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Systems of Difference Inequalities of the Elliptic Type

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§ 1. This paper is concerned with the systems I and II of difference inequalities of the elliptic type, cf. the formula (4.8) and (4.9).

The suitable assumptions being involved we shall be able to obtain estimates on the solution r_l^M $(l = 1, \ldots, p)$ of these difference inequalities that assure its convergence to zero in the limit as the mesh size h approaches zero, cf. Theorem 4 and Theorem 5.

Three estimates of that kind will be derived, cf. Theorem 6 (§ 24), Theorem 7 (§ 27) and Remark 7 (§ 27).

These results can be used to establish the convergence and the error estimate of the difference method for systems of elliptic differential equations. Denoting by u^M an approximate solution of a difference equation at the nodal point x^M, we can obtain from Theorem 6 the effective estimate for $u^M - u(x^M)$, $u(x)$ being the solution of the corresponding system of differential equations. The results will be published later.

For really computing the solution of a particular boundary problem the estimations (5.2) can easily be obtained. However, for the theoretical proof of convergence of the difference method for systems of elliptic equations the difficulty of proving the conditions (5.2) should not be overlooked also in this paper, cf. A. Pliś [2], and [4]. We shall assume here that conditions (5.2) are satisfied.

§ 2. Let us denote by Q the set of points $x \in R^n$, $x = (x_1, \ldots, x_n)$:

$$Q\colon\ 0 \leqslant x_j \leqslant \sigma \quad (j = 1, \ldots, n),\ 0 < \sigma = \text{const}\,. \tag{2.1}$$

Let us denote by M the sequence of indices

$$M = (m_1, m_2, \ldots, m_n),\ 0 \leqslant m_j \leqslant N \quad (j = 1, 2, \ldots, n)\,, \tag{2.2}$$

and by x^M the nodal point with coordinates

$$x^M = (x_1^M, x_2^M, \ldots, x_n^M)\,, \tag{2.3}$$

where $x_j^M = m_j \cdot h$ $(j = 1, \ldots, n)$, $0 < h = \sigma/N$, N being the natural number.

We shall introduce also the nodal points in the set Q characterized by the following sequences of indices:

$$(2.4)\qquad \begin{cases} j(M) = (m'_1, \ldots, m'_n),\ m'_j = m_j+1,\ m'_i = m_i \quad \text{for } i \neq j, \\ -j(M) = (m'_1, \ldots, m'_n),\ m'_j = m_j-1,\ m'_i = m_i \quad \text{for } i \neq j, \\ (i = 1, \ldots, n;\ j = 1, \ldots, n), \end{cases}$$

and for $i \neq j$:

$$(2.5)\qquad \begin{cases} ij(M) = (m'_1, \ldots, m'_n),\ m'_i = m_i+1,\ m'_j = m_j+1, \\ -ij(M) = (m'_1, \ldots, m'_n),\ m'_i = m_i-1,\ m'_j = m_j+1, \\ -i-j(M) = (m'_1, \ldots, m'_n),\ m'_i = m_i-1,\ m'_j = m_j-1, \\ i-j(M) = (m'_1, \ldots, m'_n),\ m'_i = m_i+1,\ m'_j = m_j-1, \end{cases}$$

where $m'_s = m_s$ in the formula (2.5) for $s = 1, \ldots, n;\ s \neq i,\ s \neq j$, cf. Fig. 1.

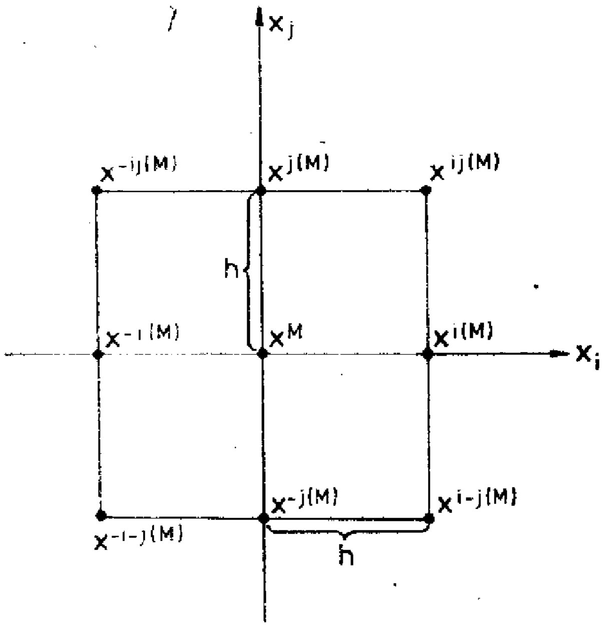

Fig. 1. The nodal points x^M, $x^{i(M)}$, $x^{ij(M)}$, ... For the sake of simplicity the nodal point x^M has been located at the origin

The nodal point $x^{ij(M)}$ can be denoted also by $x^{ji(M)}$, since we define

$$(2.6)\qquad \begin{cases} ij(M) = ji(M),\ -ij(M) = j-i(M),\ -i-j(M) = -j-i(M), \\ i-j(M) = -ji(M), \quad \text{for } i \neq j\ (i,j = 1, \ldots, n). \end{cases}$$

We denote by $\operatorname{int} Q$ the set of nodal points (2.3) which belong to the interior of the set Q, cf. (2.1), and by sym A the set of nodal points x^M such that $x^M \in \operatorname{int} Q$ and $x^{M^*} \in \operatorname{int} Q$ simultaneously, x^{M^*} and x^M being symmetric with respect to the nodal point x^A.

§ **3.** Let us denote by r_l^M $(l = 1, 2, \ldots, p)$ the value of the function r_l $(l = 1, \ldots, p)$ at the nodal point x^M.

We shall use the difference quotients

$$\left\{\begin{array}{l} r_{l+}^{Mj} = \frac{1}{h}\cdot(r_l^{j(M)} - r_l^M),\ r_{l-}^{Mj} = \frac{1}{h}\cdot(r_l^M - r_l^{-j(M)}), \\ (l = 1, \ldots, p;\ j = 1, \ldots, n), \end{array}\right. \tag{3.1}$$

$$r_l^{Mj} = \frac{1}{2h}\cdot(r_l^{j(M)} - r_l^{-j(M)}),\ (l = 1, \ldots, p;\ j = 1, \ldots, n), \tag{3.2}$$

for the first partial derivatives, and the difference quotients

$$\left\{\begin{array}{l} r_l^{Mjj} = h^{-2}\cdot(r_l^{j(M)} - 2\cdot r_l^M + r_l^{-j(M)}), \\ r_l^{Mij} = \frac{1}{4}\cdot h^{-2}(r_l^{ij(M)} - r_l^{-ij(M)} - r_l^{i-j(M)} + r_l^{-i-j(M)}), \\ (i \neq j;\ i = 1, \ldots, n;\ j = 1, \ldots, n;\ l = 1, \ldots, p), \end{array}\right. \tag{3.3}$$

for the second derivatives.

From the definitions (3.1) (3.2) (3.3) it follows that

$$\left\{\begin{array}{l} r_l^{Mj} = \frac{1}{2}\cdot(r_{l+}^{Mj} + r_{l-}^{Mj}),\ r_l^{Mjj} = \frac{1}{h}\cdot(r_{l+}^{Mj} - r_{l-}^{Mj}), \\ (j = 1, \ldots, n;\ l = 1, \ldots, p). \end{array}\right. \tag{3.4}$$

We shall use also the difference quotients r_{l++}^{Mij}, r_{l-+}^{Mij}, r_{l--}^{Mij}, r_{l+-}^{Mij}

$$(l = 1, \ldots, p;\ i \neq j;\ i = 1, \ldots, n;\ j = 1, \ldots 1, \ldots, n),$$

cf. Fig. 1:

$$\left\{\begin{array}{l} r_{l++}^{Mij} = h^{-2}\cdot(r_l^{ij(M)} - r_l^{j(M)} - r_l^{i(M)} + r_l^M), \\ r_{l-+}^{Mij} = h^{-2}\cdot(r_l^{j(M)} - r_l^{-ij(M)} - r_l^M + r_l^{-i(M)}), \\ r_{l--}^{Mij} = h^{-2}\cdot(r_l^M - r_l^{-i(M)} - r_l^{-j(M)} + r_l^{-i-j(M)}), \\ r_{l+-}^{Mij} = h^{-2}\cdot(r_l^{i(M)} - r_l^M - r_l^{i-j(M)} + r_l^{-j(M)}). \end{array}\right. \tag{3.5}$$

From the definitions (3.5) and (3.3) we obtain

$$\left\{\begin{array}{l} r_l^{Mij} = \frac{1}{4}\cdot(r_{l++}^{Mij} + r_{l-+}^{Mij} + r_{l--}^{Mij} + r_{l+-}^{Mij}), \\ (l = 1, \ldots, p;\ i = 1, \ldots, n;\ j = 1, \ldots, n;\ i \neq j). \end{array}\right. \tag{3.6}$$

§ **4.** Let us consider the following conditions W_j $(j = 1, 2, 3)$:

Condition W$_1$. The quadratic forms

$$\sum_{i,j=1}^{n} a_{lij}^M\cdot\lambda_i\cdot\lambda_j \quad (l = 1, \ldots, p)(x^M \in \operatorname{int} Q), \tag{4.1}$$

are positive definite and the characteristic roots s_{lj}^M $(l = 1, \ldots, p;\ j = 1, \ldots, n)$, $s_{lj}^M > 0$, are bounded:

$$0 < \delta_1 \leqslant s_{lj}^M \leqslant \delta_2 \quad (l = 1, \ldots, p;\ j = 1, \ldots, n), \tag{4.2}$$

the constants δ_1 and δ_2 being independent of the mesh size h.

Condition W_2. The elements of the matrix (c_{lk}^M) $(l = 1, \dots, p;\ k = 1, \dots, p)$ $(x^M \in \operatorname{int} Q)$ satisfy the inequalities

$$c_{ll}^M \leqslant \eta < 0 \quad (l = 1, \dots, p;\ \eta = \text{const}), \tag{4.3}$$

$$0 \leqslant c_{lk}^M < \delta \quad (\delta = \text{const},\ l \neq k;\ l = 1, \dots, p;\ k = 1, \dots, p), \tag{4.4}$$

where the constants η and δ does not depend on the mesh size h and

$$-\frac{1}{p-1} < \gamma < 0 \quad (p \geqslant 2), \tag{4.5}$$

the coefficient γ being defined by

$$\gamma = +\eta^{-1} \cdot \delta \quad (\gamma < 0). \tag{4.6}$$

Condition W_3. The coefficients a_{lij}^M, b_{lj}^M are bounded:

$$|a_{lij}^M| \leqslant \zeta, \quad |b_{lj}^M| \leqslant \beta, \tag{4.7}$$

for $l = 1, \dots, p;\ i = 1, \dots, n;\ j = 1, \dots, n;\ x^M \in \operatorname{int} Q$, the constants ζ and β being independent of the mesh size h.

Let us consider two systems I and II of difference inequalities for the functions r_l^M $(l = 1, \dots, p)$, defined at the nodal points x^M:

System I:

$$\sum_{i,j=1}^{n} a_{lij}^M \cdot r_l^{Mij} + \sum_{j=1}^{n} b_{lj}^M \cdot r_l^{Mj} + \sum_{k=1}^{p} c_{lk}^M \cdot r_k^M \geqslant -\varepsilon(h), \tag{4.8}$$

where $0 < \varepsilon(h) = \text{const}$ $(l = 1, \dots, p)$.

System II:

$$\sum_{i,j=1}^{n} a_{lij}^M \cdot r_l^{Mij} + \sum_{j=1}^{n} b_{lj}^M \cdot r_l^{Mj} + \sum_{k=1}^{p} c_{lk}^M \cdot r_k^M \leqslant +\varepsilon(h), \tag{4.9}$$

where $0 < \varepsilon(h) = \text{const}$ $(l = 1, \dots, p)$.

We shall use the following definition:

DEFINITION 1. We say that the systems I and II are of the *elliptic type* if the conditions W_1, W_2 and W_3 are fulfilled.

§ 5. Let us summarize now the principal assumptions:

ASSUMPTIONS H. We shall assume that 1° the functions r_l^M $(l = 1, \dots, p)$ are defined at the nodal points (2.3) of the set Q, $x^M \in Q$, cf. (2.1).

2° There exists a positive constant $\vartheta>0$ (independent of the mesh size h) such that the first order difference quotients satisfy the conditions

$$(5.1)\qquad \begin{cases} |r_{l+}^{Mj}|\leqslant h\cdot\vartheta\,, & \text{for } x^M\in\partial Q,\ x^{j(M)}\in \operatorname{int} Q\ (j=1,\dots,n)\,,\\ |r_{l-}^{Mij}|\leqslant h\cdot\vartheta\,, & \text{for } x^M\in\partial Q,\ x^{-j(M)}\in \operatorname{int} Q\ (j=1,\dots,n)\,,\\ (l=1,\dots,p)', \end{cases}$$

at the nodal points x^M on the boundary ∂Q of the set Q.

3° There exists a positive constant $L>0$ (independent of the mesh size h) such that the second order difference quotients satisfy the conditions

$$(5.2)\qquad \begin{cases} |r_l^{Mjj}-r_l^{Pjj}|\leqslant h\cdot L,\ |r_{l++}^{Mij}-r_{l++}^{Pij}|\leqslant h\cdot L\,,\\ |r_{l-+}^{Mij}-r_{l-+}^{Pij}|\leqslant h\cdot L,\ |r_{l--}^{Mij}-r_{l--}^{Pij}|\leqslant h\cdot L\,,\\ |r_{l+-}^{Mij}-r_{l+-}^{Pij}|\leqslant h\cdot L \quad (i\neq j;\ l=1,\dots,p) \end{cases}$$

at the nodal points x^M and x^P, $P=s(M)$ ($s=\pm1,\pm2,\dots,\pm n$), the distance between x^M and x^P being h in the direction of the x_s-axis.

