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DXXX PRACE MATEMATYCZNE, ZESZYT 21 1979

Continuity of invariant measures for Rényi's transformations

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1. Introduction. In this note we show that for a transformation τ: $[0, 1]\to[0, 1]$ considered by A. Rényi [9] there exists a sequence of transformations τ_N: $[0, 1]\to[0, 1]$ for which the invariant measure μ_N is given by a solution of a linear equation and the sequence of densities f_N of these measures is uniformly convergent to a density of the invariant measure under τ. This theorem gives us a computational method for finding an invariant measure under τ.

In Section 2 we recall some basic definitions and state the main theorem. In Section 3 we prove some necessary lemmas and theorem.

2. Denote by $(L^1, \|\ \|)$ the space of all integrable functions defined on the interval $[0, 1]$. The Lebesgue measure on $[0, 1]$ will be denoted by m.

Let τ: $[0, 1]\to[0, 1]$ be a measurable nonsingular transformation, that is, $m(\tau^{-1}(E)) = 0$ whenever $m(E) = 0$ for a measurable set E. Given τ we define the Frobenius–Perron operator P_τ: $L^1\to L^1$ by the formula

$$P_\tau f(x) = \frac{d}{dx} \int\limits_{\tau^{-1}([0,x])} f(s)ds .$$

It is well known that the operator P_τ is linear and continuous and satisfies the following conditions

(a) P_τ is positive: $f\geqslant 0\Rightarrow P_\tau f\geqslant 0$,

(b) P_τ preserves integrals

$$\int\limits_0^1 P_\tau f dm = \int\limits_0^1 f dm, \quad f\in L^1 ,$$

(c) $P_{\tau^n} = P_\tau^n$ (τ denotes the n-th iterate of τ),

(d) $P_\tau f = f$ if and only if the measure $d\mu = f dm$ is invariant under τ, that is $\mu(\tau^{-1}(E)) = \mu(E)$ for each measurable E.

We shall not make a distinction between functions f: $[0, 1]\to R$ defined on $[0, 1]$ and functions f: $[0, 1]\to R$ taken as elements of the space L^1. This difference will become clear in the context.

Denote by $\tau|_E$ the restriction of τ to the set $E\subset[0, 1]$.

A transformation τ: $[0, 1]\to R$ will be called piecewise C^2, if there exists a partition $0 = a_1 < a_2 < \ldots < a_k = 1$ of the unit interval such that for each integer i ($i = 1, 2, \ldots, k$) the restriction $\tau|_{(a_{i-1}, a_i)}$ is a C^2 function which can be extended to the closed interval $[a_{i-1}, a_i]$ as a C^2 function. τ need not be continuous at the points a_i.

If transformation τ: $[0, 1]\to[0, 1]$ satisfies the following conditions

(e) There exists a partition $0 = a_0 < a_1 < \ldots < a_k = 1$ of the unit interval such that for each integer l ($l = 1, 2, \ldots, k$) the restriction τ_l of τ to the open interval (a_{l-1}, a_l) is a continuous function which can be extended to the closed interval $[a_{l-1}, a_l]$ as a continuous and bijective map of interval $[a_{l-1}, a_l]$ onto $[0, 1]$.

(f) There exists p and a partition $0 = b_0^p < b_1^p < \ldots < b_{k^p}^p = 1$ of interval $[0, 1]$ such that

$$\tau^{-p+1}(\{a_0, a_1, \ldots, a_k\}) = \{b_0^p, b_1^p, \ldots, b_{k^p}^p\}$$

and satisfies the identities

$$\tau(x) = \frac{\tau(b_{i-1}^p)-\tau(b_i^p)}{b_{i-1}^p - b_i^p}(x - b_{i-1}^p) + \tau(b_{i-1}^p) \qquad \text{for } x \in (b_{i-1}^p, b_i^p) \qquad i = 1, 2, \ldots, k^p,$$

then for that transformation we can give the definition of a matrix $A = (a_{ij})$ $i, j = 1, 2, \ldots, k^p$ by formulas

