f-g))$ for every $t > 0$).*

(vi) *If E is an* MS*-space (for definition and some properties see* [1], [4], [5]*) then G is a sun.*

(vii) *If E is a smooth, uniformly convex Banach space then G is convex.*

(viii) *If E is smooth and G is a sun then G is convex.*

Proof. (i). Fix $f \in E \setminus G$ and choose $(g_n) \subset G$ such that $||g_n - f|| \to \mathrm{dist}(f, G)$. If g is p-SUBA for f then by (2.1)

$$||g - g_n||^p \leqslant \frac{1}{r} (||f - g_n||^p - ||f-g||^p) \to 0 ,$$

as $n \to \infty$.

(ii) is a consequence of Lemma 4.1. (iii) follows from Theorem 4.1, [15]. By (i), (iii) and Theorem 8, [8], we obtain (iv). (v) follows from (ii) and Theorem 4.13.b, [15]. (vi) follows from (ii) and Theorem 2.3, [4]. Theorem 4.31, [15] and (i) imply (vii). By Theorem 3.9, [15], we get (viii).

In connection with properties (v), (vi), (vii) of Theorem 4.2 we state the following

Problem 4.3. *Is every strongly Chebyshev set a sun?*

Note that the old problem of whether every Chebyshev set is a sun was negatively solved by Dunham [7], who constructed a counterexample in the space C[0, 1]. But this space is an MS-space (see [1]), therefore, by Theorem 4.2, (vi), every strongly Chebyshev set in C[0, 1] has to be a sun.

5. Examples. Some examples of 1-strongly Chebyshev sets can be found in [2] and [3]. We are going to construct some examples of p-strongly Chebyshev sets for $p > 1$.

DEFINITION 5.1. Let G be an n-dimensional subspace of a normed linear space E. If for every system of linearly independent functionals $L_1, \dots, L_n$ which are extremal points of the unit sphere in the conjugate space E^* the implication

$$h \in G, L_1(h) = \dots = L_n(h) = 0 \Rightarrow h = 0$$

holds, then G is called *an interpolating subspace*.

Example 5.2. By Corollary 3.5, [14], every interpolating subspace G of a normed linear space E is a 2-SCS if the space E is complex, and 1-SCS if the space E is real.

Example 5.3. Given a positive measure space (X, M, m), let $E = L^p(X, M, m)$. For every set $A \in M$ such that $m(A) > 0$ the subspace $G = \{h \in E\colon h|X \setminus A = 0\}$ is p-SCS. Indeed, for any $f \in E$ the element g defined by

$$g(x) := \begin{cases} f(x), & \text{if } x \in A \\ 0, & \text{if } x \in X \setminus A \end{cases}$$

is p-SUBA for f, since for every $h \in G$

$$\|f-h\|^p = \int_{X \setminus A} |f|^p dm + \int_A |f-h|^p dm = \|f-g\|^p + \|g-h\|^p.$$

Example 5.4. Let E be a Hilbert space and G its closed convex subset. Fix $f \in E \setminus G$ and put $g = P_G(f)$. We claim that for every $h \in G$

$$\|f-h\|^2 \geqslant \|f-g\|^2 + \|g-h\|^2.$$

Indeed, put $d = \|f-g\|$. Then there exists a hyperplane H which separates G and the ball $B(f, d)$. This means that there exists $L \in M(f-g)$ such that $H = \{k\colon L(k-g) = 0\}$ and $\mathrm{Re} L(h-g) \leqslant 0$ for all $h \in G$, whence

$$L(x) = \frac{(x, f-g)}{\|f-g\|}.$$

Furthermore

$$\|f-h\|^2 = \|h-g\|^2 + \|f-g\|^2 + 2\,\mathrm{Re}(h-g, g-f),$$

which proves our claiming.
By Theorem 4.2, Example 5.4 and Theorem 3.9, [15], we obtain

COROLLARY 5.5. *Let E be a Hilbert space and G its subset. The following requirements are equivalent:*

(i) *G is 2-strongly Chebyshev set,*

(ii) *G is closed and convex,*

(iii) *G is a Chebyshev set and a sun.*

The next example shows that there is a Chebyshev set which is not a strongly Chebyshev set.