4° We suppose that

$$(5.3)\qquad |r_l^{Mij}|\leqslant \Lambda\ (i=1,\dots,n;\ j=1,\dots,n;\ l=1,\dots,p)\ (x^M\in \operatorname{int} Q)\,,$$

where the constant Λ is independent of the mesh size h.

5° We suppose also that r_l^M takes on the prescribed values:

$$(5.4)\qquad r_l^M=0\,,\quad \text{for } x^M\in\partial Q\ (l=1,\dots,p)\,,$$

at the nodal points x^M on the boundary ∂Q of the set Q.

6° We suppose finally that the systems I and II of difference inequalities are of the elliptic type, cf. Definition 1 (§ 4).

§ **6.** We shall formulate now Lemma 1 on the location of the functions r_l^M ($l=1,\dots,p$) for $x^M\in Q$.

LEMMA 1. *Let us suppose that the functions* r_l^M ($l=1,\dots,p$) *satisfy part* 1°, 2°, 4° *and* 5° *of the assumptions* H, *cf.* § 5, *and let us consider the function*

$$(6.1)\qquad z=(\tfrac{1}{2}\cdot\Lambda+\vartheta)\cdot y^2,\ 0\leqslant y<+\infty\,.$$

Under these assumptions we have

$$(6.2)\qquad |r_l^M|\leqslant(\tfrac{1}{2}\cdot\Lambda+\vartheta)\cdot y^2\ (l=1,\dots,p)\,,$$

where y *denotes the Euclidean distance* $\varrho(x^M,\partial Q)$ *of the point* $x^M\in Q$ *from the boundary* ∂Q, $y=\varrho(x^M,\partial Q)$. *In the formula* (6.2) *we have the strong inequality for* $\varrho(x^M,\partial Q)>0$, *cf.* Fig. 2.

The proof of Lemma 1 can be found in the previous paper, cf. [3], § 12 (it is sufficient to repeat the proof of Lemma 1 of the paper [3] for each function r_l^M ($l=1,\dots,p$), successively).

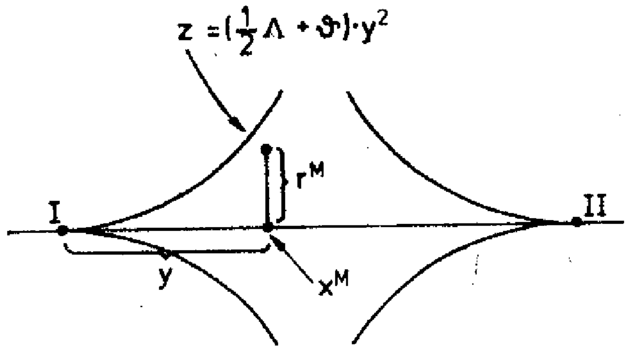

Fig. 2. The location of the function r_l^M ($l = 1, \dots, p$). The set $Q = (\mathrm{I}, \mathrm{II})$ as seen from the edge in the case $n = 2$

§ 7. Notations. Let us denote

$$r_l^{A_l} = \max_{x^M \in Q} r_l^M,\ A_l = A_l(h),\ A_l = (a_{l1}, \dots, a_{ln}),\ x^{A_l} \in \operatorname{int} Q, \tag{7.1}$$

$$r_l^{B_l} = \max_{x^M \in Q} r_l^M,\ B_l = B_l(h),\ B_l = (b_{l1}, \dots, b_{ln}),\ x^{B_l} \in \operatorname{int} Q, \tag{7.2}$$

for $l = 1, \dots, p$. Let us denote in addition, cf. [3] (§ 13):

$$\varkappa = (\tfrac{1}{2}\cdot\Lambda + \vartheta)^{-1/2}. \tag{7.3}$$

$$|m - a_l| = \sum_{j=1}^{n} |m_j - a_{lj}| \quad (l = 1, \dots, p). \tag{7.4}$$

We shall consider the sets V^{A_l} and $\operatorname{supp} V^{A_l}$ (support of the set V^{A_l}) for $l = 1, \dots, p$, connected with the nodal point x^{A_l} ($l = 1, \dots, p$):

$$V^{A_l} = \{x\colon |x_j - x_j^M| \leqslant h,\ (j = 1, \dots, n),\ h\cdot|m - a_l| \leqslant \varkappa h^{\alpha}\}, \tag{7.5}$$

$$\operatorname{supp} V^{A_l} = \{x^M\colon h\cdot|m - a_l| \leqslant \varkappa h^{\alpha}\}, \tag{7.6}$$

where $\frac{1}{2} \leqslant \alpha < 1$ ($l = 1, \dots, p$).

In a similar way for x^{B_l} ($l = 1, \dots, p$) we define

$$V^{B_l} = \{x\colon |x_j - x_j^M| \leqslant h\ (j = 1, \dots, n),\ h\cdot|m - b_l| \leqslant \varkappa h^{\alpha}\}, \tag{7.7}$$

$$\operatorname{supp} V^{B_l} = \{x^M\colon h\cdot|m - b_l| \leqslant \varkappa h^{\alpha}\}. \tag{7.8}$$

By $\varrho(x^{A_l}, \partial Q)$ we shall denote the euclidean distance of the nodal point x^{A_l} from the boundary ∂Q of the set Q, and by $\varrho(x^{A_l}, x^M)$ the Euclidean distance between the nodal points x^{A_l} and x^M.

§ 8. Lemma 2. *Let us suppose that the functions* r_l^M ($l = 1, \dots, p$ *satisfy parts* 1°, 2°, 4° *and* 5° *of the assumptions* H, cf. § 5.

We suppose also that the inequalities

$$r_l^{A_l} \geqslant h,\ r_l^{B_l} \leqslant -h, \quad \textit{for } h > 0,\ A_l = A_l(h),\ B_l = B_l(h), \tag{8.1}$$

hold for some l ($l = 1, \dots, p$), cf. *Notations* § 7.

Under these assumptions we have

$$\sup V^{A_l} \subset Q,\ \operatorname{supp} V^{B_l} \subset Q\,, \tag{8.2}$$

$$|x^{b_l} - x^{A_l}| \geqslant \varkappa h^{\alpha},\ |x^{b'_l} - x^{B_l}| \geqslant \varkappa h^{\alpha}\,, \tag{8.3}$$

where $x^{b_l} \in \partial V^{A_l}$, $x^{b'_l} \in \partial V^{B_l}$, $\frac{1}{2} \leqslant \alpha < 1$, $0 < h < 1$, *cf. Fig.* 3 *and Fig.* 4.

The proof of Lemma 2 can be found in [3], (cf. [3], § 14).

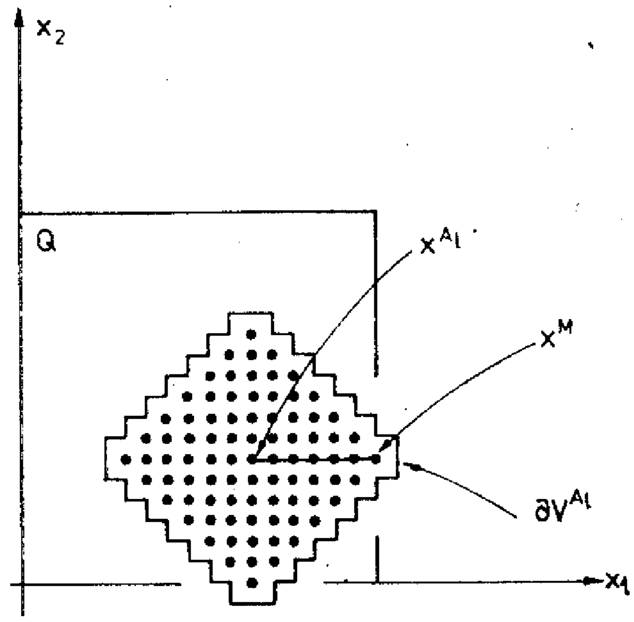

Fig. 3. The sets V^{A_l} and supp V^{A_l} $(l = 1, ..., p)$ in the two dimensional case $n = 2$. In this figure we have supp $V^{A_l} \subset Q$

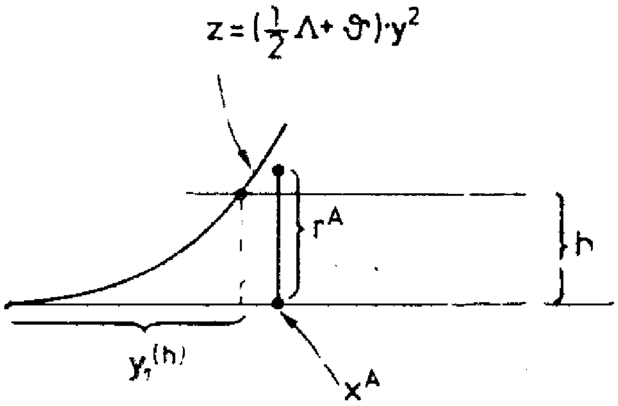

Fig. 4. $y_l(h)$ and the center x^{A_l} of the set supp V^{A_l} $(l = 1, ..., p)$

§ 9. **Remark 1.** The difference inequality (4.8) can be written in the form

$$\sum_{i,j=1}^{n} a_{lij}^{M} \cdot r_l^{Mij} + \sum_{j=1}^{n} b_{lj}^{M} \cdot r_l^{Mj} + c_{ll}^{M} \cdot r_l^{M} \geqslant -\varepsilon(h) - \sum_{k=1, k \neq l}^{p} c_{lk}^{M} \cdot r_k^{M}\,, \tag{9.1}$$

the left hand member being dependent on the function r_l^M and of the difference quotients r_l^{Mij}, r_l^{Mj}, only. The right hand side contains the remaining functions r_k^M for $k \neq l$.

Hence, (9.1) can be regarded as the difference inequality for one function r_l^M, only, the right hand side being considered for a moment as a free member.

Then, in a similar way as in [1] (cf. [1], Theorem 1, § 11) the estimate for the maximum value $r_l^{A_l}$ can be obtained. That estimate will be given in § 12.

§ **10.** To achieve the estimate for $r_l^{A_l}$, as explained in § 9, the results and notations of Lemma 3, Remark 2 and Remark 3 will be needed, all of them being placed in the next paragraph, cf. § 11. But the introduced notions are so similar to that used in [1] that we have decided to accept all the notations of the paper [1] (with minor changes) in order to facilitate the comparison between the theory for one difference inequality of the elliptic type and the corresponding theory for a system of difference inequalities.

§ **11.** LEMMA 3. *Let us suppose that the functions* r_l^M $(l = 1, \ldots, p)$ *satisfy Assumptions* H (cf. § 5 *and Notations* § 7).

Let us denote by $S_{A_l}(x)$, $S_{B_l}(x)$, $x = (x_1, \ldots, x_n) \in R^n$ *the quadratic forms*

$$(11.1)\qquad \begin{cases} S_{A_l}(x) = \frac{1}{2}\cdot \sum\limits_{i,j=1}^{n} r_l^{A_l ij}\cdot(x_i - x_l^{A_l})(x_j - x_j^{A_l}), & (x \in R^n), \\ S_{B_l}(x) = \frac{1}{2}\cdot \sum\limits_{i,j=1}^{n} r_l^{B_l ij}\cdot(x_i - x_i^{B_l})(x_j - x_j^{B_l}), & (x \in R^n) \end{cases}$$

and by $S_{A_l}^M$, $S_{B_l}^M$ *the values of* $S_{A_l}(x)$, $S_{B_l}(x)$ *at the nodal point* $x^M \in Q$, *respectively*:

$$(11.2)\qquad \begin{cases} S_{A_l}^M = \frac{1}{2} \sum\limits_{i,j=1}^{n} r_l^{A_l ij}\cdot(m_i - a_{li})(m_j - a_{lj})\cdot h^2, & (x^M \in Q), \\ S_{B_l}^M = \frac{1}{2}\cdot \sum\limits_{i,j=1}^{n} r_l^{B_l ij}\cdot(m_i - b_{li})(m_j - b_{lj})\cdot h^2, & (x^M \in Q). \end{cases}$$

Under these assumptions we have the following estimate for the quadratic forms $S_{A_l}(x)$, $S_{B_l}(x)$ *in the sets* V^{A_l} *and* V^{B_l} $(l = 1, \ldots, p)$, *respectively*:

$$(11.3)\qquad \begin{cases} S_{A_l}(x) \leqslant C_1(h), & \textit{for } x \in V^{A_l}, \\ S_{B_l}(x) \geqslant -C_1(h), & \textit{for } x \in V^{B_l}, \end{cases}$$

where $l = 1, \ldots, p$, *and*

$$(11.4)\qquad C_1(h) = 2\cdot|\theta|\cdot L\varkappa^3 h^{3\alpha} + n\cdot 2\Lambda\cdot\varkappa h^{1+\alpha} + n^2\cdot\Lambda\cdot h^2,\ |\theta| < 1.$$

The proof of Lemma 3 will be omitted, since it is similar to the proof of Lemma 5 in the paper [1].

Remark 2. The quadratic forms $S_{A_l}(x)$ and $S_{B_l}(x)$ can be estimated for $x \in R^n$. For this purpose let us denote

$$(11.5)\qquad G_1(x - x^{A_l}, h) = \begin{cases} C_1(h), & \text{for } x \in V^{A_l}, \\ \lambda^2\cdot C_1(h), & \text{for } x \in R^n \setminus V^{A_l}, \end{cases}$$

where $x - x^{A_l} = \lambda\cdot(x^{b_l} - x^{A_l})$ $(\lambda \geqslant 1)$, x^{b_l} being the intersection point of the boundary ∂V^{A_l} with a segment joining the points x^{A_l} and x $(l = 1, \ldots, p)$.

