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On the Exactness of a Class of Endomorphisms of the Real Line

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Summary. A class of piecewise monotonic and expanding transformations, defined on the whole real line, is considered. It is shown that every such transformation is an exact endomorphism in the sense of Rohlin.

1. Introduction. Let $(X, \mathfrak{M})$ be a measurable space, and let $\tau\colon X \to X$ be a measurable transformation, i.e., $\tau^{-1}(A) \in \mathfrak{M}$ for each $A \in \mathfrak{M}$. We say that a measure μ, defined on $\mathfrak{M}$, is τ-invariant (or shortly invariant) if $\mu(\tau^{-1}(A)) = \mu(A)$ for each $A \in \mathfrak{M}$. A measurable transformation τ for which there exists an invariant measure μ is called a measure-preserving transformation, or an endomorphism of a measure space $(X, \mathfrak{M}, \mu)$.

Let τ be an endomorphism of a measure space $(X, \mathfrak{M}, \mu)$. The endomorphism τ is called an exact endomorphism ([6]) if the σ-algebra $\mathfrak{M}_\infty = \bigcap_{n=0}^{\infty} \tau^{-n}(\mathfrak{M})$ contains only sets of measure zero and their complements.

In ([6; p. 525]), V. A. Rohlin has given a criterion for the exactness of some measure-preserving transformations and gives many applications of this criterion. Namely he proved, among other things, that some number-theoretic transformations of the unit interval onto itself are exact endomorphisms.

In the case of transformations defined on the whole real line it is rather difficult to decide, by applying Rohlin's criterion, whether a given transformation (possessing an absolutely continuous invariant measure) is an exact endomorphism. This is because each absolutely continuous measure, invariant under transformation defined on the whole real line, has a density which vanish at infinity.

Some conditions for the exactness of the transformations defined on the whole real line were proposed by J. H. B. Kemperman ([1]), and M. Lin ([8]).

Recently A. Lasota ([3; Theorem 2]) has given a different type of criterion (based on a fixed point theorem) for the exactness of some class of nonsingular transformations.

Using the criterion just mentioned, and applying a technique introduced by A. Lasota, G. Pianigiani and J. A. Yorke ([4], [5]) we shall show, inter alia, the exactness of some transformations defined on the whole real line. In particular, we shall show the exactness of the transformations of the form $\varphi(x) = A\tan(Bx+C)$, where $|AB|>1$. The ergodicity of these transformations was proved by J. H. B. Kemperman ([1], [2]) and F. Schweiger ([7]).

Our paper is divided into four sections. In Section 2, we state a theorem which contains the principal results of this paper. In Section 3 we lay the foundation for the proof of this theorem. Section 4 contains the proof of our theorem.

2. Statement of the Main Results. We shall start with the definition of the Frobenius-Perron operator. Let $(L^1, \|\cdot\|)$ be the space of all integrable (with respect to the Lebesgue measure m on the whole real line R) functions defined on the whole real line R, and let $\tau\colon R \to R$ be a nonsingular transformation.

The Frobenius-Perron operator P_τ, corresponding to τ, is defined by the formula

$$P_\tau f = \frac{d}{dm}(\mu_f \circ \tau^{-1}),$$

where $d\mu_f = f dm$.

From the definition it follows that the measure μ_f is invariant under τ (τ preserves the measure μ_f) if and only if the function f is a fixed point for P_τ, i.e., $P_\tau f = f$.

Let D denote a set of all densities, that is all $f \in L^1$ such that $f \geqslant 0$ and $\|f\| = 1$. It should be noted that there holds the inclusion $P_\tau(D) \subset D$. This inclusion follows from the fact that P_τ is a positive isometry.

Now, we shall select a class of densities which will be needed in our further considerations. A density $f \in D$ will be called regular if it is locally Lipschitzean. The regularity of f (see: [5]) is defined by

$$\operatorname{Reg}(f) = \sup\left\{\frac{|f'(x)|}{f(x)} : x \in R, \text{ and } f'(x) \text{ is defined, and } f(x) > 0\right\}.$$

Let D_1 be the set of all $f \in D$ which are regular, and which satisfy the following conditions:

$$\operatorname{Reg}(f) < \infty, \quad \text{and} \quad \bigvee_{-\infty}^{+\infty} f < \infty$$

(here and in what follows the symbol $\bigvee_b^a f$ as well as $\bigvee_I f$ denotes the variation of f over the closed interval $I = [a, b]$).

It should be noted that the set D_1 is dense in D.

Now we describe a class of (piecewise monotonic) transformations defined on the whole real line R (except for countably many points) which will be the subject of our study.

