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Continuity Properties of Conditionally Positive Definite Functions on Linear Spaces

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The purpose of the present paper is the presentation of some generalization of some results of the author presented in [2]. Roughly speaking, we show that some continuity properties of functions of class $\mathrm{CP}(\boldsymbol{R}^m; S)$ (see the definition below) are preserved by the strict inductive limit passage of the spaces S.

Let S be a complex linear and $b_x(f, g)$, where $x \in \boldsymbol{R}^m$, a bilinear form in $f, g \in S$ i.e. for each $x \in \boldsymbol{R}^m$ $b_x(f, g) \in \boldsymbol{C}$ and $b_x(f, g)$ is linear in f and antilinear in g.

We say that $b = \{b_x(f, g)\}$ is weakly conditionally positive definite if the following property holds true:

For every $f \in S$, every $n = 1, 2, 3, \dots$ and $a_1 \dots a_n \in \boldsymbol{C}$ such that $\sum_{j|1}^{n} a_j = 0$ and arbitrary $x_1 \dots x_n \in R^m$, the inequality

$$\sum_{j,k|1}^{n} b_{x_j - x_k}(f, f) a_j \bar{a}_k \geqslant 0$$

holds true.

The weakly conditionally positive definite function $b = \{b_x(f, g)\}$ will be called of class $\mathrm{CP}(\boldsymbol{R}^m; S)$ if it satisfies the following three conditions:

(1) $\quad b_0(f,f) = 0 \quad$ for every $f \in S$.

(2) $\quad \overline{b_x(f,f)} = b_{-x}(f,f) \quad$ for $x \in \boldsymbol{R}^m$, $f \in S$.

(3) $\quad$ For every $f \in S$ $b_x(f,f)$ is continuous in x at $x = 0$.

Notice that by polarization formula (1) implies that $b_0(f, g) = 0$ for all $f, g \in S$. Condition (2) says that the function $(x, y) \to b_{x-y}(f,f)$ is hermitian symmetric. Also (3) implies that $b_x(f, g)$ is continuous in x all over the space $\boldsymbol{R}^m$ for arbitrary $f, g \in S$ — see [2].

The inner product of $x, y \in \boldsymbol{R}^m$ is denoted by $\langle x, y\rangle$ and the related norm by $|x|$. The spectral form of generalized Levy-Khintchine formula for $b = \{b_x(f, g)\}$ of class $\mathrm{CP}(\boldsymbol{R}^m; S)$ is the following one (see [2] for the proof):

(LK) $\quad b_x(f, g) = u(f, g; x) - \frac{1}{2}G(f, g: x, x) + P(f, g; x)$

where the functions u, G and P are determined in the unique way by b. These functions have the following properties:

(4) $u(f, g; x)$ is bilinear in f, g and real linear in x;

(5) $G(f, g; x, y)$ is bilinear in f, g, real linear in x and y and symmetric in x, y i.e. $G(f, g; x, y) = G(f, g; y, x)$ and for every $f \in S$ the function $(x, y) \to G(f, f; x, y)$ is positive definite;

(6) $P(f, g; x)$ has the unique, up to unitary equivalence representation

$$P(f, g; x) = \int_{R^m} \left(e^{i\langle x, y \rangle} - 1 - \frac{i\langle x, y \rangle}{1+|y|^2} \right) \frac{1+|y|^2}{|y|^2} \, d(E_y Rf, Rg)$$

where E is a spectral measure on the Borel sets of $\boldsymbol{R}^m$ in some Hilbert space K (with the inner product $(\cdot, \cdot)$), vanishing on the singleton $\{0\}$ i.e. at the zero vector of $\boldsymbol{R}^m$, R: $S \to K$ is a linear operator and K is spanned by $E(\sigma)RS$, where σ are Borel sets of $\boldsymbol{R}^m$.

In connection with (6), we recall that the uniqueness up to unitary equivalence means that if K_α is some other Hilbert space with inner product $(\cdot, \cdot)'$ and E' a spectral measure on Borel sets of $\boldsymbol{R}^m$, R': $S \to K'$ a linear operator and moreover K' is spanned by sets $E'(\sigma)R'S$, and $(E(\sigma)Rf, g) = (E'(\sigma)R'f, g)'$ for all σ, f, g then there is a unitary map U: $K \to K'$ which establishes the unitary equivalence of E and E' and moreover $UR = R'$.