From the fact that $S_{A_l}(x)$ is the quadratic form and from the estimate (11.3) it follows that

$$(11.6)\qquad S_{A_l}(x) \leqslant G_1(x - x^{A_l}, h), \quad \text{for } x \in R^n \ (l = 1, \ldots, p).$$

Let us denote also

$$(11.7)\qquad G_1(x-x^{B_l}, h) = \begin{cases} C_1(h), & \text{for } x \in V^{B_l}, \\ \lambda'^2 \cdot C_1(h), & \text{for } x \in R^n \setminus V^{B_l}, \end{cases}$$

where $x-x^{B_l} = \lambda' \cdot (x^{b_l'} - x^{B_l})$ $(\lambda' \geqslant 1)$, $x^{b_l'}$ being the intersection point of the boundary ∂V^{B_l} with the segment joining the points x^{B_l} and x $(l = 1, \dots, p)$.

Then we have

$$(11.8)\qquad S_{B_l}(x) \geqslant -G_1(x-x^{B_l}, h), \quad \text{for } x \in R^n.$$

Remark 3. Let us consider the expressions

$$(11.9)\qquad \sum_{i,j=1}^{n} a_{lij}^{A_l} \cdot r_l^{A_l ij}, \quad \sum_{i,j=1}^{n} a_{lij}^{B_l} \cdot r_l^{B_l ij} \quad (l = 1, \dots, p),$$

the coefficients $a_{lij}^{A_l}$, $a_{lij}^{B_l}$ being taken from the quadratic form $\sum_{i,j=1}^{n} a_{lij}^{M} \cdot \lambda_i \lambda_j$, cf. § 4, the formula (4.1).

We shall verify that the expressions (11.9) have the following estimates:

$$(11.10)\qquad \sum_{i,j=1}^{n} a_{lij}^{A_l} \cdot r_l^{A_l ij} \leqslant E^{A_l}(h), \quad \sum_{i,j=1}^{n} a_{lij}^{B_l} \cdot r_l^{B_l ij} \geqslant -E^{B_l}(h),$$

where

$$(11.11)\qquad \begin{cases} E^{A_l}(h) = \sum_{k=1}^{n} G_1\big((s_{lk}^{A_l})^{1/2} \cdot \alpha_{lk}^{A_l};\ h\big), \\ E^{B_l}(h) = \sum_{k=1}^{n} G_1\big((s_{lk}^{B_l})^{1/2} \cdot \alpha_{lk}^{B_l};\ h\big), \end{cases}$$

and the functions $E^{A_l}(h)$ and $E^{B_l}(h)$ $(l = 1, \dots, p)$ are positives $E^{A_l}(h) > 0$, $E^{B_l}(h) > 0$, for $h > 0$.

In the formula (11.11) $s_{l1}^{A_l}, s_{l2}^{A_l}, \dots, s_{ln}^{A_l}$ denote positive characteristic roots of the form $f_1 = \sum_{i,j=1}^{n} a_{lij}^{A_l} \cdot \lambda_i \lambda_j$, cf. Condition W_1 (§ 4), $(\alpha_{lkj}^{A_l})$ $(k, j = 1, \dots, n)$ is the orthogonal matrix transforming f_1 to the canonical form, and $\alpha_{lk}^{A_l}$ denotes the unit vector

$$\alpha_{lk}^{A_l} = (\alpha_{lk1}^{A_l}, \alpha_{lk2}^{A_l}, \dots, \alpha_{lkn}^{A_l}) \ (k = 1, \dots, n).$$

We introduce similar notations for the form $\sum_{i,j=1}^{n} a_{lij}^{B_l} \cdot \lambda_i \lambda_j$.

In order to deduce the first formula (11.10) it is sufficient to observe that f_1 is the positive definite form, cf. Condition W_1 (§ 4), and the form $S_{A_l}(x)$:

$$(11.12)\qquad S_{A_l}(x) = \sum_{i,j=1}^{n} r_l^{A_l ij} \cdot (x_i - x_i^{A_l})(x_j - x_j^{A_l}) =$$

$$= \sum_{i,j=1}^{n} r_l^{A_l ij} \cdot \mu_i \mu_j, \quad \text{for } \mu_i = x_i - x_i^{A_l} \ (i = 1, 2, \dots, n),$$

possesses the estimate (11.6), hence from the known lemma on quadratic forms, cf. Lemma 3 in the paper [1], we obtain the first part of (11.10).

The proof of the second part of (11.10) is similar.

§ 12. Now we shall give the estimate for the maximum value $r_l^{A_l}$ and the minimum value $r_l^{B_l}$.

Lemma 4. *Let us suppose that the functions* r_l^M $(l = 1, \ldots, p)$ *satisfy the Assumptions* H. *We shall use the notations of* § 7.

Let us suppose in addition that

$$\sum_{i,j=1}^{n} a_{lij}^{M}\cdot r_l^{Mij} + \sum_{j=1}^{n} b_{lj}^{M}\cdot r_l^{Mj} + \sum_{k=1}^{p} c_{lk}^{M}\cdot r_k^{M} \geqslant -\varepsilon(h)\,, \tag{12.1}$$

for $\varepsilon(h)>0$, *if* $r_l^M>0$ $(l = 1, \ldots, p)$ *at the nodal point* x^M, $x^M \in \operatorname{int} Q$, *and*

$$\sum_{i,j=1}^{n} a_{lij}^{M}\cdot r_l^{Mij} + \sum_{j=1}^{n} b_{lj}^{M}\cdot r_l^{Mj} + \sum_{k=1}^{p} c_{lk}^{M}\cdot r_k^{M} \leqslant +\varepsilon(h)\,, \tag{12.2}$$

if $r_l^M<0$ $(l = 1, \ldots, p)$ *at the nodal point* x^M, $x^M \in \operatorname{int} Q$.

Under these assumptions we have

$$\begin{cases} r_l^{A_l} \leqslant g^{A_l}(h) - \eta^{-1}\cdot \displaystyle\sum_{k=1,\,k\neq l}^{p} c_{lk}^{A_l}\cdot r_k^{A_l}\,, \\ r_l^{B_l} \geqslant -g^{B_l}(h) - \eta^{-1}\cdot \displaystyle\sum_{k=1,\,k\neq l}^{p} c_{lk}^{B_l}\cdot r_k^{B_l}\,, \end{cases} \tag{12.3}$$

where

$$\begin{cases} 0<g^{A_l}(h) = -\eta^{-1}\cdot[E^{A_l}(h)+D(h)+\varepsilon(h)]\,, \\ 0<g^{B_l}(h) = -\eta^{-1}\cdot[E^{B_l}(h)+D(h)+\varepsilon(h)]\,. \end{cases} \tag{12.4}$$

In the formula (12.4) *the quantities* $E^{A_l}(h)$, $E^{B_l}(h)$ *are defined by* (11.11), *and* $0<D(h) = n\beta h\Lambda$.

Proof. Assuming the contrary, suppose that

$$r_l^{A_l} > g^{A_l}(h) - \eta^{-1}\cdot \sum_{k=1,\,k\neq l}^{p} c_{lk}^{A_l}\cdot r_k^{A_l}\,. \tag{12.5}$$

With (12.5) in mind we shall verify that

$$\sum_{i,j=1}^{n} a_{lij}^{A_l}\cdot r_l^{A_lij} \leqslant E^{A_l}(h)\,, \tag{12.6}$$

$$\sum_{j=1}^{n} b_{lj}^{A_l}\cdot r_l^{A_lj} \leqslant D(h)\,, \tag{12.7}$$

$$c_{ll}^{A_l}\cdot r_l^{A_l} \leqslant \eta\cdot r_l^{A_l}\,. \tag{12.8}$$

In fact, (12,6) follows from Remark 3, cf. (11.10).

The difference quotients at the nodal point x^{A_l} satisfy the inequalities $|r_l^{A_l j}| \leqslant h$, $|r_l^{A_l jj}| \leqslant h \cdot \Lambda$, since x^{A_l} is the nodal point where the maximum value is attained, cf. Lemma 2 in the paper [1] (the formula (5.3)). Therefore, from (4.7) it follows that

$$\sum_{j=1}^{n} b_{lj}^{A_l} \cdot r_l^{A_l j} \leqslant n \cdot \beta \cdot h \cdot \Lambda = D(h), \tag{12.9}$$

which completes the proof of (12.7)

Inequality (12.8) follows immediately from the assumption (4.3): $c_{ll}^{A_l} \leqslant \eta < 0$, since $r_l^{A_l} \geqslant 0$ because of (5.4).

From (12.6) (12.7) (12.8) we obtain first by summation

$$\begin{aligned} \sum_{i,j=1}^{n} a_{lij}^{A_l} \cdot r_l^{A_l ij} + \sum_{j=1}^{n} b_{lj}^{A_l} \cdot r_l^{A_l j} + c_{ll}^{A_l} \cdot r_l^{A_l} + \sum_{k=1,k\neq l}^{p} c_{lk}^{A_l} \cdot r_k^{A_l} & \\ \leqslant E^{A_l}(h) + D(h) + \eta \cdot r_l^{A_l} + \sum_{k=1,k\neq l}^{p} c_{lk}^{A_l} \cdot r_k^{A_l} & . \end{aligned} \tag{12.10}$$

But the assumption (12.5) and the definition of $g^{A_l}(h)$, cf. (12.4), yield

$$E^{A_l}(h) + D(h) + \eta r_l^{A_l} + \sum_{k=1,k\neq l}^{p} c_{lk}^{A_l} \cdot r_k^{A_l} < -\varepsilon(h), \tag{12.11}$$

hence, (12.10) and (12.11) imply that

$$\sum_{i,j=1}^{n} a_{lij}^{A_l} \cdot r_l^{A_l ij} + \sum_{j=1}^{n} b_{lj}^{A_l} \cdot r_l^{A_l j} + c_{ll}^{A_l} \cdot r_l^{A_l} + \sum_{k=1,k\neq 1}^{p} c_{lk}^{A_l} \cdot r_k^{A_l} < -\varepsilon(h). \tag{12.12}$$

Since inequalities (12.12) and (12.1) are contradictory, we conclude that the maximum value $r_l^{A_l}$ satisfies the first formula of (12.3).

The proof of the second part of (12.3) can be obtained in a similar way.

This ends the proof of Lemma 4.

§ **13.** THEOREM 1. *Let us suppose that the functions* r_l^M $(l = 1, \ldots, p)$ *satisfy the assumptions of Lemma* 4; *cf.* § 12.

Under these assumptions 1° *the maximum values* $r_l^{A_l}$ $(l = 1, \ldots, p)$ *satisfy the linear system of algebraic inequalities*

$$r_l^{A_l} + \gamma \cdot \sum_{k=1,k\neq l}^{p} r_k^{A_k} \leqslant +g^{A_l}(h) \qquad (l = 1, \ldots, p); \tag{13.1}$$

2° *The minimum values* $r_l^{B_l}$ $(l = 1, \ldots, p)$ *satisfy the linear system of algebraic inequalities*

$$r_l^{B_l} + \gamma \cdot \sum_{k=1,k\neq l}^{p} r_k^{B_k} \geqslant -g^{B_l}(h) \qquad (l = 1, \ldots, p). \tag{13.2}$$

In the formula (13.1) (13.2) γ *denotes a negative number*:

$$\gamma = +\eta^{-1} \cdot \delta \;\; (\gamma < 0), \tag{13.3}$$

cf. Condition W$_2$ (§ 4); *and the functions* $g^{A_l}(h)$, $g^{B_l}(h)$ *are defined by* (12.4).

Proof. 1) Let us evaluate first the sum on the right hand side of the first formula (12.3).

$r_k^{A_k}$ denotes the maximum value for the function r_k^M $(x^M \in Q)$, hence

(13.4) $$r_k^{A_l} \leqslant r_k^{A_k}.$$

The difference inequalities we consider are of the elliptic type, therefore $c_{lk}^{A_l} \geqslant 0$ $(l \neq k)$, cf. (4.4), and

(13.5) $$c_{lk}^{A_l} \cdot r_k^{A_l} \leqslant c_{lk}^{A_l} \cdot r_k^{A_k}.$$

In addition we have $0 \leqslant c_{lk}^{A_l} < \delta$ $(k \neq l)$, cf. (4.4), and $r_k^{A_k} \geqslant 0$, since the function r_k^M takes on zero value on the boundary ∂Q, and $r_k^{A_k}$ denotes the maximum value of r_k^M, cf. (5.4) and (7.1).

Multiplying (4.4) by $r_k^{A_k}$ we obtain therefore

(13.6) $$0 \leqslant c_{lk}^{A_l} \cdot r_k^{A_k} \leqslant \delta \cdot r_k^{A_k}.$$

From (13.5) and (13.6) follows the inequality

(13.7) $$c_{lk}^{A_l} \cdot r_k^{A_l} \leqslant c_{lk}^{A_l} \cdot r_k^{A_k} \leqslant \delta \cdot r_k^{A_k}$$

But $-\eta^{-1} > 0$, hence from (13.7) we obtain by summation

(13.8) $$-\eta^{-1} \cdot \sum_{k=1, k \neq l}^{p} c_{lk}^{A_l} \cdot r_k^{A_l} \leqslant -\eta^{-1} \cdot \sum_{k=1, k \neq l}^{p} r_k^{A_k}.$$

So, the formula (12.3) implies that

(13.9) $$r_l^{A_l} \leqslant +g^{A_l}(h) - \eta^{-1} \cdot \delta \cdot \sum_{k=1, k \neq l}^{p} r_k^{A_k},$$

because of (13.8), hence, with the aid of the definition (13.3) we obtain the desired formula (13.1).