Let $\{I_k\}_{k=-\infty}^{+\infty}$ be a doubly infinite sequence of the open intervals I_k of R such that

(2.1) $\left\{ c_1 = \inf_k |I_k| > 0, \ c_2 = \sup_k |I_k| < \infty, \text{ for each integer } k = 0, \pm 1, \pm 2, \ldots \right.$ (here and in what follows the symbol $|I|$ denotes the length of the interval I);

(2.2) $$I_k \cap I_j = \emptyset \text{ for } k \neq j, \text{ and } \bigcup_{k=-\infty}^{+\infty} \operatorname{cl} I_k = R.$$

We shall say that a transformation $\varphi\colon R_0 \to R$, where $R_0 = \bigcup_{k=-\infty}^{+\infty} I_k$, belongs to the class Φ if it satisfies the following conditions:

(2.3) $\left\{\begin{array}{l}\text{the restriction } \varphi_k \text{ of } \varphi \text{ to the interval } I_k \text{ is differentiable and its} \\ \text{derivative } \varphi_k' \text{ is locally Lipschitzean;}\end{array}\right.$

(2.4) there is a constant $c_3 > 1$ such that $|\varphi_k'(x)| \geq c_3$ for $x \in I_k$;

(2.5) $\varphi_k(I_k) = R$ for $k = 0, \pm 1, \pm 2, \ldots$;

(2.6) $\left\{\begin{array}{l}\text{there is a constant } c_4 > 0, \text{ and a function } \sigma \in L^1,\ \sigma \geq 0 \text{ such that} \\ c_4^{-1}\sigma(x) \leq \sigma_k(x) \leq c_4\sigma(x) \text{ for } x \in R,\ k = 0, \pm 1, \pm 2, \ldots, \text{ where } \sigma_k(x) \\ = |(\varphi_k^{-1})'(x)|;\end{array}\right.$

(2.7) $$c_5 = \sup_k \sup_x \big(|\sigma_k'(x)|/\sigma(x)\big) < +\infty .$$

The following theorem contains our main result concerning some ergodic properties of the transformations belonging to the class under consideration.

THEOREM. *Let $\varphi \in \Phi$, and let P_φ be the corresponding Frobenius-Perron operator. Then*

(2.8) $\left\{\begin{array}{l}\textit{there exists a unique } f_0 \in D \textit{ such that } f_0 = \lim_{n\to\infty} P_\varphi^n f \textit{ for all } f \in D, \textit{ and} \\ \textit{consequently;}\end{array}\right.$

(2.9) *the measure $d\mu = f_0 dm$ is φ-invariant.*

(2.10) $\left\{\begin{array}{l}\textit{If } f \in D_1, \textit{ then } \{P_\varphi^n f\} \textit{ is a sequence of Lipschitzean functions which} \\ \textit{uniformly converges to } f_0, \textit{ consequently;}\end{array}\right.$

(2.11) *the density f_0 is Lipschitzean.*

(2.12) *The endomorphism φ of the measure space $(R, \mathscr{B}, \mu)$ is exact.*

It may be worthwhile to stress that the condition (2.3) involved in the definition of the class Φ cannot be weakened without affecting the truth of (2.10) and (2.11) (see: Appendix).

We also note that the thesis of Theorem remains valid if we replace the condition (2.6) by the following condition:

(2.13) $\left\{\begin{array}{l}\text{there exists a subset } Z_0 \text{ of the integers such that,} \\ \int \inf_j \big(\sum_{k\in Z_0} |I_k|^{-1}/\sigma_k(y) \int_{I_k} |I_j|^{-1}\sigma_j(x)m(dx)\big)m(dy) > 0.\end{array}\right.$

This result will be shown in a subsequent paper.

We shall need a few auxiliary results before proving the Theorem.

3. A few auxiliary results. We first state the result concerning the differentiability (almost everywhere) of a functional series. Namely, we shall need the following consequence of the Fubini theorem.

PROPOSITION 3.1. *Let* $\{h_j\}_{j=-\infty}^{+\infty}$ *be a doubly infinite sequence of non-negative real valued functions, defined on the real line, which satisfy*

(3.1) $$\sum_{j=-\infty}^{+\infty} h_j < \infty \quad \text{(a.e.)};$$

(3.2) $$\bigvee_{-\infty}^{+\infty} h_j \leqslant \alpha_j \quad \textit{for} \quad j = 0, \pm 1, \pm 2, \ldots, \quad \textit{and} \quad \sum_{j=-\infty}^{+\infty} \alpha_j < \infty;$$

(3.3) $$\lim_{x \to -\infty} h_j(x) = 0 \quad \textit{for} \quad j = 0, \pm 1, \pm 2, \ldots$$

Then the function $h = \sum_{j=-\infty}^{+\infty} h_j$ *is differentiable* (*almost everywhere*), *and* $h' = \sum_{j=-\infty}^{+\infty} h'_j$ (a.e.).