$$\text{(1)} \qquad a_{ij} = \frac{1}{|\tau'(x)|} \qquad \text{for } x \in \tau_l^{-1}((b_{j-1}^p, b_j^p))$$

if there exists $l \in \{1, 2, \ldots, k\}$ such that

$$\tau_l^{-1}((b_{j-1}^p, b_j^p)) \subset (b_{i-1}^p, b_i^p).$$

$$\text{(2)} \qquad a_{ij} = 0$$

if for any $l \in \{1, 2, \ldots, k\}$

$$\tau_l^{-1}((b_{j-1}^p, b_j^p)) \cap (b_{i-1}^p, b_i^p) = \emptyset,$$

where

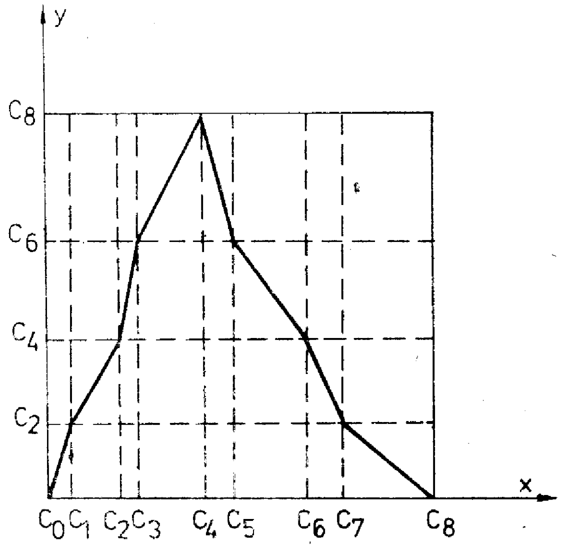
$$\tau_l = \tau_{(a_{l-1}, a_l)} \quad \text{and } 0 = b_0^p < b_1^p < \ldots < b_{k^p}^p = 1$$

the partition of $[0, 1]$ such that

$$\tau^{-p+1}(\{a_0, a_1, \ldots, a_k\}) = \{b_0^p, b_1^p, \ldots, b_{k^p}^p\}.$$

To illustrate this definition we take for example $k = 2$, $p = 2$ and the transformation given by the figure, where $a_0 = c_0$, $a_1 = c_4$, $a_2 = c_8$, $b_0^2 = c_0$, $b_1^2 = c_2$, $b_2^2 = c_4$, $b_3^2 = c_6$, $b_4^2 = c_8$ and $b_0^3 = c_0$, $b_1^3 = c_1$, $b_2^3 = c_2$, $b_3^3 = c_3$, $\ldots$, $b_8^3 = c_8$. For this transformation matrix A is given by

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ a_{41} & a_{42} & 0 & 0 \end{bmatrix}.$$



where

$$a_{11} = \frac{1}{|\tau'(x)|} \quad \text{for } x \in (c_0, c_1), \qquad a_{12} = \frac{1}{|\tau'(x)|} \quad \text{foi } x \in (c_1, c_2),$$

$$a_{23} = \frac{1}{|\tau'(x)|} \quad \text{for } x \in (c_2, c_3), \qquad a_{24} = \frac{1}{|\tau'(x)|} \quad \text{for } x \in (c_3, c_4),$$

$$a_{34} = \frac{1}{|\tau'(x)|} \quad \text{for } x \in (c_4, c_5), \qquad a_{33} = \frac{1}{|\tau'(x)|} \quad \text{for } x \in (c_5, c_6),$$

$$a_{42} = \frac{1}{|\tau'(x)|} \quad \text{for } x \in (c_6, c_7), \qquad a_{41} = \frac{1}{|\tau'(x)|} \quad \text{for } x \in (c_7, c_8).$$