2) Now we shall evaluate the sum on the right hand side of the second formula (12.3).

The value $r_k^{B_k}$ is the minimum value of the function r_k^M $(x^M \in Q)$, hence

(13.10) $$r_k^{B_l} \geqslant r_k^{B_k}.$$

The difference inequalities just considered are of the elliptic type, therefore $c_{kl}^{B_l} \geqslant 0$ $(l \neq k)$, cf. (4.4), and

(13.11) $$c_{lk}^{B_l} \cdot r_k^{B_l} \geqslant c_{lk}^{B_l} \cdot r_k^{B_k}.$$

In addition, we have $0 \leqslant c_{lk}^{B_l} < \delta$ $(k \neq l)$, cf. (4.4), and $r_k^{B_k} \leqslant 0$, since the function r_k^M takes on zero values on the boundary ∂Q, and $r_k^{B_k}$ denotes the minimum value of r_k^M, cf. (5.4) and (7.2).

Multiplying both sides of (4.4) by $r_k^{B_k}$ we obtain therefore

(13.12) $$0 \geqslant c_{lk}^{B_l} \cdot r_k^{B_k} \geqslant \delta \cdot r_k^{B_k}.$$

We see that the inequality (13.11) can be rewritten in the form

(13.13) $$c_{lk}^{B_l} \cdot r_k^{B_l} \geqslant c_{lk}^{B_l} \cdot r_k^{B_k} \geqslant \delta \cdot r_k^{B_k}.$$

Multiplying (13.13) by $-\eta^{-1}>0$ we obtain by summation

$$-\eta^{-1}\cdot\sum_{k=1,k\neq l}^{p} c_{lk}^{B_l}\cdot r_k^{B_l}\geqslant -\eta^{-1}\cdot\delta\cdot\sum_{k=1,k\neq l}^{p} r_k^{B_k}. \tag{13.14}$$

So, the second formula (12.3) becomes

$$r_l^{B_l}\geqslant -g^{B_l}(h)-\eta^{-1}\cdot\delta\cdot\sum_{k=1,k\neq l}^{p} r_k^{B_k}, \tag{13.15}$$

because of (13.14), hence by the definition (13.3) we obtain the desired formula (13.2). This ends the proof of the Theorem 1.

§ **14.** Remark 4. The results of § 13 and of the following paragraphs possess the simply geometrical interpretation.

Let us consider the hyperplane π_l $(l = 1, \dots, p)$ in the p-dimensional space $y = (y_1, \dots, y_p) \in R^p$:

$$\pi_l\colon\ y_l+\gamma\cdot\sum_{k=1,k\neq l}^{p} y_k = +g^{A_l}(h) \qquad (l = 1, \dots, p). \tag{14.1}$$

Since π_l is orthogonal to the vector $e_l = (e_{l1}, \dots, e_{lp})$, $e_{ll} = 1$, $e_{lk} = \gamma$ $(l \neq k)$, the point R:

$$R = (r_1^{A_1}, r_2^{A_2}, \dots, r_p^{A_p}), \tag{14.2}$$

with non-negative coordinates $r_l^{A_l}$ $(l = 1, \dots, p)$ is in the set S:

$$S = \Big\{y\colon\ y_l\geqslant 0,\ y_l+\gamma\cdot\sum_{k=1,k\neq l}^{p} y_k\leqslant +g^{A_l}(h)\ (l = 1, \dots, p)\Big\}, \tag{14.3}$$

cf. Fig. 5 (the case $p = 2$) and the Theorem 1.

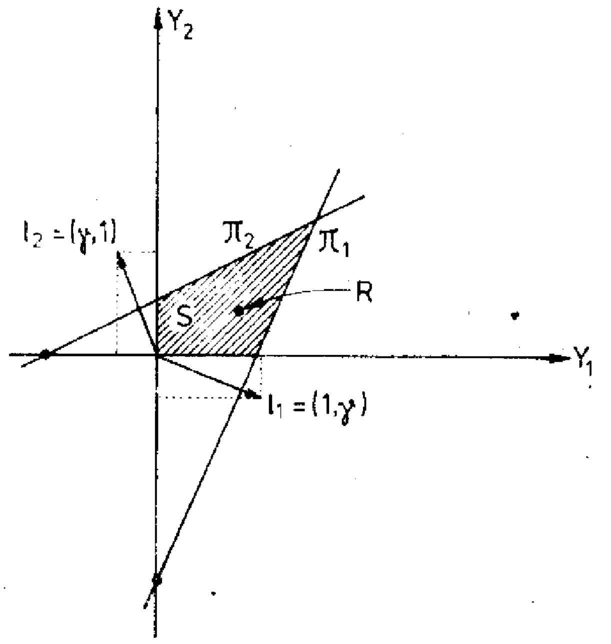

Fig. 5. The set S in the two dimensional case ($p = 2$) and the point R, $R \in S$, cf. Remark 4, § 14

We shall prove in the following paragraphs that under suitable assumptions the right hand sides $g^{A_l}(h)$ in (14.1) converge to zero, as $h \to 0$:

(14.4) $$0 < g^{A_l}(h) \to 0, \quad \text{as} \quad h \to 0 \qquad (l = 1, \ldots, p).$$

As a consequence, the hyperplane π_l tends toward the hyperplane $y_l + \gamma \cdot \sum_{k=1, k \neq l}^{p} y_k = 0$ $(l = 1, \ldots, p)$, as $h \to 0$, and the domain S bounded by π_l, tends toward the origin, as well as the point R, $R \in S$:

(14.5) $$0 \leqslant r_l^{A_l} \to 0, \quad \text{as} \quad h \to 0 \ (l, 1, \ldots, p).$$

§ **15.** LEMMA 5. *Let us suppose that the functions* r_l^M $(l = 1, \ldots, p)$ *satisfy the assumptions of Lemma* 4, *cf.* § 12.

Suppose in addition that for some fixed l $(l = 1, \ldots, p)$ *we have*

(15.1) $$r_l^{A_l} \geqslant h, \quad \text{for } h > 0, \ A_l = A_l(h),$$

and

(15.2) $$0 < \varepsilon(h) \to 0, \quad \text{as} \quad h \to 0.$$

Under these assumptions we have

(15.3) $$0 < g^{A_l}(h) \to 0, \quad \text{as} \quad h \to 0,$$

the function $g^{A_l}(h)$ *being defined by* (12.4).

Proof. From the assumption (15.1) and Lemma 2, cf. § 8, it follows that

(15.4) $$\operatorname{supp} V^{A_l} \subset Q, \ |x^{b_l} - x^{A_l}| \geqslant \varkappa h^{\alpha},$$

where $x^{b_l} \in \partial V^{A_l}$, $\frac{1}{2} \leqslant \alpha < 1$, $0 < h < 1$.

We shall prove first that

(15.5) $$G_1(x - x^{A_l}, h) \to 0, \quad \text{as} \quad h \to 0, \ A_l = A_l(h),$$

the convergence being uniform with respect to x in every closed and bounded set in the n-dimensional space R^n.

In fact, from the definition (11.5) we have

(15.6) $$G_1(x - x^{A_l}, h) = C_1(h), \quad \text{for } x \in V^{A_l}.$$

The value $C_1(h)$ of the function $G_1(x - x^{A_l}, h)$ at the point $x^{b_l} \in \partial V^{A_l}$ divided by $|x^{b_l} - x^{A_l}|^2$ approaches zero, as $h \to 0$. In fact, from the second formula (15.4) and the definition of $C_1(h)$, cf. (11.4), it follows that

(15.7) $$\frac{C_1(h)}{|x^{b_l} - x^{A_l}|^2} \leqslant \frac{C_1(h)}{\varkappa^2 \cdot h^{2\alpha}} \to 0, \quad \text{as} \quad h \to 0,$$

since

$$(15.8)\qquad \begin{cases} \dfrac{h^{3\alpha}}{h^{2\alpha}} = h^{\alpha} \to 0, \quad \dfrac{h^{1+\alpha}}{h^{2\alpha}} = h^{1-\alpha} \to 0\,, \\[2ex] \dfrac{h^{2}}{h^{2\alpha}} = h^{2(1-\alpha)} \to 0, \quad \text{as} \quad h \to 0\,. \end{cases}$$

In the set $R^n \setminus V^{A_l}$ we have, cf. (11.5):

$$(15.9)\qquad G_1(x-x^{A_l}, h) = \frac{|x-x^{A_l}|^2}{|x^{b_l}-x^{A_l}|^2} \cdot C_1(h)\,, \qquad \text{for } x \in R^n \setminus V^{A_l},$$

hence from (15.9) and (15.7) we obtain

$$(15.10)\qquad G_1(x-x^{A_l}, h) \to 0, \text{ as } h \to 0\,,$$

the convergence being uniform with respect to x in every closed and bounded set in the n-dimensional space R^n.

In the second part of the proof we shall verify that the function $E^{A_l}(h)$, cf. (11.11), tends toward zero, as $h \to 0$:

$$(15.11)\qquad E^{A_l}(h) \to 0\,, \quad \text{as} \quad h \to 0\,.$$

To establish this let us observe that the points $(s_{lk}^{A_l})^{1/2} \cdot \alpha_{lk}^{A_l} \in R^n$ $(k = 1, \ldots, n)$, cf. (11.11), are in some bounded set Q_2 for every $h>0$:

$$(15.12)\qquad (s_{lk}^{A_l})^{1/2} \cdot \alpha_{lk}^{A_l} \in Q_2\,, \qquad \text{for } h>0 \ (k = 1, \ldots, n)\,.$$

In fact, from Remark 3, cf. § 11, it follows that the points

$$(15.13)\qquad \alpha_{lk}^{A_l} = (\alpha_{lk1}^{A_l}, \alpha_{lk2}^{A_l}, \ldots, \alpha_{lkn}^{A_l}) \in R^n,$$

are on the unit sphere, since $(\alpha_{lkj}^{A_l})$ $(k, j = 1, \ldots, n)$ is the orthogonal matrix. In addition, the characteristic roots $s_{lk}^{A_l}$ $(k = 1, \ldots, n)$ are bounded:

$$(15.14)\qquad 0 < \delta_1 \leqslant s_{lk}^{A_l} \leqslant \delta_2\,, \qquad \text{for } h>0 \ (k = 1, \ldots, n)\,,$$

cf. (4.2).

From the formula (15.12) and (15.5) it follows that in particular

$$(15.15)\qquad G_1((s_{lk}^{A_l})^{1/2} \cdot \alpha_{lk}^{A_l};\ h) \to 0, \quad \text{as} \quad h \to 0 \ (k = 1, \ldots, n)$$

Hence, from the definition of the function $E^{A_l}(h)$, cf. (11.11), and (15.15) we obtain the desired formula (15.11).

In the third part of the proof we shall verify the relation (15.3). In fact, let us consider the function $g^{A_l}(h)$, cf. the definition (12.4). On the right hand side of the first formula (12.4) we have

$$(15.16)\qquad 0 < D(h) = n \cdot \beta \cdot h \cdot A \to 0, \quad \text{as} \quad h \to 0.$$

furthermore, we have the relation (15.2) for the function $\varepsilon(h)$ and (15.11) for the function $E^{A_l}(h)$. Hence, we have

$$0<g^{A_l}(h)\to 0, \quad \text{as} \quad h\to 0\,. \tag{15.17}$$

This ends the proof of the Lemma 5.

§ **16.** Lemma 6. *Let us suppose that the functions* r_l^M $(l=1,\ldots,p)$ *satisfy the assumptions of Lemma* 4, *cf.* § 12.

Let us suppose in addition that for some fixed value l $(l=1,\ldots,p)$ *we have*

$$r_l^{B_l}\leqslant -h\,, \quad \textit{for } h>0,\ B_l=B_l(h)\,, \tag{16.1}$$

and

$$0<\varepsilon(h)\to 0, \quad \text{as} \quad h\to 0\,. \tag{16.2}$$

Under these assumptions we have

$$0<g^{B_l}(h)\to 0, \quad \text{as} \quad h\to 0\,, \tag{16.3}$$

the function $g^{B_l}(h)$ *being defined by* (12.4).

The proof of Lemma 6 is similar to the proof of Lemma 5 for the function $g^{A_l}(h)$ and will be omitted.

§ **17.** Theorem 2. *Let us suppose that the functions* r_l^M $(l=1,\ldots,p)$ *satisfy the assumptions of Lemma* 4, *cf.* § 12.