Proof. Applying the Jordan decomposition to h_j, we obtain

$$h_j = h_{j1} - h_{j2},$$

where

$$h_{j1}(x) = 1/2\Big(\bigvee_{-\infty}^{x} h_j + h_j(x)\Big), \quad h_{j2}(x) = 1/2\Big(\bigvee_{-\infty}^{x} h_j - h_j(x)\Big),$$

and both the functions h_{j1} and h_{12} are increasing.

From the definition of h_{j1} it follows that $h_{j1} \geqslant 0$. Also, h_{j2} is a non-negative function, since the assumption (3.3) implies $h_j(x) \leqslant \bigvee_{-\infty}^{x} h_j$.

Now, by the Fubini theorem we get the equalities

$$\Big(\sum_{j=-\infty}^{+\infty} h_{jk}\Big)' = \sum_{j=-\infty}^{+\infty} h'_{jk} \quad \text{for} \quad k = 1, 2.$$

This two equalities imply that $h' = \sum_{j=-\infty}^{+\infty} h'_j$ (a.e.). The proposition has been proved.

We now turn to a problem of differentiation of the Frobenius-Perron operator, corresponding to $\varphi \in \Phi$. A simple computation shows that the Frobenius-Perron operator, corresponding to φ, can be written in the form:

(3.4) $$P_\varphi f(x) = \sum_{j=-\infty}^{+\infty} g_j(x) \text{ (a.e.)} \quad \text{for each } f \in L^1,$$

where $g_j(x) = \sigma_j(x) f \circ \varphi_j^{-1} x)$.

By its very definition the operator P_φ is a mapping from L^1 into L^1, but the last formula enables us to consider P_φ as a map from the space of functions defined on R into itself. Below we shall prove the following

LEMMA 3.1. *Let* $\varphi \in \Phi$, *and let* P_φ *be the corresponding Frobenius-Perron operator. If* f *is a regular density of bounded variation, then the function* $P_\varphi f$ *is differentiable* (a.e.), *and*

$$(P_\varphi f)' = \sum_{j=-\infty}^{+\infty} g'_j.$$

Proof. We shall prove this lemma by showing that the functions g_j ($j = 0, \pm 1, \pm 2, \ldots$) satisfy all the hypotheses of the Proposition 3.1.

First we show the following inequalities

(3.5) $$\bigvee_{-\infty}^{+\infty} g_j \leqslant \alpha_j \quad \text{for} \quad j = 0, \pm 1, \pm 2, \ldots,$$

where $\alpha_j = c_5 \int_{I_j} f dm + c_3^{-1} \bigvee_{I_j} f$.

To prove these inequalities we note that for each $j = 0, \pm 1, \pm 2, \ldots$ we have

$$\bigvee_{-\infty}^{+\infty} g_j = \int_{-\infty}^{+\infty} g_j' dm \leqslant \int_{-\infty}^{+\infty} |\sigma_j'| f \circ \varphi_j^{-1} dm + \int_{-\infty}^{+\infty} \sigma_j f' \circ \varphi_j^{-1} \sigma_j dm .$$

Taking into account that

$$\sigma_j \leqslant c_3^{-1} \quad \text{and} \quad |\sigma_j'|/\sigma_j \leqslant c_5$$

(these inequalities follow from the conditions (2.4) and (2.7)) we get

$$\bigvee_{-\infty}^{+\infty} g_j \leqslant c_5 \int_{-\infty}^{+\infty} f \circ \varphi_j^{-1} \sigma_j dm + c_3^{-1} \int_{-\infty}^{+\infty} f' \circ \varphi_j^{-1} \sigma_j dm \leqslant c_5 \int f dm + c_3^{-1} \bigvee_{I_j} f,$$

which was to be shown.

It follows from the estimates (3.5) that

(3.6) $$\sum_{j=-\infty}^{+\infty} \alpha_j \leqslant c_5 \|f\| + c_3^{-1} \bigvee_{-\infty}^{+\infty} f .$$

Thus we see that the condition (3.2) of the Proposition 3.1 is fulfilled.

Obviously the series $\sum_{j=-\infty}^{+\infty} g_j(x)$ is absolutely convergent for almost all x (see: formula (3.4)). Thus the condition (3.1) is also fulfilled.

It remains only to show that g_j tends to zero as $x \to -\infty$, for each $j = 0, \pm 1, \pm 2, \ldots$.