Example 5.6. Let $E = R^2$ be normed by $||(x, y)|| = \max(|x|, |y|)$. Put

$$G = \{(x, y) \in R^2 \colon \exp(-4/x^2) \leqslant y \leqslant 1-|x|\}.$$

Then G is a convex Chebyshev set. Take $f = (0, -1)$. It is obvious that $P_G(f) = g = (0, 0)$. For $h = \big(x, \exp(-4/x^2)\big)$

$$||f-h|| - ||f-g|| = \exp(-4/x^2).$$

But $||g-h|| = |x|$ and there are no $p \geqslant 1$ and $r > 0$ satisfying $r|x|^p \leqslant \exp(-4/x^2)$ when $x \to 0$.

References

[1] D. Amir, F. Deutsch, *Suns, Moons and Quasi-Polyhedra*, J. Approximation Theory 6 (1972), 176—201.

[2] D. A. Ault, F. Deutsch, P. D. Morris, J. E. Olson, *Interpolating Subspaces in Approximation Theory*, J. Approximation Theory 3 (1970), 164—182.

[3] D. Braess, *Kritische Punkte bei der nichtlinearen Tschebyscheff-Approximation*, Math. Zeitschrift 132 (1973), 327—341.

[4] B. Brosowski, B. Deutsch, *Radial Continuity of Set-Valued Metric Projection*, Bericht GWD MBH Goettingen 3 (1972), 1—25.

[5] B. Brosowski, B. Deutsch, *On Some Geometrical Properties of Suns*, Bericht GWD MBH Goettingen 2 (1972), 1—35.

[6] B. Brosowski, R. Wegmann, *Charakterisierung bester Approximationen in Normierten Vektorräumen*, J. Approximation Theory 3 (1970), 369—397.

[7] Ch. B. Dunham, *Chebyshev Sets in* $C[0, 1]$ *which are not Suns*, Canad. Math. Bull. 18 (1975), 35—37.

[8] B. A. Кощеев, *Связность и некоторые аппроксимативные свойства множеств в линейных нормированных пространствах*, Мат. Заметки 17 (1975), 193—204.

[9] H. W. McLaughlin, K. B. Somers, *Another Characterization of Haar Subspaces*, J. Approximation Theory 14 (1975), 93—102.

[10] H. W. McLaughlin, K. B. Somers, *A Generalization of Strong Unicity*, Amer. Math. Soc. Notices 20 (1973).

[11] D. J. Newman, H. S. Shapiro, *Some Theorems on Chebyshev Approximation*, Duke Math. J. 30 (1963), 673—681.

[12] H. S. Shapiro, Topics in Approximation Theory, Lecture Notes in Math. 187, Springer-Verlag, Berlin—Heidelberg—New York 1971.

[13] I. M. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Springer-Verlag, Berlin—Heidelberg—New York 1970.

[14] J. Sudolski, A. Wójcik, *Strong Unicity in a Quotient Space*, Univ. Jagell. Acta Math. 24 (1984), 221—225.

[15] Л. П. Власов, *Аппроксимативные свойства множеств в линейных нормированных пространствах*, Успехи М. Н. 28 (1973), 3—66.

[16] A. Wójcik, *Characterization of Strong Unicity by Tangent Cones*, Proceedings of the Conference „Approximation and Function Spaces" held in Gdańsk, 1979, Z. Ciesielski (ed.), PWN Warszawa, North-Holland Amsterdam (1981), 854—866.

[17] D. E. Wulbert, *Uniqueness and Differential Characterizations of Approximations from Manifolds of Functions*, American J. Math. 93 (1971), 350—366.

INSTYTUT MATEMATYKI
AKADEMIA GÓRNICZO-HUTNICZA
AL. MICKIEWICZA 30, 30-059 KRAKÓW

Received January 11, 1982.