Let us suppose also that

$$r_l^{A_l}\geqslant h\,, \quad \textit{for } h>0,\ A_l=A_l(h)\ (l=1,\ldots,p)\,, \tag{17.1}$$

$$0<\varepsilon(h)\to 0 \quad \textit{as} \quad h\to 0\,. \tag{17.2}$$

If in addition h is a sufficiently small positive number, i.e. h satisfies the condition

$$(\varkappa\cdot h^{\alpha}+h)^2+(n-1)\cdot h^2<\delta_1,\ \tfrac{1}{2}\leqslant\alpha<1,\ 0<h<1\,, \tag{17.3}$$

where δ_1 *stands for an lower bound on the characteristic values, cf.* (4.2), *then*

1° *the maximum values* $r_l^{A_l}$ $(l=1,\ldots,p)$ *satisfy the linear system of algebraic inequalities*

$$r_l^{A_l}+\gamma\cdot\sum_{k=1,\,k\neq l}^{p} r_k^{A_k}\leqslant\Omega(h)\ \ (l=1,\ldots,p)\,, \tag{17.4}$$

where

$$\Omega(h)=-\eta^{-1}\cdot[\omega(h)+D(h)+\varepsilon(h)]\,, \tag{17.5}$$

$$\omega(h)=n^2\cdot\delta_2\cdot(2\cdot L\varkappa h^{\alpha}+n\cdot 2\Lambda\cdot\varkappa^{-1}\cdot h^{1-\alpha}+n^2\cdot\Lambda\cdot\varkappa^{-2}\cdot h^{2(1-\alpha)})\,, \tag{17.6}$$

$$0<g^{A_l}(h)<\Omega(h)\to 0, \quad \textit{as} \quad h\to 0\ \ (l=1,\ldots,p)\,. \tag{17.7}$$

In the formula (17.6) δ_2 *stands for an upper bound on characteristic values, cf.* (4.2).
2° *We have the convergence:*

$$0 \leqslant r_l^{A_l} \to 0, \quad \textit{as} \quad h \to 0, \tag{17.8}$$

and the estimate

$$0 \leqslant r_l^{A_l} < \frac{\Omega(h)}{1+(p-1)\gamma} \quad (l = 1, \ldots, p). \tag{17.9}$$

Proof. We shall prove first that the functions $E^{A_l}(h)$ $(l = 1, \ldots, p)$ on the right hand side of the formula (12.4) satisfy the following inequalities:

$$E^{A_l}(h) \leqslant \omega(h) \ (l = 1, \ldots, p). \tag{17.10}$$

In fact, from the assumptions (17.1) and Lemma 2, cf. § 8, it follows that the diameter $2 \cdot d$ of the set V^{A_l} $(l = 1, \ldots, p)$ fulfills the relation

$$(2 \cdot d)^2 \leqslant [2 \cdot (\varkappa h^\alpha + h)]^2 + (n-1) \cdot (2h)^2. \tag{17.11}$$

From (17.11) and from the assumption (17.3) we obtain $d < \delta_1^{1/2}$. But $\delta_1^{1/2} \leqslant (s_{lk}^{A_l})^{1/2}$, cf. (4.2), therefore the points

$$(s_{lk}^{A_l})^{1/2} \cdot \alpha_{lk}^{A_l} \in R^n \quad (k = 1, \ldots, n), \tag{17.12}$$

are in the set $R^n \setminus V^{A_l}$. This and the second part of the definition of the function $G_1(x - x^{A_l}, h)$, cf. (11.5), yields

$$G_1((s_{lk}^{A_l})^{1/2} \cdot \alpha_{lk}^{A_l};\ h) = |x^{k_l} - x^{A_l}|^{-2} \cdot |(s_{lk}^{A_l})^{1/2} \cdot \alpha_{lk}^{A_l}|^2 \cdot C_1(h) \leqslant s_{lk}^{A_l} \cdot n \cdot C_1(h) \cdot |x^{k_l} - x^{A_l}|^{-2}, \tag{17.13}$$

where $x^{k_l} \in \partial V^{A_l}$, since (α_{lkj}) is the orthogonal matrix, $1 \leqslant |\alpha_{lk}^{A_l}| \leqslant n^{1/2}$ and $s_{lk}^{A_l}$ are positive numbers, cf. (4.2) and Remark 3 (§ 11).

In addition, from the assumption (17.1) and Lemma 2, cf. § 8, it follows that

$$\operatorname{supp} V^{A_l} \subset Q, \ |x^{k_l} - x^{A_l}| \geqslant \varkappa h^\alpha, \tag{17.14}$$

and

$$C_1(h) \cdot |x^{k_l} - x^{A_l}|^{-2} \leqslant C_1(h) \cdot \varkappa^{-2} \cdot h^{-2\alpha}. \tag{17.15}$$

Hence, (17.15) (17.13) and the boundedness of $s_{lk}^{A_l}$ $(k = 1, \ldots, n)$, cf. (4.2), imply that

$$G_1((s_{lk}^{A_l})^{1/2} \cdot \alpha_{lk}^{A_l};\ h) \leqslant n \cdot \delta_2 \cdot C_1(h) \cdot \varkappa^{-2} \cdot h^{-2\alpha}. \tag{17.16}$$

We can use now the definition of $E^{A_l}(h)$, cf. (11.11), and (17.16) and we obtain first the estimate

$$E^{A_l}(h) \leqslant n^2 \cdot \delta_2 \cdot C_1(h) \cdot \varkappa^{-2} \cdot h^{-2\alpha}. \tag{17.17}$$

Furthermore, from the definition of $C_1(h)$, cf. (11.4), it follows that

$$\begin{aligned} C_1(h) \cdot \varkappa^{-2} \cdot h^{-2\alpha} &\leqslant \varkappa^{-2} \cdot h^{-2\alpha} \cdot (2 \cdot L\varkappa^3 \cdot h^{3\alpha} + n \cdot 2\Lambda \cdot \varkappa h^{1+\alpha} + n^2 \cdot \Lambda \cdot h^2) \\ &= 2 \cdot L\varkappa h^\alpha + n \cdot 2\Lambda \cdot \varkappa^{-1} \cdot h^{1-\alpha} + n^2 \cdot \Lambda \cdot \varkappa^{-2} \cdot h^{2(1-\alpha)}. \end{aligned} \tag{17.18}$$

Finally, we observe that (17.17) (17.18) and the definition of the function $\omega(h)$, cf. (17.6), yield

$$E^{A_l}(h) \leqslant \omega(h) \quad (l = 1, \ldots, p). \tag{17.19}$$

This completes the proof of the formula (17.10).

Now we shall prove the relations (17.7) and (17.4). The relation (17.7) follows from the definition of $g^{A_l}(h)$, cf. (12.4), the estimate (17.19) and from the definition of $\Omega(h)$, cf. (17.5) (cf. also Lemma 5, § 15, the formula (15.3)).

The system of inequalities (17.4) can be obtained from the estimate (17.7) and the Theorem 1, § 13, the formula (13.1). It is sufficient only to replace $g^{A_l}(h)$ on the right hand side of (13.1) by the greater quantity $\Omega(h)$, cf. (17.7).

This ends the proof of the formula (17.7) and (17.4).

Our last objective is to prove the convergence (17.8) and the estimate (17.9).

The convergence (17.8) follows from Lemma 5, cf. (15.3), and the system of algebraic inequalities for maximum values $r_l^{A_l} \geqslant 0$, cf. Theorem 1 and the system (13.1).

The estimate (17.9) can be obtained in the following manner. Let us consider the system of linear algebraic equations for p unknowns y_l $(l = 1, \ldots, p)$:

$$y_l + \gamma \cdot \sum_{k=1, k \neq l}^{p} y_k = \Omega(h) \quad (l = 1, \ldots, p). \tag{17.20}$$

The matrix Δ of the system (17.20) is symmetric: $\Delta = (\Delta_{lk})$, $\Delta_{ll} = 1$, $\Delta_{lk} = \gamma$ $(l \neq k)$ and the determinant $\mathrm{Det}\Delta$ of that matrix differs from zero:

$$\mathrm{Det}\Delta = (1-\gamma)^{p-1} \cdot [1 + (p-1) \cdot \gamma] > 0, \tag{17.21}$$

cf. the assumption (4.5).

Consequently, the system (17.20) possesses only one solution

$$y_l = \frac{\Omega(h)}{1 + (p-1) \cdot \gamma} \quad (l = 1, \ldots, p), \tag{17.22}$$

as can be verified by substituting (17.22) into (17.20).

The system of inequalities (17.4) being satisfied by the maximum values $r_l^{A_l} \geqslant 0$ $(l = 1, \ldots, p)$, we obtain the desired estimate (17.9) from the system of equalities (17.20) and from the formula (17.22).

This ends the proof of the Theorem 2.

§ 18. Remark 5. It seems that the best possible way to obtain the value of the determinant $\mathrm{Det}\Delta$, cf. (17.21), is to find the triangular matrices Δ' and Δ'' such that

$$\Delta = \Delta' \cdot \Delta''. \tag{18.1}$$

The matrix Δ being given there are infinitely many triangular matrices Δ' and Δ'' such that (18,1) holds. Notwithstanding this fact the matrices Δ' and Δ'' can be chosen so as to obtain a simple proof of (17.21) by induction.

Remark 6. The results of § 17, cf. Theorem 2, possess the following geometrical interpretation. Let us consider the hyperplanes Σ_l $(l = 1, \ldots, p)$:

$$\Sigma_l\colon\ y_l+\gamma\cdot\sum_{k=1,k\neq l}^{p} y_k = \Omega(h) \quad (l = 1, \ldots, p)\,, \tag{18.2}$$

cf. (17.20), and the set S_1:

$$S_1 = \{y\colon\ y_l\geqslant 0,\ y_l+\gamma\cdot\sum_{k=1,k\neq l}^{p} y_k\leqslant\Omega(h)\ (l = 1, \ldots, p)\}. \tag{18.3}$$

Since the scalar product $e\cdot e_l$ is constant and does not depend of l: $e\cdot e_l = 1+(p-1)\cdot\gamma$ for $e = (e^1, \ldots, e^p)$, $e^k = 1$ $(k = 1, \ldots, p)$; $e_l = (e_{l1}, \ldots, e_{lp})$, $e_{ll} = 1$, $e_{lk} = \gamma$ $(l \neq k)$, the set S_1 is symmetric with respect to the straight line $y_l = t$ $(0\leqslant t<+\infty;\ l = 1, \ldots, p)$ and we have only one point R_1 of intersection of the hyperplanes Σ_l, cf. Fig. 6, R_1 being on the straight line $y_l = t$ $(l = 1, \ldots, p)$.

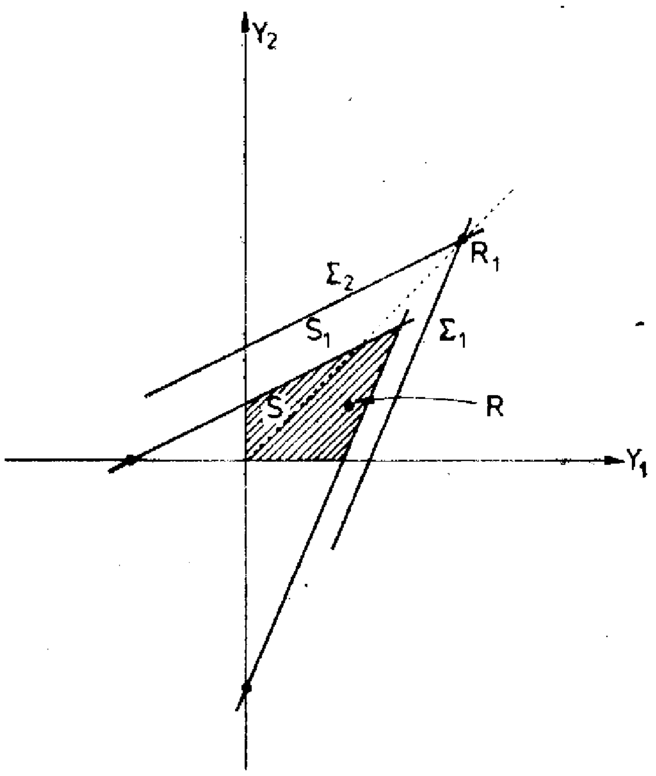

Fig. 6. The sets S and S_1, $S \subset S_1$, in the two dimensional case $(p = 2)$ and the point R_1, cf. Remark 6, § 18

We can calculate the coordinates of the point R_1 from (18.2) by substituting $y_l = t$ $(l = 1, \ldots, p)$ into (18.2). Then we obtain

$$t = \frac{\Omega(h)}{1+(p-1)\cdot\gamma}\,. \tag{18.4}$$

Hence, the point R_1 has equal coordinates:

$$R_1 = \left(\frac{\Omega(h)}{1+(p-1)\cdot\gamma}, \frac{\Omega(h)}{1+(p-1)\cdot\gamma}, \ldots, \frac{\Omega(h)}{1+(p-1\cdot\gamma)}\right) \tag{18.5}$$

and they are positive, if

$$1+(p-1)\cdot\gamma>0\,. \tag{18.6}$$

So we see that (18.6) as well as (4.5) is the condition for the intersection of hyperplanes Σ_l at the point R_1 with positive coordinates.

The similar interpretation can be given for the minimum values $r_l^{B_l} \leqslant 0$, cf. Fig. 7 and Theorem 3 (§ 19).

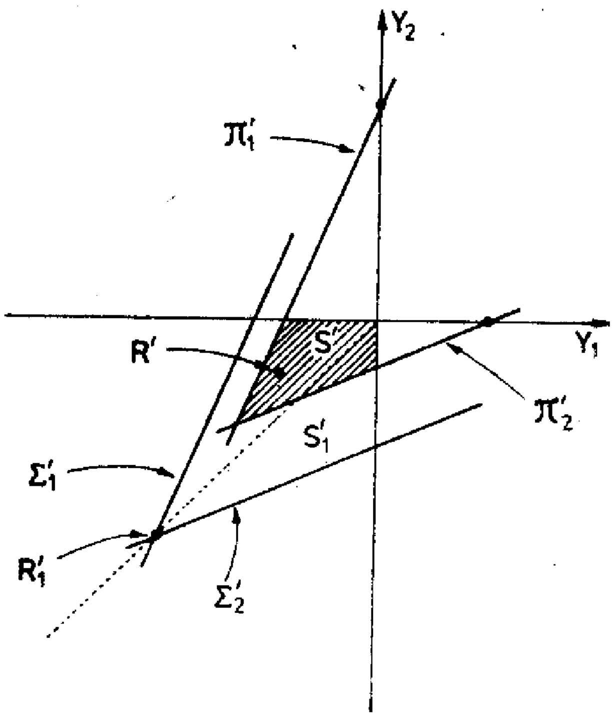

Fig. 7. The sets S' and S_1', $S' \subset S_1'$, in the two dimensional case ($p = 2$). The hyperplanes π_l', Σ_l', and the point R' can be defined in a similar way as π_l, Σ_l and the point R

§ **19.** THEOREM 3. *Let us suppose that the functions* r_l^M $(l = 1, \dots, p)$ *satisfy the assumptions of Lemma* 4, *cf.* § 12.