Suppose that $\liminf_{x \to -\infty} g_j(x) < \limsup_{x \to -\infty} g_j(x)$ for some j. Then $\bigvee_{-\infty}^{+\infty} g_j = \infty$. This contradiction (see: formula (3.5)) shows that our assumption is incorrect. Thus $\liminf_{x \to -\infty} g_j(x) = \limsup_{x \to -\infty} g_j(x)$ for each $j = 0, \pm 1, \pm 2, \ldots$, and since σ_j is integrable, $\lim_{x \to -\infty} g_j(x) = 0$. Thus the condition (3.3) is fulfilled. This finishes the proof of the lemma.

We now show that the set D_1 is invariant under $P_\varphi^n = P_\varphi P_\varphi^{n-1}$ for each natural number $n \geqslant 1$, i.e., $P_\varphi^n(D_1) \subset D_1$. We do this in three stages. First we show that the regularity of $P_\varphi f$ is finite, if $f \in D_1$. Next we show that $P_\varphi f$ is Lipschitzean, if $f \in D_1$. Then we show that these two properties has each iterate $P_\varphi^n f$ of $P_\varphi f$ and, that it is a function of bounded variation.

LEMMA 3.2. *If* $f \in D_1$, *then* $\mathrm{Reg}(P_\varphi f) \leqslant c_5 + c_3^{-1} \mathrm{Reg}(f)$.

Proof. By the Lemma 3.1 we have

$$(P_\varphi f)' = \sum_{j=-\infty}^{+\infty} (\sigma_j' f \circ \varphi_j^{-1} + \sigma_j f' \circ \varphi_j^{-1} \sigma_j) \quad \text{(a.e.)},$$

and hence

$$\frac{|(P_\varphi f)'|}{P_\varphi f} \leqslant \frac{|\sum_j \sigma_j' f \circ \varphi_j^{-1}|}{Pf} + \frac{|\sum_j \sigma_j f' \circ \varphi_j^{-1} \sigma_j|}{Pf}$$

$$\leqslant \sup_j \frac{|\sigma_j'|}{\sigma_j} + \sup_j \sigma_j \frac{f' \circ \varphi_j^{-1}}{f \circ \varphi_j^{-1}}$$

$$\leqslant c_5 + c_3^{-1} \operatorname{Reg}(f) \quad \text{(a.e.)}.$$

Thus $\operatorname{Reg}(P_\varphi f) \leqslant c_5 + c_3^{-1} \operatorname{Reg}(f)$. The lemma has been proved.

LEMMA 3.3. *If $f \in D_1$, then*

$$|P_\varphi f(x_1) - P_\varphi f(x_2)| \leqslant C(f)\left(c_4 c_6^2 \operatorname{Reg}(f) + c_5 c_6\right)|x_1 - x_2|$$

for each $x_1, x_2 \in R$; where $c_6 = \sup_{x \in R} \sigma(x)$, and

$$C(f) = \bigvee_{-\infty}^{+\infty} f + c_1^{-1} \|f\|.$$

Proof. Let f be an arbitrary function belonging to D_1. For $x_1, x_2 \in R$ by formula (3.4), we have

$$|P_\varphi f(x_1) - P_\varphi f(x_2)| \leqslant |A(x_1, x_2)| + |B(x_1, x_2)|,$$

where

$$A(x_1, x_2) = \sum_{j=-\infty}^{+\infty} \sigma_j(x_1)\left(f \circ \varphi_j^{-1}(x_1) - f \circ \varphi_j^{-1}(x_2)\right),$$

and

$$B(x_1, x_2) = \sum_{j=-\infty}^{+\infty} (\sigma_j(x_1) - \sigma_j(x_2)) f \circ \varphi_j^{-1}(x_2).$$

We show first that the following inequality holds true

$$|A(x_1, x_2)| \leqslant c_6^2 c_4 \operatorname{Reg}(f) C(f) |x_1 - x_2|.$$

It may easily be checked that for every integer j the following three inequalities hold true:

$$|f \circ \varphi_j^{-1}(x_1) - f \circ \varphi_j^{-1}(x_2)| \leqslant f(s_j) \operatorname{Reg}(f) |\varphi_j^{-1}(x_1) - \varphi_j^{-1}(x_2)|,$$

where

$$f(s_j) = \max_{s \in I_j} f(s);$$

$$|\varphi_j^{-1}(x_1) - \varphi_j^{-1}(x_2)| \leqslant c_4 c_6 |x_2 - x_1|;$$

and

$$f(t_j) \leqslant |I_j|^{-1} \int_{I_j} f\,dm,$$

where $f(t_j) = \min_{s \in I_j} f(s)$.

The first two of these inequalities give

$$|A(x_1, x_2)| \leqslant c_6^2 c_4 (\operatorname{Reg}(f)) |x_1 - x_2| \sum_{j=-\infty}^{+\infty} f(s_j);$$

while the third inequality implies that

(3.7) $$\sum_{j=-\infty}^{+\infty} f(s_j) \leqslant \sum_{j=-\infty}^{+\infty} (f(s_j)-f(t_j)) + c_1^{-1}\|f\| \leqslant C(f).$$

Thus we see that the desired inequality holds true.