The $u(f, g; x)$ is called the elementary part of b, G the Gaussian part of b and P the Poisson one.

Let S be a locally convex linear space and $b = \{b_x(f, g)\}$ of class $\mathbb{C}$P$(\boldsymbol{R}^m; S)$ with (LK) representation. We say that b is of class (UC) $(\boldsymbol{R}^m; S)$ if the following implication holds true:

If $b_x(f, g)$ is jointly continuous in $f, g \in S$ for each $x \in \boldsymbol{R}^m$ then:

(a) The operator R: $S \to K$ is continuous;

(b) $u(f, g; x)$ is jointly continuous in f, g uniformly in x for x varying over an arbitrary compact subset of $\boldsymbol{R}^m$.

(c) $G(f, g; x, y)$ is jointly continuous in f, g uniformly in $(x, y) \in \boldsymbol{R}^m \times \boldsymbol{R}^m$ for (x, y) varying over an arbitrary compact set.

Using the arguments as in [2] (8.4), (8,5), one proves easily what follows:

(P) If R: $S \to K$ is continuous, then the Poisson part $P(f, g; x)$ is jointly continuous in f, g, uniformly in x on every compact subset of $\boldsymbol{R}^m$.

We now recall some properties of strict inductive limits of sequences of locally convex spaces. We refer in this matter to [1].

Suppose we are given the sequence $\{S_n\}$ of locally convex complex spaces such that $S_n \subset S_{n+1}$ for all n and let $S = \bigcup_{n|1}^{\infty} S_n$.

We suppose that the identity embedding id_n: $S_n \to S_{n+1}$ is a strict morphism for every n. Let τ be the finest locally convex topology in S for which id'_n: $S_n \to S$ is continuous for every n. This topology is called the topology of the strict inductive limit of $\{S_n\}$ and the fact that S is considered as a locally convex space with this topology is shortly written as $\operatorname{limind} S_n = S$. By Schwartz-Dieudonné theorem ([1], p. 159) τ induces in each S_n the initial locally convex topology of S_n itself. The following lemma is now a consequence of Prop. 1, [1], p. 159.

LEMMA 1. *If $S = \operatorname{limind} S_n$ and Z is a locally convex space, $R\colon S \to Z$ a linear operator such that for every n its restriction R_n to S_n is continuous, then R is continuous.*
Next we will prove the following:

LEMMA 2. *Suppose $S = \operatorname{limind} S_n$ and let $b(f, g)$ be a bilinear form on $S \times S$. If for every n the restriction b_n of b to the topological product $S_n \times S_n$ is jointly continuous, then $b(f, g)$ is jointly continuous in the product topology of $S \times S$.*

Proof: Let $\varepsilon > 0$ and define

$$N = \{(f, g) \in S \times S\colon |b(f, g)| < \varepsilon\}$$

and

$$N_n = \{(f, g) \in S_n \times S_n\colon |b(f, g)| < \varepsilon\} = (\mathrm{id}_n' \times \mathrm{id}_n')^{-1}(N) .$$

By our assumption N_n is open in the product topology of $S_n \times S_n$. Moreover, N_n is balanced and convex, and consequently N, being absorbing, balanced and convex is open in $\operatorname{limind}(S_n \times S_n)$. On the other hand the product locally convex topology ξ of $S \times S = (\operatorname{limind} S_n) \times (\operatorname{limind} S_n)$ has the following property: for every n $\mathrm{id}_n' \times \mathrm{id}_n'\colon S_n \times S_n \to S \times S$ is a continuous embedding. It follows that ξ is finer than the topology of $\operatorname{limind}(S_n \times S_n)$. Hence N is open in the ξ topology, which completes the proof, because b being continuous at zero of $S \times S$ is continuous — see [1], Prop. 1. p. 356. Our basic theorem is the following one (we use the (LK) notation):

THEOREM 1. *Suppose that S_n are locally convex spaces and $S = \operatorname{limind} S_n$ makes sense. Then if $b = \{b_x(f, g)\}$ is of class* CP$(\boldsymbol{R}^m;\ S)$ *the following implication holds true: the condition*

(a)′ *The restriction $b^{(n)}$ of b to $S_n \times S_n$ is of class* (UC)$(\boldsymbol{R}^m;\ S_n)$ *for each n,*

implies that

(b)′ *b is of class* (UC)$(\boldsymbol{R}^m;\ S)$.