Let us suppose also that

$$r_l^{B_l} \leqslant -h\,, \quad \text{for } h>0,\ B_l = B_l(h)\ (l = 1, \dots, p)\,, \tag{19.1}$$

$$0<\varepsilon(h) \to 0, \quad as \quad h \to 0\,. \tag{19.2}$$

If in addition h is a sufficiently small positive number, i.e. h satisfies the condition

$$(\varkappa \cdot h^{\alpha}+h)^2+(n-1)\cdot h^2<\delta_1,\ \tfrac{1}{2} \leqslant \alpha<1,\ 0<h<1\,, \tag{19.3}$$

where δ_1 *stands for an lower bound on characteristic values, cf.* (4.2), *then*

1° *the minimum values* $r_l^{B_l}$ $(l = 1, \dots, p)$ *satisfy the linear system of algebraic inequalities*

$$r_l^{B_l}+\gamma \cdot \sum_{k=1, k \neq l}^{p} r_k^{B_k} \geqslant -\Omega(h) \quad (l = 1, \dots, p)\,, \tag{19.4}$$

where

$$\Omega(h) = -\eta^{-1} \cdot [\omega(h)+D(h)+\varepsilon(h)]\,, \tag{19.5}$$

$$\omega(h) = n^2 \cdot \delta_2 \cdot (2L\varkappa h^{\alpha}+n \cdot 2\Lambda \cdot \varkappa^{-1} \cdot h^{1-\alpha}+n^2 \cdot \Lambda \cdot \varkappa^{-2} \cdot h^{2(1-\alpha)})\,, \tag{19.6}$$

$$0<g^{B_l}(h)<\Omega(h) \to 0,\ as\ h \to 0 \quad (l = 1, \dots, p)\,. \tag{19.7}$$

In the formula (19.6) δ_2 *stands for an upper bound on characteristic values, cf.* (4.2).

2° *We have the convergence*

$$0 \geqslant r_l^{B_l} \to 0, \text{ as } h \to 0 \quad (l = 1, \ldots, p), \tag{19.8}$$

and the estimate:

$$-\frac{\Omega(h)}{1+(p-1)\cdot\gamma} < r_l^{B_l} \leqslant 0 \quad (l = 1, \ldots, p). \tag{19.9}$$

The proof of Theorem 3 is similar to the proof of Theorem 2 and will be omitted.

§ 20. LEMMA 7. *Let us suppose that the functions r_l^M $(l = 1, \ldots, p)$, satisfy the assumptions of Lemma* 4, *cf.* § 12, *and the relation* (17.2) *for the function $\varepsilon(h)$.*

Let us denote by $R(h)$ the point of the p-dimensional space with coordinates:

$$R(h) = (r_1^{A_1}, r_2^{A_2}, \ldots, r_p^{A_p}),\ A_l = A_l(h)\ (l = 1, \ldots, p), \tag{20.1}$$

$r_l^{A_l}$ being the maximum value $r_l^{A_l} \geqslant 0$ $(l = 1, \ldots, p)$.

Let us suppose that $0 < h_\nu \to 0$, as $\nu \to +\infty$, and

$$R(h_\nu) \to R_0, \quad \text{as} \quad \nu \to +\infty, \tag{20.2}$$

where the point R_0 has the coordinates $R_0 = (R_{01}, \ldots, R_{0p})$ and R_0 is on the s-dimensional hyperplane (subspace) generated by coordinate axes $y_{l_1}, y_{l_2}, \ldots, y_{l_s}$ $(1 \leqslant l_k \leqslant p,\ k = 1, \ldots, s;\ s < p)$:

$$R_{0l} = 0, \text{ for } l \neq l_k\ (k = 1, \ldots, s;\ s < p). \tag{20.3}$$

Hence, we have the relations

$$r_l^{A_l} \to R_{0l}, \text{ as } \nu \to +\infty,\ l = l_k\ (k = 1, \ldots, s;\ s < p,\ A_l = A_l(h_\nu), \tag{20.4}$$

$$r_l^{A_l} \to 0, \text{ as } \nu \to +\infty,\ l \neq l_k\ (k = 1, \ldots, s;\ s < p),\ A_l = A_l(h_\nu). \tag{20.5}$$

Let us suppose in addition that

$$r_l^{A_l} \geqslant h_\nu, \text{ for } l = l_k\ (k = 1, \ldots, s;\ s < p)(\nu = 1, 2, \ldots),\ A_l = A_l(h_\nu). \tag{20.6}$$

Under these assumptions

1° *we have the convergence:*

$$R(h_\nu) \to R_0 = 0, \quad \text{as} \quad \nu \to +\infty, \tag{20.7}$$

that is we have the convergence toward zero for the values $l = l_k$ $(k = 1, \ldots, s;\ s < p)$, too:

$$r_l^{A_l} \to 0, \quad \text{as} \quad \nu \to +\infty,\ l = l_k\ (k = 1, \ldots, s;\ s < p),\ A_l = A_l(h), \tag{20.8}$$

and

$$R_{0l} = 0, \quad \text{for } l = l_k\ (k = 1, \ldots, s;\ s < p). \tag{20.9}$$

2° *If* $h = h_\nu$ *is a sufficiently small positive number, i.e.* $h = h_\nu$ *satisfies the condition* (17.3), *then we have the estimate*

$$(20.10) \qquad \begin{cases} 0 \leqslant r_l^{A_l} < \dfrac{1}{1+(s-1)\cdot\gamma} \cdot \Big[\Omega(h_\nu) - \gamma \cdot \sum\limits_{l \neq l_1, \dots, l_s} r_l^{A_l}\Big], \\ (l = l_k;\ k = 1, \dots, s;\ s<p), \end{cases}$$

the function $\Omega(h)$ *being defined by* (17.5) *and* (17.6).

Proof. Let us select from (13.1) the algebraic inequalities corresponding to the indices $l = l_k$ $(k = 1, \dots, s;\ s<p)$:

$$(20.11) \qquad r_{l_k}^{A_{l_k}} + \gamma \cdot \sum_{l \neq l_k, l = l_1, \dots, l_s} r_l^{A_l} \leqslant +g^{A_{lk}}(h_\nu) - \gamma \cdot \sum_{l \neq l_1, \dots, l_s} r_l^{A_l}.$$

In view of (20.6) and Lemma 5, cf. § 15, we have the convergence:

$$(20.12) \qquad 0 < g^{A_{l_k}}(h_\nu) \to 0, \text{ as } \nu \to +\infty,\ A_{l_k} = A_{l_k}(h_\nu)\ (k = 1, \dots, s;\ s<p).$$

Let us denote the right hand side of (20.11) by $\bar{g}^{A_{l_k}}(h_\nu)$:

$$(20.13) \qquad \bar{g}^{A_{l_k}}(h_\nu) = +g^{A_{l_k}}(h_\nu) - \gamma \cdot \sum_{l \neq l_1, \dots, l_s} r_l^{A_l}.$$

Then the system (20.11) can be rewritten in the form which is similar to the system (13.1):

$$(20.14) \qquad \{r_{l_k}^{A_{l_k}} + \gamma \cdot \sum_{l \neq l_k, l = l_1, \dots, l_s} r_l^{A_l} \leqslant +\bar{g}^{A_{l_k}}(h_\nu), \quad (k = 1, \dots, s;\ s<p),$$

where the right hand members of (20.14) tend toward zero, too, because of (20.12) and the assumption (20,5):

$$(20.15) \qquad 0 < \bar{g}^{A_{l_k}}(h_\nu) \to 0, \text{ as } \nu \to +\infty.$$

We shall prove that (20.15) and (20.14) enable us to establish the convergence toward zero of the maximum values $r_l^{A_l}$ on the left hand side of (20,14), too:

$$(20.16) \qquad r_l^{A_l} \to 0, \quad \text{as} \quad \nu \to +\infty,\ l = l_k\ (k = 1, \dots, s;\ s<p),\ A_l = A_l(h_\nu).$$

To establish this let us consider the linear system of s algebraic equations for the unknowns y_l, $l = l_k$ $(k = 1, \dots, s;\ s<p)$:

$$(20.17) \qquad y_{l_k} + \gamma \cdot \sum_{l \neq l_k, l = l_1, \dots, l_s} y_l = +\bar{g}^{A_{l_k}}(h_\nu),\ (k = 1, \dots, s;\ s<p).$$

The matrix $\bar{\Delta}$ of coefficients of the system (20.17) is symmetric, $\bar{\Delta} = (\bar{\Delta}_{jk})$, $\bar{\Delta}_{jj} = \gamma$ $(j \neq k)(j, k = 1, \dots, s)$, and the determinant of $\bar{\Delta}$ differs from zero:

$$(20.18) \qquad \mathrm{Det}\bar{\Delta} = (1-\gamma)^{s-1} \cdot [1+(s-1)\cdot\gamma] > 0,$$

cf. the assumption (4.5) and the inequality $1 \leqslant s < p$.

Hence, the system (20.17) possesses only one solution:

$$(20.19) \qquad y_{l_k} = y_{l_k}(h_\nu)\ (k = 1, \dots, s;\ s<p).$$

But the system (20.14) is satisfied by the maximum values $r_l^{A_l}\geqslant 0$; $l = l_k$ ($k = 1, .., s$; $s<p$), therefore from (20.14) and (20.17) we obtain the inequalities:

$$0\leqslant r_{l_k}^{A_{l_k}}\leqslant y_{l_k}(h_\nu) \quad (k = 1, \dots, s;\ s<p)\,. \tag{20.20}$$

From Cramer's formula for the solution (20.19) and from the formula (20.15) it follows that the solution $y_{l_k}(h_\nu)$ tends toward zero:

$$y_{l_k}(h_\nu) \to 0, \quad \text{as} \quad \nu \to +\infty \ (k = 1,, \dots, s;\ s<p)\,, \tag{20.21}$$

hence, from (20.21) and (20.20) we obtain

$$0\leqslant r_{l_k}^{A_{l_k}} \to 0, \quad \text{as} \quad \nu \to +\infty \ (k = 1, \dots, s;\ s<p)\,. \tag{20.22}$$

This ends the proof of the relations (20.16).

We shall prove now the estimate (20.10). For this purpose it is sufficient first to repeat part of the proof of Theorem 2 for $l = l_k$ ($k = 1, \dots, s$; $s<p$), cf. § 17, in particular the formula (17.10) up to (17.19). Then we obtain the relation similar to (17.17):

$$0<g^{A_l}(h_\nu)<\Omega(h_\nu) \to 0, \ \text{as} \ \nu \to +\infty, l = l_k\, (k = 1, \dots, s; s<p)\,. \tag{20.23}$$

We can use (20.23), the definition (20.13) and the inequality (20.14) to derive the following linear system of s algebraic inequalities for s maximum values $r_l^{A_l}$, $l = l_k$ $k = 1, \dots, s$; $s<p$):

$$r_{l_k}^{A_{l_k}}+\gamma\cdot \sum_{l\neq l_k,\, l=l_1, \dots, l_s} r_l^{A_l} \leqslant \Omega(h_\nu)-\gamma\cdot \sum_{l\neq l_1, \dots, l_s} r_l^{A_l}\,. \tag{20.24}$$

Let us consider the linear system of s algebraic equations for s unknowns y_{l_k} ($k = 1, \dots, s$; $s<p$):

$$y_{l_k}+\gamma\cdot \sum_{l\neq l_k,\, l=l_1, \dots, l_s} y_l = \Omega(h_\nu)-\gamma\cdot \sum_{l\neq l_1, \dots, l_s} r_l^{A_l}\,. \tag{20.25}$$

The matrix $\bar{\Delta}$ of the system (20.25) is symmetric and the determinant of the matrix $\bar{\Delta}$ differs from zero, cf. (20.18). Consequently, the system (20.25) possesses only one solution:

$$\begin{cases} y_{l_k} = \dfrac{1}{1+(s-1)\cdot\gamma}\cdot\left[\Omega(h_\nu)-\gamma\cdot \displaystyle\sum_{l\neq l_1, \dots, l_s} r_l^{A_l}\right], \\ (k = 1, \dots, s;\ s<p)\,, \end{cases} \tag{20.26}$$

as can be verified by substituting (20.26) into (20.25).

The system of inequalities (20.24) being satisfied by the maximum values $r_l^{A_l}$, $l = l_k$ ($k = 1, \dots, s$; $s<p$), we obtain the desired estimate (20.10) from the system of equalities (20.25) and from the formula (20.26).

This ends the proof of Lemma 7 (cf. Fig. 7).

§ 21. A similar lemma can be proved for the minimum values $r_l^{B_l}$:

LEMMA 8. *Let us suppose that the functions* r_l^M ($l = 1, \dots, p$) *satisfy the assumptions of Lemma* 4, cf. § 12, *and the relation* (17.2) *for the function* $\varepsilon(h)$.

Let us denote by $R'(h)$ the point of the p-dimensional space with coordinates:

$$R'(h) = (r_1^{B_1}, r_2^{B_2}, \dots, r_p^{B_p}),\quad B_l = B_l(h)\ (l = 1, \dots, p)\,, \tag{21.1}$$

$r_l^{B_l}$ being the minimum values $r_l^{B_l} \leqslant 0$ $(l = 1, \dots, p)$.