Now we pass to the proof of the following inequality

$$|B(x_1, x_2)| \leqslant c_5 c_6 C(f)|x_1 - x_2|.$$

Our starting point for the proof is the following inequality:

$$|\sigma_j(x_1) - \sigma_j(x_2)| \leqslant c_5 c_6 |x_1 - x_2|,$$

which is valid for any integer j. Immediately we obtain from this

$$|B(x_1, x_2)| \leqslant c_5 c_6 |x_1 - x_2| \sum_{j=-\infty}^{+\infty} f \circ \varphi_j^{-1}(x_2).$$

Repeating the same calculations as is (3.7) we obtain

(3.8) $$\sum_{j=-\infty}^{+\infty} f \circ \varphi_j^{-1}(x_2) \leqslant C(f),$$

which finishes the proof of the required inequality. The lemma has been proved.

We are now ready to prove an important (for further considerations) statement.

PROPOSITION 3.2. *The set D_1 is invariant under P_φ^{n-1} for every natural $n \geqslant 1$,* i.e., $f_{n-1} = P_\varphi^{n-1} f \in D_1$ *for every $n \geqslant 1$, and $f \in D_1$. Moreover, if $f \in D_1$, then there are constants $c_7, c_8 > 0$ such that $\limsup_{n\to\infty} \mathrm{Reg}(f_n) \leqslant c_7$, $\limsup_{n\to\infty} \bigvee_{-\infty}^{+\infty} f_n \leqslant c_7$, and $\limsup_{n\to\infty} |f_n(x_1) - f_n(x_2)| \leqslant c_8 |x_1 - x_2|$ for any $x_1, x_2 \in R$. The numbers c_7, c_8 are independent of $f \in D_1$.*

Proof. Let f be an arbitrary function belonging to D_1. If $f_k \in D_1$ for $k = 0, 1, \ldots, n-1$, then by Lemma 3.2 we have

$$\mathrm{Reg}(f_{k+1}) \leqslant c_5 + c_3^{-1}\mathrm{Reg}(f_k) \quad \text{for} \quad k = 0, 1, \ldots, n-1;\ f_0 = f.$$

Hence

$$\mathrm{Reg}(f_n) \leqslant c_5(c_3^{-1} + \ldots + c_3^{-(n-1)}) + c_3^{-n}\mathrm{Reg}(f),$$

so that

(3.9) $$\limsup_{n\to\infty} \mathrm{Reg}(f_n) \leqslant c_7, \quad \text{where} \quad c_7 = c_3 c_5 (c_3 - 1)^{-1}.$$

Next, the formula (3.4), and the inequalities (3.5), (3.6) together give

$$\bigvee_{-\infty}^{+\infty} f_{k+1} \leqslant c_5 \|f_k\| + c_3^{-1} \bigvee_{-\infty}^{+\infty} f_k \quad \text{for} \quad k = 0, 1, \ldots, n-1;\ f_0 = f$$

These inequalities, together with the equality $\|f_k\| = \|f\|$ $(k = 1, 2, \ldots)$ give

$$\bigvee_{-\infty}^{+\infty} f_{k+1} \leqslant c_3^{-n} \bigvee_{-\infty}^{+\infty} f + c_3 c_5 (c_3 - 1)^{-1} \|f\|,$$

so that

(3.10) $$\limsup_{n\to\infty} \bigvee_{-\infty}^{+\infty} f_n \leqslant c_7.$$

Finally, by Lemma 3.3 we have

$$|f_n(x_1)-f_n(x_2)| \leqslant C(f_{n-1})(c_4 c_6^2 \operatorname{Reg}(f_{n-1})+c_5 c_6)|x_1-x_2|,$$

for $x_1, x_2 \in R$, $n = 1, 2, \ldots$; $f_0 = f$.

This and inequalities (3.9), (3.10) imply that

$$\limsup_{n\to\infty} |f_n(x_1)-f_n(x_2)| \leqslant c_8|x_1-x_2|, \quad \text{for each } x_1, x_2 \in R,$$

where $c_8 = (c_7+c_1^{-1})(c_4 c_6^2 c_7+c_5 c_6)$.
The proposition has been proved.

We close this section with a result which will serve as a test for exactness of the transformations under consideration.

A closed convex set $Y \subset L^1$ is said to be imbedded in $Z \subset L^1$ (see: [3]) if for every two different functions $h_1, h_2 \in Y$ the closed interval $[0, 1]$ is contained in the interior of the set $\{r \in R\colon rh_1+(1-r)h_2 \in Z\}$.