Proof: (b)′ includes an implication. Suppose (a)′ holds true and $b_x(f, g)$ is jointly continuous in $f, g \in S$ for $x \in \boldsymbol{R}^m$. Then $b_x(f, g)$ is jointly continuous in $f, g \in S_n$, $x \in \boldsymbol{R}^m$ for each n. By Lemma 1 and (a)′ $R\colon S \to K$ is a continuous operator. Next, since $b^{(n)} \in$ (UC) $(\boldsymbol{R}^m;\ S_n)$ (any n), $u(f, g;\ e_p)$ is jointly continuous in $f, g \in S_n$; e_p $(p = 1, \ldots, m)$ form a basis for $\boldsymbol{R}^m$. By Lemma 2 and (a)′ it follows then that $u(f, g;\ e_p)$ $(p = 1, \ldots, m)$ are jointly continuous in $f, g \in S$.

Let $\varepsilon > 0$ and suppose that $x = \sum_{j|1}^{m} x_j e_j$ and $|x_j| \leqslant \eta$, $\eta > 0$ for $j = 1, \ldots, m$, η arbitrary. Let N be a neighbourhood of $(0, 0) \in S \times S$ which is the intersection of neighbourhoods

$\left\{(f, g)\colon |u(f, g; e_p)| < \dfrac{\varepsilon}{m(1+\eta)}\right\}$. It follows that if $(f, g) \in N$ then

$$|u(f, g; x)| \leqslant \sum_{j|1}^{m} |x_j|\,|u(f, g; e_j)| \leqslant \sum_{j|1}^{m} |x_j| \frac{\varepsilon}{m(1+\eta)}$$

if $|x_j| \leqslant \eta$ for $j = 1, \ldots, m$ which proves that u satisfies (b) of the definition of class $(\mathrm{UC})(\boldsymbol{R}^m; S)$. By similar token when using Lemma 2 we show that G satisfies (c) of this definition, which completes the proof.

It follows from Thm. 8.1 of [2] that if S is a metric linear space, then the function $b = \{b_x(f, g)\}$ of class $\mathrm{CP}(\boldsymbol{R}^m; S)$ which is jointly continuous in f, g for every $x \in \boldsymbol{R}^m$, is in class $(\mathrm{UC})(\boldsymbol{R}^m; S)$. We get therefore by using Theorem 1 and Lemma 2 the following theorem.

THEOREM 2. *Let $\{S_n\}$ be an increasing sequence of linear metric locally convex spaces such $S = \operatorname{limind} S_n$ makes sense. If the function $b = \{b_x(f, g)\} \in \mathrm{CP}(\boldsymbol{R}^m; S)$ is jointly continuous in $f, g \in S_n$ for $x \in \boldsymbol{R}^m$, for each n, then b is jointly continuous in $f, g \in S$ and of class* $(\mathrm{UC})(\boldsymbol{R}^m; S)$, *and consequently its Poisson $P(f, g; x)$ part is jointly continuous in $f, g \in S$, uniformly in x varying on an arbitrary compact in $\boldsymbol{R}^m$, and u and G satisfy* (b) *and* (c) *respectively.*

COROLLARY. *If S_n are F-spaces i.e. complete metric locally convex spaces with translation invariant metrics, then Th. 2 holds true if $b_x(f, g)$ is separately continuous in $f, g \in S_n$ for every n.*