Let us suppose that $0 < h_\nu \to 0$, as $\nu \to +\infty$, and

$$R'(h_\nu) \to R_0', \quad as \quad \nu \to +\infty\,, \tag{21.2}$$

where the point R_0' has the coordinates $R_0' = (R_{01}', \dots, R_{0p}')$ and R_0' is on the s-dimensional hyperplane (subspace) generated by coordinate axes $y_{l_1}, y_{l_2}, \dots, y_{l_s}$ $(1 \leqslant l_k \leqslant p;\ k = 1, \dots, s;\ s < p)$:

$$R_{0l}' = 0\,, \quad for\ l \neq l_k\ (k = 1, \dots, s;\ s < p)\,. \tag{21.3}$$

Hence, we have the relations:

$$r_l^{B_l} \to R_{0l}', \quad as \quad \nu \to +\infty,\ (l = l_k;\ k = 1, \dots, s;\ s < p),\ B_l = B_l(h_\nu)\,, \tag{21.4}$$

$$r_l^{B_l} \to 0,\ as \quad \nu \to +\infty,\ (l \neq l_k;\ k = 1, \dots, s;\ s < p),\ B_l = B_l(h_\nu)\,. \tag{21.5}$$

Let us suppose in addition that

$$r_l^{B_l} \leqslant -h_\nu, \quad for\ l = l_k\ (k = 1, \dots, s;\ s < p)(\nu = 1, 2, \dots),\ B_l = B_l(h_\nu)\,. \tag{21.6}$$

Under these assumptions 1° *we have the convergence:*

$$R'(h_\nu) \to 0, \quad as \quad \nu \to +\infty\,, \tag{21.7}$$

that is we have the convergence toward zero for the values $l = l_k$, $(k = 1, \dots, s;\ s < p)$, too:

$$r_l^{B_l} \to 0, \quad as \quad \nu \to +\infty,\ l = l_k\ (k = 1, \dots, s;\ s < p),\ B_l = B_l(h_\nu) \tag{21.8}$$

and

$$R_{0l}' = 0\,, \quad for\ l = l_k\ (k = 1, \dots, s;\ s < p)\,. \tag{21.9}$$

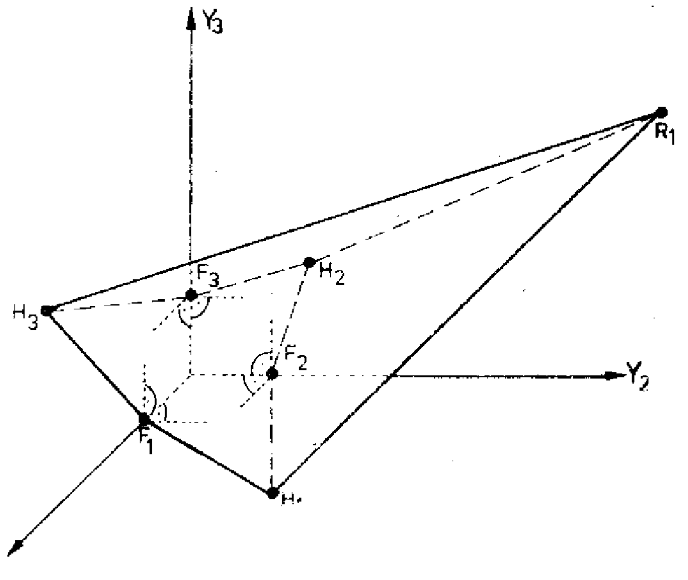

Fig. 8. The set S_1 in the three dimensional case ($p = 3$) as bounded by the planes $\Sigma_1 = F_1H_1R_1H_3$, $\Sigma_2 = F_2H_2R_1H_1$, $\Sigma_3 = F_3H_3R_1H_2$. The points H_j ($j = 1, 2, 3$) are on the bisectrices of the angles $y_1 0 y_2$, $y_2 0 y_3$, $y_3 0 y_1$, respectively. The point R_1 is on the straight line $y_l = t$ ($0 \leqslant t < +\infty$; $l = 1, 2, 3$). The set S is inside the set S_1; $S \subset S_1$, the faces of the set S being parallel to the corresponding faces of the set S_1. This figure can be used to visualize the proof of the Lemma 7 (cf. § 20)

2° *If* $h = h_\nu$ *is a sufficiently small positive number, i.e.* $h = h_\nu$ *satisfies the condition* (17.3), *then we have the estimate*:

$$\text{(21.10)} \qquad \begin{cases} 0 \geqslant r_l^{B_l} \geqslant -\dfrac{1}{1+(s-1)\cdot\gamma}\cdot\left[\Omega(h_\nu)+\gamma\cdot\displaystyle\sum_{l\neq l_1,\dots,l_s} r_l^{B_l}\right], \\ l = l_k \ (k = 1, \dots, s;\ s<p), \end{cases}$$

the function $\Omega(h)$ *being defined by* (17.5) *and* (17.6).

The proof of Lemma 8 is similar to the proof of Lemma 7 and will be omitted.

§ **22.** Theorem 4. *Let us suppose that the functions* r_l^M $(l = 1, \dots, p)$ *satisfy the assumptions of Lemma* 4, *cf.* § 12, *and the relation* (17.2) *for the function* $\varepsilon(h)$.

Under these assumptions 1° *the point* $R(h)$ *with coordinates* $R(h) = (r_1^{A_1}, \dots, r_p^{A_p})$ *converges toward the origin, as* $h \to 0$:

$$\text{(22.1)} \qquad R(h) \to 0, \quad \textit{as} \quad h \to 0,$$

that is the maximum values $r_l^{A_l}$ $(l = 1, \dots, p)$ *converge toward zero*:

$$\text{(22.2)} \qquad r_l^{A_l} \to 0, \quad \textit{as} \quad h \to 0,\ A_l = A_l(h)\ (l = 1, \ .., p).$$

2° *If* h *is a sufficiently small positive number, i.e.* h *satisfies the condition* (17.3), *then we have the estimate*:

a) $r_l^M \leqslant \dfrac{\Omega(h)}{1+(p-1)\cdot\gamma}$, *for* $x^M \in Q$, *if* $r_l^{A_l} \geqslant h$ $(l = 1, \dots, p)$,

b) $r_l^M \leqslant h$, *for* $x^M \in Q$, *if* $r_l^{A_l} \leqslant h$.

In the formula a) *we have* $A_l = A_l(h)$ *and the function* $\Omega(h)$ *is defined by* (17.5) *and* (17.6).

Proof. Let us observe first that the sequence $R(h)$ is bounded since $r_l^{A_l}$ $(l = 1, \dots, p)$ satisfy the linear system of algebraic inequalities (13.1), and the right hand sides $g^{A_l}(h)$ $(l = 1, \dots, p)$ are bounded, cf. the formula (15.3).

Proceeding by the contrary let us suppose that $R(h)$ does not converge toward the origin, as $h \to 0$. Then there exists the constant $\varepsilon_0 > 0$ and a sequence $0 \leqslant h_\nu \to 0$, as $\nu \to +\infty$, such that

$$\text{(22.3)} \qquad R(h_\nu) \to R_0, \quad \text{as} \quad \nu \to +\infty,$$

the distance of the point R_0 with coordinates $R_0 = (R_{01}, \dots, R_{0p})$, $R_{0l} > 0$ $(l = 1, \dots, p)$, from the origin being greater than ε_0:

$$\text{(22.4)} \qquad \varrho(R_0, 0) > \varepsilon_0,$$

where ϱ denotes the Euclidean distance between the points R_0 and 0.

There are only two possible cases:

1) the point R_0 is at a positive distance from all coordinate hyperplanes and from the origin:

$$R_{0l} \geqslant \varepsilon_0', \; \varepsilon_0' > 0 \quad (l = 1, \ldots, p) . \tag{22.5}$$

Hence, we have the relations

$$r_l^{A_l} \to R_{0l} > \varepsilon_0', \quad \text{as} \quad \nu \to +\infty \; (l = 1, \ldots, p), \; A_l = A_l(h_\nu) . \tag{22.6}$$

2) The point R_0 is on some coordinate hyperplane but at a positive distance from the origin, greater then ε_0:

$$R_{0l} > \varepsilon_0'', \quad \text{for } l = l_k \; (k = 1, \ldots, s; \; s < p)(\varepsilon_0'' > 0) , \tag{22.7}$$

$$R_{0l} = 0, \quad \text{for } l \neq l_k \; (k = 1, \ldots, s; \; s < p) . \tag{22.8}$$

Hence, we have the relations

$$\begin{cases} r_l^{A_l} \to R_{0l} \geqslant \varepsilon_0'', \quad \text{as} \quad \nu \to +\infty , \quad \text{for } l = l_k \; (k = 1, \ldots, s; \; s < p) , \\ A_l = A_l(h_\nu) , \end{cases} \tag{22.9}$$

$$r_l^{A_l} \to 0, \text{ as } \nu \to +\infty , \quad \text{for } l \neq l_k \; (k = 1, \ldots, s; \; s < p), \; A_l = A_l(h_\nu) . \tag{22.10}$$

We shall prove that both cases lead to contradiction.

In fact, according to what has been said in connection with the case 1), cf. (22.6), we have, for sufficiently great values of ν $(\nu \geqslant \nu_0)$

$$r_l^{A_l} \geqslant h_\nu , \quad \text{for } \nu \geqslant \nu_0 \; (l = 1, \ldots, p), \; A_l = A_l(h_\nu) . \tag{22.11}$$

But from the inequality (22.11) and from the Theorem 2, cf. § 17, formula (17.8), it follows that

$$0 < r_l^{A_l} \to 0, \quad \text{as} \quad \nu \to +\infty \; (l = 1, \ldots, p), \; A_l = A_l(h_\nu) , \tag{22.12}$$

hence, we have $R(h_\nu) \to R_0 = 0$, as $\nu \to +\infty$, and this contradicts the assumption (22.4).

In the case 2), cf. (22.7) (22.8), we have

$$r_l^{A_l} \geqslant h_\nu , \quad \text{for } \nu \geqslant \nu_1, \; l = l_k \; (k = 1, \ldots, s; \; s < p), \; A_l = A_l(h_\nu) , \tag{22.13}$$

ν_1 being a sufficiently great natural number.

However, from the inequality (22.13) and from Lemma 7, cf. § 20, the formula (20.7), we obtain the convergence:

$$R(h_\nu) \to R_0 = 0, \quad \text{as} \quad \nu \to +\infty , \tag{22.14}$$

and this again contradicts the assumption (22.4).

This ends the proof of the formula (22.1) and (22.2).

We shall derive now the estimates a) and b). Indeed, $r_l^{A_l}$ $(l = 1, \ldots, p)$ denotes the maximum value, hence, we have

$$r_l^M \leqslant r_l^{A_l}, \quad \text{for } x^M \in Q \; (l = 1, \ldots, p) . \tag{22.15}$$

From the formula (22.15) and from the Theorem 2, cf. § 17, the relation (17.9), it follows part a) of the estimate.

Part b) of the estimate follows also from (22.15).

This ends the proof of the Theorem 4.

§ **23.** The similar theorem can be proved for the minimum values $r_l^{B_l}$ $(l = 1, \dots, p)$:

THEOREM 5. *Let us suppose that the functions* r_l^M $(l = 1, \dots, p)$ *satisfy the assumptions of Lemma* 4, *cf.* § 12, *and the relation* (17.2) *for the function* $\varepsilon(h)$.

Under these assumptions 1° *the point* $R'(h)$ *with coordinates* $R'(h) = (r_1^{B_1}, \dots, r_p^{B_p})$ *converges toward the origin, as* $h \to 0$:

$$R'(h) \to 0, \quad as \quad h \to 0\,, \tag{23.1}$$

that is the minimum values $r_l^{B_l}$ $(l = 1, \dots, p)$ *converge toward zero*:

$$r_l^{B_l} \to 0, \quad as \quad h \to 0, \ B_l = B_l(h) \ (l = 1, \dots, p)\,. \tag{23.2}$$

2° *If h is a sufficiently small positive number, i.e. h satisfies the condition* (17.3), *then we have the estimate*:

a) $$r_l^M \geqslant -\frac{\Omega(h)}{1+(p-1)\cdot\gamma}, \quad for\ x^M \in Q,\ if\ r_l^{B_l} \leqslant -h \ (l = 1, \dots, p)\,,$$

b) $$r_l^M \geqslant -h, \quad for\ x^M \in Q, \quad if\ r_l^{B_l} \geqslant -h\,.$$

In the formula a) *we have* $B_l = B_l(h)$, *and the function* $\Omega(h)$ *is defined by* (17.5) *and* (17.6).

The proof of the Theorem 5 is similar to the proof of the Theorem 4, cf. § 22, and is based on the Theorem 3, cf. § 19, and on Lemma 8, cf. § 21.

This proof will be omitted.

§ **24.** We intend to study later the difference methods for the systems of differential equations of the elliptic type. In this connection the following Theorem 6 will be useful:

THEOREM 6. *Let us suppose that the systems* I *and* II *of difference inequalities, cf.* (4.8) (4.9), *are of the elliptic type, cf. the definition* 1 (§ 4).

Let us suppose also that the functions r_l^M $(l = 1, \dots, p)(x^M \in Q)$, *satisfy the assumptions of Lemma* 4, *cf.* § 12, *and the relation*

$$0 < \varepsilon = \varepsilon(h) \to 0, \quad as \quad h \to 0\,. \tag{24.1}$$

Under these assumptions 1° *we have the convergence*:

$$r_l^M \to 0, \quad as \quad h \to 0 \ (l = 1, \dots, p)(x^M \in Q); \tag{24.2}$$

2° *If h is sufficiently small positive number, i.e. h satisfies the condition*

$$(\varkappa h^{\alpha}+h)^2+(n-1)\cdot h^2<\delta_1,\ \tfrac{1}{2}\leqslant\alpha<1,\ 0<h<1\,, \tag{24.3}$$

where δ_1 *stands for an lower bound on the characteristic values, cf.* (4.2), *then we have the following estimate:*

a) $|r_l^M|\leqslant\Omega_1(h)$, *for* $x^M\in Q$, *if* $r_l^{A_l}\geqslant h$, $r_l^{B_l}\leqslant -h$ $(l=1,\dots,p)$;
b) $-h\leqslant r_l^M\leqslant\Omega_1(h)$, *for* $x^M\in Q$, *if* $r_l^{A_l}\geqslant h$, $r_l^{B_l}\geqslant -h$ $(l=1,\dots,p)$;
c) $-\Omega_1(h)\leqslant r_l^M\leqslant h$, *for* $x^M\in Q$, *if* $r_l^{A_l}\leqslant h$, $r_l^{B_l}\leqslant -h$ $(l=1,\dots,p)$;
d) $-h\leqslant r_l^M\leqslant h$, *for* $x^M\in Q$, *if* $r_l^{A_l}\leqslant h$, $r_l^{B_l}\geqslant -h$ $(l=1,\dots,p)$.