The following proposition is a particular case of Theorem 2 in paper [3]:

PROPOSITION 3.3. *Let $\tau\colon R \to R$ be a nonsingular transformation. Assume that there exists a set $K \subset D$ which satisfies the following conditions:*

(3.11) *K is convex, and compact;*

(3.12) *K is imbedded in D;*

(3.13) *the family $H = \{h \in D\colon \lim_{n\to\infty} \varrho(P_\tau^n h, K) = 0\}$,*

*where $\varrho(P_\tau^n h, K) = \inf\{\|P_\tau^n h-g\|\colon g \in K\}$,
is dense in D. Then there exists a unique $h_0 \in D$ such that*

(3.14) *$h_0 = \lim_{n\to\infty} P_\tau^n h$ for all $h \in D$ and, consequently;*

(3.15) *the measure $dv = h_0 dm$ is τ-invariant;*

(3.16) *the endomorphism τ of the measure space $(R, \mathfrak{B}(R), v)$*

is exact.

4. Proof of the Theorem. The idea of the proof is arrange things so that the Proposition 3.3 may be applied. To this end we begin by proving the following

CLAIM: *Let $\varphi \in \Phi$, and let P_φ be the corresponding Frobenius-Perron operator. Then there are two constants $c_9, c_{10} > 0$ such that*

$$c_9\sigma \leqslant \liminf_{n\to\infty} f_n \leqslant \limsup_{n\to\infty} f_n \leqslant c_{10}\sigma \quad \textit{for all } f \in D_1,$$

where, as before, $f_n = P_\varphi^n f$. The numbers c_9, c_{10} are independent of $f \in D_1$.

Proof of the Claim. Since the proof of the Claim is rather long, it will be convenient to divide it into three steps.

STEP 1. *If f is a regular density of bounded variation, then*

$$\limsup_{n\to\infty} f_n \leqslant c_{10}\sigma .$$

Proof. Let f be an arbitrary regular density of bounded variation over R. From the formula (3.4), and the inequalities (2.6), (3.8) it follows that

$$f_1(x) \leqslant c_4 \sigma(x) C(f) .$$

Since f_n are regular densities of bounded variation (see: Proposition 3.2), from the last inequality we obtain for any $n \geqslant 1$

$$f_n(x) \leqslant c_4 \sigma(x) C(f_{n-1}), \quad f_0 = f .$$

Because $\|f_n\| = \|f\|$ for any $n \geqslant 1$, and $\limsup_{n\to\infty} \bigvee_{-\infty}^{+\infty} f_n \leqslant c_7$ (see: Proposition 3.2), therefore

$$\limsup_{n\to\infty} f_n \leqslant c_{10}\sigma , \quad \text{where } c_{10} = c_4(c_7 + c_1^{-1}) .$$

The Step 1 is proved.

STEP 2. *If f is a regular density of bounded variation, then there is a compact interval I, a sequence $\{x_n\}$ of real numbers, and a natural number n_0 (n_0 depends on f) such that*

(4.1) $$I_{k_0} \subset I \quad \textit{for some } k_0 ,$$

(4.2) $$x_n \in I \quad \textit{for each } n \geqslant 1 ,$$

(4.3) $$f(x_n) \geqslant c_{11} \quad \textit{for } n \geqslant n_0 ,$$

where $c_{11} > 0$ is a constant independent of f.

Proof. By Step 1, we can choose a real number $0 < c_{12} < 1$ such that

(4.4) $$\int_{|x| \geqslant c_{13}} f_n(s)\, dm(s) \leqslant c_{12} \quad \text{for } n \geqslant n_0 ,$$

where $c_{13} > 0$ is some constant independent of f, and n_0 is a natural number (n_0 depends on f). Moreover, the interval $I = [-c_{13}, c_{13}]$ contains an interval I_{k_0}.

Note that there are numbers $x_n \in I$ such that

$$f_n(x_n) \geqslant |I|^{-1} \int_I f_n(s)\, dm(s) \quad \text{for } n = 1, 2, \ldots .$$

This and the inequality (4.4) imply (4.3) with $c_{11} = (1 - c_{12})|I|^{-1}$. Thus the Step 2 is proved.

STEP 3. *If $f \in D_1$, then there is a constant $c_9 > 0$ such that*

$$\liminf_{n\to\infty} f_n \geqslant c_9 \sigma .$$

Proof. Note first that for an arbitrary compact interval $I \subset R$, and for a large enough n the image $f_n(I)$ is a compact interval without zero. This is because (by Proposition 3.3) f_n is a continuous function (in fact, Lipschitzean) and $f_n > 0$ for sufficiently large n (if f does not vanish identically).

