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DCLXI UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XXIV 1984

Some Remarks on Processes III

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We give theorem which says that for a limit pseudoprocess v of the pseudoprocess generated by an ordinary differential equation $x' = f(t, x)$ there is a differential equation $x' = g(t, x)$ generating v. This is an inverse problem of the problem discussed in [5]. We shall use the notation and terminology from [3], [4], [5].

Let $f\colon R \times R^n \to R^n$ be a continuous mapping. We consider a differential equation

$$x' = f(t, x)\,. \tag{1}$$

We assume that for every $(t_0, x_0) \in R \times R^n$ there is the solution $\varphi(t_0, x_0, t)$ of equation (1) with the initial condition $x(t_0) = x_0$, defined for $t \in [t_0, \infty)$. We define the function $u\colon R \times R^n \times R_* \to R^n$ by $u^t(x, \tau) = \varphi(t, x, t+\tau)$ (we will use the notation $u^t(x, \tau)$ instead $u(t, x, \tau)$).

In the case if the problem $x' = f(t, x)$, $x(t_0) = x_0$ has exactly one solution defined for $t \in [t_0, \infty)$, for all $(t_0, x_0) \in R \times R^n$ we call u a *pseudoprocess* generated by the equation (1) (see [3], [4], [5]). For $T \in R$ we define the function $u_T\colon R \times R^n \times R_* \ni (t, x, \tau) \to u^{t+T}(x, \tau) \in R$. If u is the pseudoprocess, u_T is said to be a *T-translation* of u.

THEOREM. *Suppose that the mapping f is bounded and uniformly continuous. Let $v\colon R \times R^n \times R_* \to R^n$ be a function such that for some sequence $\{T_n\}$, $T_n \in R$ we have $u_{T_n} \to v$, as $n \to \infty$ at every point $(t, x, \tau) \in R \times R^n \times R_*$. Then there exists a continuous function $g\colon R \times R^n \to R^n$ and a solution $\psi(t_0, x_0, t)$, $t \in [t_0, \infty)$ of the initial problem*

$$\begin{aligned} x' &= g(t, x) \\ x(t_0) &= x_0 \end{aligned} \tag{2}$$

such that $v^t(x, \tau) = \psi(t, x, t+\tau)$ for $(t, x, \tau) \in R \times R^n \times R_*$ and $f(t+T_n, x) \to g(t, x)$ as $n \to \infty$, uniformly on compact sets in $R \times R^n$.

If we assume additionally that f fulfils the local Lipschitz condition with respect to x, then u and v are the pseudoprocesses obtained from the differential equations (1) *and* (2), *respectively, and in the case $T_n \to \infty$, v is the limit pseudoprocess of u (see* [5], [1], [2]).

First we shall prove the following:

LEMMA. *Let $p_n\colon R \to R$ for $n = 1, 2, \ldots$, be a sequence of the differentiable functions such that for some function $p\colon R \to R$*

$$p_n(s) \to p(s), \quad \text{as } n \to \infty$$

at each point $s \in R$. If the sequence $\{p'_n\}_{n=1,2,\ldots}$ is a commonly bounded and uniformly equicontinuous family of the functions, then the derivative p' exists, moreover $p'_n \to p'$ uniformly on compact sets in R.

Proof. The sequence $\{p'_n\}$ fulfils the assumptions of the Arzeli Theorem, therefore there is a subsequence $\{p_{n_k}\}$ and a function q such that $p'_{n_k} \to q$ uniformly on compact sets in R. Hence, by the known theorem, the derivative p' exists and $p' = q$. By the same argument we can show that for every sequence $\{l_k\}$ there exists a subsequence $\{l_{n_k}\}$ such that $p'_{l_{n_k}} \to q$ uniformly on compact sets in R. The Lemma is proved.

Now we prove the Theorem. Because $\varphi(t_0, x_0, t)$, $t \geqslant t_0$ is the solution of (1) with the initial condition $x(t_0) = x_0$, then we have

$$\frac{\delta u^{t_0}(x_0, \tau)}{\partial \tau} = f(t_0+\tau, \varphi(t_0, x_0, t_0+\tau)).$$

We put

$$p_n(\tau) = u^{t_0+T_n}(x_0, \tau).$$

Since f is a bounded function, the function $\varphi(t_0, x_0, t)$ fulfils the Lipschitz condition with the same constant, with respect to t. Thus by the uniform continuity of f and previous property of the function $\varphi(t_0, x_0, t)$ we obtain that the sequence $\{p'_n\}$ is uniformly equicontinuous family. Because f is bounded, the family $\{p'_n\}$ is commonly bounded. Now, using Lemma we see that there exists the derivative $\dfrac{\partial v^{t_0}(x_0, \tau)}{\partial \tau}$ and

$$\frac{\partial u^{t_0+T_n}(x_0, \tau)}{\partial \tau} \to \frac{\partial v^{t_0}(x_0, \tau)}{\partial \tau}$$

uniformly on compact sets in R. We define

$$g(t, x) = \left.\frac{\partial v^t(x, \tau)}{\delta \tau}\right|_{\tau=0}.$$

It is obvious that $f(t+T_n, y) \to g(t, y)$, as $n \to \infty$, for each $(t, y) \in R \times R^n$. Moreover, by the Arzeli Theorem this convergence is uniform on compact sets in $R \times R^n$. Then g is a continuous and g fulfils the local Lipschitz condition while f its fulfils. By inequality

$$\begin{aligned} &|f(t_0+T_n, u^{t_0+T_n}(x_0, \tau)) - g(t_0, v^{t_0}(x_0, \tau))| \\ &\quad \leqslant |f(t_0+T_n, u^{t_0+T_n}(x_0, \tau)) - f(t_0+T_n, v^{t_0}(x_0, \tau))| + \\ &\qquad + |f(t_0+T_n, v^{t_0}(x_0, \tau)) - g(t_0, v^{t_0}(x_0, \tau))| \end{aligned}$$

and by the uniform continuity of the function f we can pass to the limit in the equality

$$\frac{\partial u^{t_0+T_n}(x_0, \tau)}{\partial \tau} = f(t_0+T_n, u^{t_0+T_n}(x_0, \tau))$$

as $n \to \infty$. Hence the function

$$\psi(t_0, x_0, t) = v^{t_0}(x_0, t-t_0), \qquad t \geqslant t_0$$

is a solution of the problem (2). The proof is completed.

References

[1] C. M. Dafermos, *An invariance principle for compact processes*, J. Differential Equations 9 (1971), 239-252.

[2] —, *Uniform processes and semicontinuous Liapunov functionals*, J. Differential Equations 11 (1972), 401-415.

[3] J. Ombach, A. Trzepizur, *Some remarks on processes*, Zesz. Nauk. Uniw. Jagiell. Prace Matem. 16 (1974).

[4] A. Pelczar, *Stability Question in Pseudoprocesses and Generalized Pseudodynamical Systems*, Bull. Acad. Polon. Sci. Sér. Math. Astronom. Phys. 21 (1973), 541-549.

[5] A. Trzepizur, J. Ombach, *Some remarks on processes II*, in press.

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Received September 1, 1980.