In the formulas a) b) c) d) *we have* $A_l=A_l(h)$, $B_l=B_l(h)$ $(l=1,\dots,p)$ *and*

$$\Omega_1(h)=\frac{\Omega(h)}{1+(p-1)\cdot\gamma}, \tag{24.4}$$

$$\Omega(h)=-\eta^{-1}\cdot[\omega(h)+D(h)+\varepsilon(h)]\,, \tag{24.5}$$

$$\omega(h)=n^2\cdot\delta_2\cdot[2\cdot L\varkappa h^{\alpha}+n\cdot 2\Lambda\cdot\varkappa^{-1}h^{1-\alpha}+n^2\cdot\Lambda\cdot\varkappa^{-2}\cdot h^{2(1-\alpha)}]\,, \tag{24.6}$$

where δ_2 *stands for an upper bound on the characteristic values, cf.* (4.2).

Proof. $r_l^{A_l}\geqslant 0$ $(l=1,\dots,p)$ and $r_l^{B_l}\leqslant 0$ $(l=1,\dots,p)$ denote the maximum and the minimum values for r_l^M, respectively, cf. (5.4), therefore we have

$$r_l^{B_l}\leqslant r_l^M\leqslant r_l^{A_l},\quad \text{for } x^M\in Q,\ A_l=A_l(h),\ B_l=B_l(h)\ (l=1,\dots,p)\,. \tag{24.7}$$

From the Theorem 4, cf. § 22, and the Theorem 5, cf. § 23, we obtain

$$r_l^{A_l}\to 0,\quad \text{as}\quad h\to 0,\ A_l=A_l(h),\ (l=1,\dots,p)\,, \tag{24.8}$$

$$r_l^{B_l}\to 0,\quad \text{as}\quad h\to 0,\ B_l=B_l(h)\ (l=1,\dots,p)\,, \tag{24.9}$$

consequently, from (24.9) (24.8) and (24.7) the desired relation (24.2) follows.

We shall verify the estimate for r_l^M in the case a). In this case we have $r_l^{A_l}\geqslant h$ $(l=1,\dots,p)$, therefore

$$r_l^M\leqslant\frac{\Omega(h)}{1+(p-1)\cdot\gamma},\quad \text{for } x^M\in Q\ (l=1,\dots,p)\,, \tag{24.10}$$

because of the Theorem 4, cf. § 22.

Moreover, we have $r_l^{B_l}\leqslant -h$ $(l=1,\dots,p)$, hence, from Theorem 5, cf. § 23, it follows that

$$r_l^M\geqslant-\frac{\Omega(h)}{1+(p-1)\cdot\gamma},\quad \text{for } x^M\in Q\ (l=1,\dots,p)\,. \tag{24.11}$$

The formulas (24.11) (14.10) and the definition of $\Omega_1(h)$, cf. (24.4), enable us to write:

$$|r_l^M|\leqslant\Omega_1(h)\,,\quad \text{for } x^M\in Q\ (l=1,\dots,p)\,. \tag{24.12}$$

The proof in the remaining cases b) c) d) is similar.

This ends the proof of the Theorem 6.

§ 25. Another error estimate can be obtained with the use of Lemma 7, cf. § 20, and Lemma 8, cf. § 21. That estimate will be given in the Theorem 7, cf. § 27. First we shall prove the following lemma:

LEMMA 9. *Let us suppose that the functions* r_l^M $(l = 1, \dots, p)$ *satisfy the assumptions of the Lemma* 4, *cf.* § 12, *and the relation* (17.2) *for the function* $\varepsilon(h)$.

Let us suppose in addition that for the natural numbers l_k, $1 \leqslant l_k \leqslant p$, *we have*

$$r_l^{A_l} \geqslant h, \quad \text{for } l = l_k \ (k = 1, \dots, s;\ s < p),\ A_l = A_l(h) \tag{25.1}$$

$$r_l^{A_l} \leqslant h, \quad \text{for } l \neq l_k \ (k = 1, \dots, s;\ s < p),\ A_l = A_l(h)\,. \tag{25.2}$$

Under these assumptions we have the following estimate

$$r_l^M \leqslant \frac{\Omega(h) - \gamma(p-s)\cdot h}{1 + (s-1)\cdot\gamma}, \quad \text{for } x^M \in Q,\ l = l_k \ (k = 1, \dots, s;\ s < p), \tag{25.3}$$

$$r_l^M \leqslant h, \quad \text{for } x^M \in Q,\ l \neq l_k \ (k = 1, \dots, s;\ s < p), \tag{25.4}$$

the function $\Omega(h)$ *being by* (24.5) *and* (24.6).

Proof. We shall verify the first formula (25.3). For this purpose we shall use the estimate (20.10) in the following manner:

Since we have $\gamma < 0$, hence from (25.2) we obtain the following inequality for $l \neq l_k$ $(k = 1, \dots, s;\ s < p)$

$$-\gamma \cdot \sum_{l \neq l_1, \dots, l_s} r_l^{A_l} \leqslant -\gamma \cdot (p-s) \cdot h\,. \tag{25.5}$$

According to (25.5) and (20.10), we have

$$0 \leqslant r_l^{A_l} \leqslant \frac{\Omega(h) - \gamma(p-s)h}{1 + (s-1)\cdot\gamma}, \quad \text{for } l = l_k \ (k = 1, \dots, s;\ s < p)\,. \tag{25.6}$$

But $r_l^{A_l}$ $(l = 1, \dots, p)$ denotes the maximum value for r_l^M, $r_l^M \leqslant r_l^{A_l}$, for $x^M \in Q$, hence from 25.6) we obtain the desired estimate (25.3).

The formula (25.4) follows immediately from the inequality $r_l^M \leqslant r_l^{A_l}$ and from the assumption (25.6).

This ends the proof of the Lemma 9.

§ 26. A similar lemma can be proved for the maximum values $r_l^{B_l}$:

LEMMA 10. *Let us suppose that the functions* r_l^M $(l = 1, \dots, p)$ *satisfy the assumptions of the Lemma* 4, *cf.* § 12, *and the relation* (17.2) *for the function* $\varepsilon(h)$.

Let us suppose also that for the natural numbers λ_k, $1 \leqslant \lambda_k \leqslant p$, *we have*

(26.1) $$r_l^{B_l} \leqslant -h, \quad \text{for } l = \lambda_k \ (k = 1, \ldots, s;\ s < p),\ B_l = B_l(h),$$

(26.2) $$r_l^{B_l} \geqslant -h, \quad \text{for } l \neq \lambda_k \ (k = 1, \ldots, s;\ s < p),\ B_l = B_l(h).$$

Under these assumptions we have the following estimate:

(26.3) $$r_l^M \geqslant \frac{-\Omega(h) + \gamma(p-s)h}{1+(s-1)\cdot\gamma}, \quad \text{for } x^M \in Q,\ l = \lambda_k \ (k = 1, \ldots, s;\ s < p),$$

(26.4) $$r_l^M \geqslant -h, \quad \text{for } x^M \in Q,\ l \neq \lambda_k \ (k = 1, \ldots, s;\ s < p),$$

the function $\Omega(h)$ *being defined by* (24.5) *and* (24.6).

Proof. We shall use the estimate (21.10). Since $\gamma < 0$, hence from (26.2) we obtain for $l \neq \lambda_k$ $(k = 1, \ldots, s;\ s < p)$:

(26.5) $$-\gamma \cdot \sum_{l \neq \lambda_1, \ldots, \lambda_s} r_l^{B_l} \geqslant +\gamma \cdot (p-s)h.$$

From (26.5) and (21.10) it follows that

(26.6) $$0 \geqslant r_l^{B_l} \geqslant \frac{-\Omega(h) + \gamma(p-s)h}{1+(s-1)\cdot\gamma}, \quad \text{for } l \neq \lambda_k \ (k = 1, \ldots, s;\ s < p).$$

But $r_l^{B_l}$ denotes the minimum value for r_l^M, $r_l^M \geqslant r_l^{B_l}$, for $x^M \in Q$ $(l = 1, \ldots, p)$, therefore from (26.6) we obtain the desired formula (26.3).

The formula (26.4) follows immediately from the inequality $r_l^M \geqslant r_l^{B_l}$ and from the assumption (26.2).

This ends the proof of the Lemma 10.

§ 27. As indicated earlier, cf. § 25, the results of Lemma 9 and Lemma 10 can be applied to obtain an another error estimate. The question is settled in the following Theorem 7:

THEOREM 7. *Let us suppose that the systems* I *and* II *of difference inequalities, cf.* (4.8) *and* (4.9), *are of the elliptic type, cf. the definition* 1 (§ 4).

Let us suppose in addition that the functions r_l^M $(l = 1, \ldots, p)$ *satisfy the assumptions of the Lemma* 4, *cf.* § 12, *and the relation* (24.1) *for the function* $\varepsilon(h)$.

Let us suppose also that for the natural numbers l_k $(1 \leqslant l_k \leqslant p)$ *and* λ_k $(1 \leqslant \lambda_k \leqslant p)$ *we have:*

(27.1) $$\begin{cases} r_l^{A_l} \geqslant h, & \text{for } l = l_k;\ r_l^{A_l} \leqslant h, \quad \text{for } l \neq l_k, \\ r_l^{B_l} \leqslant -h, & \text{for } l = \lambda_k;\ r_l^{B_l} \geqslant -h, \quad \text{for } l \neq \lambda_k, \end{cases}$$

where $k = 1, \ldots, s;\ s < p$, $A_l = A_l(h)$, $B_l = B_l(h)$.

If in addition h is a sufficiently small positive number, i.e. h satisfies the condition (24.3), *then we have the following estimate*:

a$_1$) $|r_l^M| \leqslant \Omega_2(h)$, *for* $x^M \in Q$, *if* $l = l_k$, $l = \lambda_k$,

b$_1$) $-h \leqslant r_l^M \leqslant \Omega_2(h)$, *for* $x^M \in Q$, *if* $l = l_k$, $l \neq \lambda_k$,

c$_1$) $-\Omega_2(h) \leqslant r_l^M \leqslant h$, *for* $x^M \in Q$, *if* $l \neq l_k$, $l = \lambda_k$,

d$_1$) $-h \leqslant r_l^M \leqslant h$, *for* $x^M \in Q$, *if* $l \neq l_k$, $l \neq \lambda_k$,

the function $\Omega_2(h)$ *being defined by*

$$\Omega_2(h) = \frac{\Omega(h) - \gamma(p-s)h}{1+(s-1)\cdot\gamma}, \tag{27.2}$$

cf. the definition of $\Omega(h)$ (24.5) *and* (24.6).

The proof of the Theorem 7 follows immediately from Lemma 9, cf. § 25, and from Lemma 10, cf. § 26.

Remark 7. We wish to emphasize that we have in fact three error estimates. The best one follows from the linear system of algebraic inequalities (13.1) and (13.2), and does not involve any assumption of the form $r_l^{A_l} \geqslant h$ or $r_l^{B_l} \leqslant -h$. It is sufficient only to consider an auxiliary system of linear algebraic equations:

$$y_l + \gamma \cdot \sum_{k=1, k\neq l}^{p} y_k = +g^{A_l}(h) \quad (l = 1, \ldots, p), \tag{27.3}$$

the determinant of (27.3) being different from zero, cf. (17.21), and denote the solution of (27.3) by $y_l = y_l(h)$ $(l = 1, \ldots, p)$.

Then we have

$$r_l^{A_l} \leqslant y_l(h) \;\; (l = 1, \ldots, p),\;\; A_l = A_l(h), \tag{27.4}$$

the shape of the set S, cf. (14.3), being exploited in an essential way.

Similarly with the aid of the solution $y_l = \bar{y}_l(h)$ $(l = 1, \ldots, p)$ of the system

$$y_l + \gamma \cdot \sum_{k=1, k\neq l}^{p} y_k = -g^{B_l}(h) \quad (l = 1, \ldots, p), \tag{27.5}$$

we can write

$$r_l^{B_l} \geqslant \bar{y}_l(h) \quad (l = 1, \ldots, p),\;\; B_l = B_l(h). \tag{27.6}$$

Since we have $r_l^{B_l} \leqslant r_l^M \leqslant r_l^{A_l}$ $(l = 1, \ldots, p)$ for $x^M \in Q$, we obtain the estimate

$$\bar{y}_l(h) \leqslant r_l^M \leqslant y_l(h) \quad (l = 1, \ldots, p),\;\; x^M \in Q. \tag{27.7}$$

It is the best possible estimate, but unsatisfactory from the numerical point of view.

A better situation arises when we use the second estimate depending on the function $\Omega_1(h)$, cf. Theorem 6, § 24.

The third estimate depends on the function $\Omega_2(h)$, cf. Theorem 7, § 27. It should be observed in this connection that for small enough choices of h there can be $\Omega_2(h) \leqslant \Omega_1(h)$.

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