Now, in the last section (Proposition 3.3), we have proved that $\limsup_{n\to\infty} \operatorname{Reg}(f_n) \leqslant c_7$. This inequality, restricted to a compact interval I, leads to the following inequality

$$f_n(y) \geqslant f_n(x)\exp(-c_{07}|x-y|) \text{ for each } x, y \in I \text{ and, } n \geqslant n_0;$$

where $c_{07} > c_7$ is arbitrary (but fixed).

By virtue of this inequality and the preceding step we see that for any $y \in I$ and, $n \geqslant n_0$

$$f_n(y) \geqslant c_{14} \quad \text{for each } y \in I, \tag{4.5}$$

where $c_{14} = c_{11}\exp(-c_{07}|I|)$.

For $n > n_0$ we have

$$f_n = P_\varphi f_{n-1} = \textstyle\sum_{j=-\infty}^{+\infty} f_{n-1} \circ \varphi_j^{-1} \sigma_j \geqslant c_4^{-1} \sigma \sum_{j=-\infty}^{+\infty} f_{n-1} \circ \varphi_j^{-1},$$

and hence, by (4.5), and the fact $\varphi_{k_0}^{-1}(R) = I_{k_0} \subset I$ we have

$$f_n \geqslant c_4^{-1} \sigma f_{n-1} \circ \varphi_{k_0}^{-1} \geqslant c_9 \sigma \quad \text{for} \quad n > n_0,$$

where $c_9 = c_4^{-1} c_{14}$. This finishes the proof of the Step 3 and completes the proof of the Claim.

We have at last made all the preparations necessary for the definition of a set which satisfies all three conditions of the Proposition 3.3.

To prove parts: (2.8), (2.9), and (2.12) of the theorem's thesis we shall define the set

$$D_2 = \{f \in D_1 : \operatorname{Reg} f \leqslant a,\ b\sigma \leqslant f \leqslant c\sigma\},$$

where $a > c_7$, $c_9 > b > 0$, and $c > c_{10}$ are arbitrary (but fixed).

By Proposition 3.2 and Claim, we have: $f_n = P_\varphi^n f \in D_2$ for all $f \in D_1$ and, for a large enough n (which depends on f), i.e., the condition (3.13) of the Proposition 3.3 is fulfilled (D_1 is dense in D).

It is not hard to see that the set D_2 is convex, compact and imbedded in D. Thus the set D_2 fulfils all three conditions of the Proposition 3.3, whose thesis completes the proof of parts: (2.8), (2.9), and (2.12) of the theorem's thesis.

Now we shall consider the uniform convergence of the sequence $\{f_n\}$. From the inequalities $\limsup_{n\to\infty} f_n \leqslant c_{10}\sigma$ (see: Claim) and,

$$\limsup_{n\to\infty} |f_n(x_1) - f_n(x_2)| \leqslant c_8 |x_1 - x_2|$$

for each $x_1, x_2 \in R$ (see: Proposition 3.2) it follows that the family $F = \{f_n : n = 1, 2, \ldots\}$ is equibounded and equicontinuous. Thus the family F is relatively compact (by the Ascoli-Arzela theorem).

Since every convergent subsequence of F converges to a fixed point of P_φ and since P_φ has the unique fixed point f_0, the sequence $\{f_n\}$ converges to the density f_0. This proves parts: (2.10) and (2.11) of the theorem's thesis and completes the proof of the Theorem.

Appendix. As we already mentioned, at the end of the Section 2, the condition (2.3) involved in the definition of the class Φ is essential for the parts: (2.10) and (2.11) of the theorem's thesis. To illustrate this, we shall construct a transformation $\psi: R_0 \to R$ which satisfies all conditions of the definition of the class Φ, except the condition above mentioned. Instead of this condition, the following condition will be fulfilled:

(A. 1) the restriction ψ_k of ψ to the interval I_k is differentiable on I_k except for some countable set of points.

We shall show that the Frobenius-Perron operator corresponding to ψ has a (unique) fixed point which is some discontinuous function.

We now turn to the construction of the transformation ψ. To this end, let us take into account a doubly infinite matrix, i.e., a real-valued function on $Z \times Z$ (here $Z = \{0, \pm 1, \pm 2, \ldots\}$), denoted $T = (t_{ij})_{i,j=-\infty}^{+\infty}$, such that

(A. 2) $$\inf_{i \in Z} t_{ij} = t_j > 0, \quad \sum_{k \in Z} t_{jk} = 1 \quad \text{for each } j.$$

First, for each $i = 0, \pm 1, \pm 2, \ldots$, let us put:

$$\bar{x}_i = (i-1) + \sum_{j=0}^{-\infty} t_{ij};$$

and next:

$$x_{i1} = \bar{x}_i,\ x_{ik} = \bar{x}_i + \sum_{r=1}^{k-1} t_{ir} \quad \text{for} \quad k = 2, 3, \ldots;$$

$$x_{ik} = \bar{x}_i - \sum_{r=0}^{k} t_{ir} \quad \text{for} \quad k = 0, -1, -2, \ldots.$$

Now, for each pair i, k of integers, denote by ψ_{ik} a linear mapping (decreasing or increasing) from $J_{ik} = [x_{ik}, x_{ik+1})$ onto whole interval $J_k = [k-1, k)$.