The classical model of spaces appearing as strict inductive limits of S_n such as in Theorem 2 are the $\mathscr{D}$ spaces of Schwartz. Suppose namely that Ω is an open subset of R^q. Let K be a compact set such that $K \subset \Omega$ and consider the totality $\mathscr{D}(K)$ of all $C^{\infty}(\Omega)$ complex functions with supports included in K. With usual algebraic operations $\mathscr{D}(K)$ becomes a linear complex space. We define the multiindex $p \cdot = (p_1 \ldots p_q)$ (p_j natural numbers) and $|p| = \sum_{j|1}^{q} p_j$. For $f \in \mathscr{D}(K)$ we define $(D^p f)(x) = \left(\dfrac{\partial^{|p|}}{\partial^{p_1} x_1, \ldots, \partial^{p_n} x_n} f\right)(x)$ and the seminorm $S_{p,K}(f) = \sup_{x \in K} |D_p f)(x)|$. Let K_n be a sequence such that K_n is a compact included in Ω, $K_n \subset K_{a+1}^{0}$ = interior of K_{n+1}, and every compact $K \subset \Omega$ is included in some K_n. Let $\mathscr{D}(K_n)$ be the Fréchet space with metric

$$\varrho_n(f, g) = \sum_{j|0}^{\infty} \frac{1}{2^j} \frac{S_{j,K_n}(f-g)}{1+S_{j,K_n}(f-g)}.$$

The identity embedings $\mathrm{id}_n\colon \mathscr{D}(K_n) \to \mathscr{D}(K_{n+1})$ are continuous morphisms. Suppose that $\Omega = \bigcup_{n|1}^{\infty} K_n$. We define $\mathscr{D}(\Omega) = \operatorname{limind} \mathscr{D}(K_n)$. Since every compact $K \subset \Omega$ is included in some K_n, every space $\mathscr{D}(K)$ is included in $\mathscr{D}(\Omega)$. Since $\mathscr{D}(\Omega) = \bigcup_{n|1}^{\infty} \mathscr{D}(K_n)$ we get that the set theoretic equality $\bigcup_{K} \mathscr{D}(K) = D(\Omega)$, ($K$ compact). Moreover, the topology of $\mathscr{D}(\Omega)$ is independent of the choice of K_n and is the finest locally convex topology for which all the canonical injections $\mathscr{D}(K) \to \mathscr{D}(\Omega)$ are continuous for compacts $K \subset \Omega$, when we equip $\mathscr{D}(K)$ with topology defined by $\varrho_K(f, g) = \sum_{n|0}^{\infty} \frac{1}{2^n} \frac{S_{n,K}(f-g)}{1+S_{n,K}(f-g)}$. $\mathscr{D}(\Omega)$ is the Schwartz space of test functions and continuous linear functionals on $\mathscr{D}(\Omega)$ are just the distributions over Ω.

It follows now from our Theorem 2, having in view Lem. 1,2, that, since $\mathscr{D}(\Omega)$ is a strict inductive limit of Fréchet spaces, the following theorem holds true:

THEOREM 3. *Let $b_x(f, g)$ be of class* CP$(\boldsymbol{R}^m, \mathscr{D}(\Omega))$ *such that for every $x \in \boldsymbol{R}^m$ and every compact $K \subset \Omega$ $b_x(f, g)$ is separately continuous in $f, g \in \mathscr{D}(K) \times \mathscr{D}(K)$. Then $b_x(f, g)$ jointly continuous on $\mathscr{D}(\Omega) \times \mathscr{D}(\Omega)$ and is of class* (UC)$(\boldsymbol{R}^m; \mathscr{D}(\Omega))$, *hence u satisfies* (b), *G satisfies* (c) *and P the property of* (P).

The equivalent formulation of the above theorem is the following:

THEOREM 4. *Suppose the function $b = \{b_x(f, g)\}$ is of class* CP$(\boldsymbol{R}^m; \mathscr{D}(\Omega))$. *If $b_x(f, g)$ separately continuous in $f, g \in \mathscr{D}(\Omega)$, then b is jointly continuous in $f, g \in \mathscr{D}(\Omega)$, is of class* (UC)$(\boldsymbol{R}^m; \mathscr{D}(\Omega))$ *and consequently b satisfies* (a) *and* (b), *G satisfies* (c), *the Poisson part $P(f, g; x)$ is jointly continuous in $f, g \in \mathscr{D}(\Omega)$, uniformly in x on compact subsets of $\boldsymbol{R}^m$.*

References

[1] J. Horváth, *Topological vector spaces and distributions*, Vol. I Addison-Wesley P. Comp. London 1966.
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