Clearly, $J_i = \bigcup_{k=-\infty}^{+\infty} J_{ik}$. Also, since $J_{ik} \cap J_{ir} = \emptyset$ for $k \neq r$, for each $x \in J_i$ there exists a unique J_{ik} such that $x \in J_{ik}$. Hence, by setting $\psi_i(x) = \psi_{ik}(x)$ for $x \in J_i$, we define a (piecewise linear) bijection ψ_i: $J_i \to R$.

Finally, let us define a mapping ψ: $R_0 \to R$ as follows: $\psi(x) = \psi_k(x)$ for $x \in J_k$.

It can be shown, with the aid of (A.2), that ψ is an expanding transformation, i.e., $\inf_{x \in R} |\psi'(x)| > 1$.

Further, there is a close connection between the existence of a fixed point of the matrix T and the existence of a fixed point of the Frobenius-Perron operator P_ψ. Namely, we claim that the following conditions are equivalent:

(i) *there exists a (row) vector* $v = (v_i)_{i=-\infty}^{+\infty}$ *such that*

$$v_i \geq 0, \quad \sum_{i=-\infty}^{+\infty} v_i = 1 \quad \textit{and} \quad vT = v;$$

(ii) *there exists a function* $f \in D$ *such that* $f = \sum_{i=-\infty}^{+\infty} v_i \mathbf{1}_{J_i}$ *and* $P_\psi f = f$.

To prove this equivalence, let us observe that for an arbitrary $x \in J_i$ we have

(A.3) $$I_x = \psi^{-1}((-\infty, x)) = \bigcup_{r=-\infty}^{+\infty} (J_{ri}^x \cup \bigcup_{j=-\infty}^{i-1} J_{rj}),$$

where

$$J_{ri}^x = [x_{ri}, \psi_{ri}^{-1}(x)) = [x_{ri}, t_{ri}(x-i+1)+x_{ri})$$

if ψ_{ri} is increasing; or else

$$J_{ri}^x = [\psi_{ri}^{-1}(x), x_{ri}) \quad \text{if} \quad \psi_{ri} \text{ is decreasing.}$$

Now, let us take an arbitrary $f \in D$ such that $f = \sum_{i=-\infty}^{+\infty} v_i \mathbf{1}_{J_i}$. If $x \in J_i$, then

(A.4) $$f(x) = v_i.$$

On the other hand, from the definition of the Frobenius-Perron operator and the equality (A.3) we conclude that

$$\begin{aligned} P_\psi f(x) &= \frac{d}{dx}\int_{I_x} f(s)dm(s) = \frac{d}{dx}\Big(\sum_{k=-\infty}^{+\infty} v_k \int_{I_x} \mathbf{1}_{J_k}(s)dm(s)\Big) \\ &= \frac{d}{dx}\sum_{k=-\infty}^{+\infty} v_k m(I_x \cap J_k) = \frac{d}{dx}\sum_{k=-\infty}^{+\infty} v_k m(J_{ki}^x \cup \bigcup_{j=-\infty}^{i-1} J_{kj}) \\ &= \sum_{k=-\infty}^{+\infty} v_k dt_{ki}(x-i+1)/dx = (vT)_i. \end{aligned}$$

We have thus proved the equality

$$P_\psi f(x) = (vT)_i \quad \text{for each } x \in J_i.$$

This last equality together with (A.4) proves the desired equivalence.

To get the transformation which has the desired properties, take an arbitrary doubly infinite sequence $v = (v_i)_{i=-\infty}^{+\infty}$ such that $v_i > 0$, $\sum_{i=-\infty}^{+\infty} v_i = 1$. Setting $t_{ki} = v_i$ for $i, k = 0, \pm 1, \pm 2, \ldots$ we then get $vT = v$. Hence it follows that $P_\psi f = f$, where ψ is the transformation determined by the matrix T and $f = \sum_{i=-\infty}^{+\infty} v_i \mathbf{1}_{J_i}$, which was to be shown.

It can be shown that ψ is an exact endomorphism and that $f = \lim_{n\to\infty} P_\psi h$ for each $h \in D$; here we shall not present the proof of these facts.

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