Let τ: $[0, 1] \to [0, 1]$ be a piecewise C^2 function for which there exists a partition $0 = a_0 < a_1 < \ldots < a_k = 1$ of the unit interval such that for each integer l ($l = 1, 2, \ldots, k$) the restriction τ_l of τ to the open interval (a_{l-1}, a_l) is a C^2 function which can be extended to the closed interval $[a_{l-1}, a_l]$ as a C^2 and bijective map of interval $[a_{l-1}, a_l]$ onto $[0, 1]$. Let

$$0 = b_0^N < b_1^N < \ldots < b_{k^N}^N = 1 \quad \text{and} \quad 0 = b_0^{N+1} < b_1^{N+1} < \ldots < b_{k^{N+1}}^{N+1} = 1$$

be new partitions of interval $[0, 1]$ such that

$$\tau^{-N+1}(\{a_0, a_1, \ldots, a_k\}) = \{b_0^N, b_1^N, \ldots, b_{k^N}^N\}$$

and

$$\tau^{-N}(\{a_0, a_1, \ldots, a_k\}) = \{b_0^{N+1}, b_1^{N+1}, \ldots, b_{k^{N+1}}^{N+1}\}.$$

Denote by τ_N transformations of [0, 1] in [0, 1] given by the formulas

$$(3) \qquad \tau_N(x) = \frac{\tau(b_{i-1}^{N+1}) - \tau(b_i^{N+1})}{b_{i-1}^{N+1} - b_i^{N+1}} (x - b_{i-1}^{N+1}) + \tau(b_{i-1}^{N+1})$$

for $x \in (b_{i-1}^{N+1}, b_i^{N+1})$ and

$$(3') \qquad \tau_N(x) = \tau(x) \quad \text{for } x \in \{b_0^{N+1}, b_1^{N+1}, \ldots, b_{k^{N+1}}^{N+1}\}.$$

Transformation τ_N satisfy the conditions (e) and (f), therefore by formulas (1) and (2) a matrix which we will denote by A_N corresponds to τ_N.

Let A' denote the transpose of the matrix A.

THEOREM 1. *If transformation τ: $[0, 1] \to [0, 1]$ is a piecewise C^2 function for which there exists a partition $0 = a_0 < a_1 < \ldots < a_k = 1$ of the $[0, 1]$ such that for each integer l ($l = 1, 2, \ldots, k$) $\tau_l = \tau|_{(a_{l-1}, a_l)}$ is a C^2 function which can be extended to the closed interval $[a_{l-1}, a_l]$ as a C^2 and bijective map of interval $[a_{l-1}, a_l]$ onto $[0, 1]$ and $s = \min_{x \in [0,1]} |\tau'(x)| > 1$ then*

(i) *for any $N = 1, 2, \ldots$ there exists exactly one function $f_N \in L^1$ such that $\|f_N\| = 1$, $f_N \geq 0$ and measure $d\mu_N = f_N dm$ is invariant under τ_N, where τ_N is given by formulas* (3), (3'),

(ii) *function f_N is constant on the interval (b_{i-1}^N, b_i^N) $i = 1, 2, \ldots, k^N$ and vector $y^N = (y_1^N, y_2^N, \ldots, y_{k^N}^N)$ $y_i^N = f_N(x)$ for $x \in (b_{i-1}^N, b_i^N)$ is a solution of linear equation*

$$A_N' y = y,$$

(iii) *the sequence of functions f_N is uniformly convergent to a continuous function $f \in L^1$,*

(iv) *measure $d\mu = f dm$ is invariant under τ.*

In the case when the transformation τ is given by function

$$\tau(x) = \varphi(x) (\operatorname{mod} 1),$$

where $\varphi(x)$ is a bijective map of [0, 1] onto $[0, n]$, $n \in \{N \cup \infty\}$ the continuity of function f has also been proved by M. Halfant (see [13]).

3. Now we give the lemmas and theorems which we shall use in proof of Theorem 1.

We say that the matrix $A = (a_{ij}) \geq 0$ $i, j = 1, 2, \ldots, m$ if $a_{ij} \geq 0$ for $i, j = 1, 2, \ldots, m$ and $A > 0$ if $a_{ij} > 0$ for $i, j = 1, 2, \ldots, m$.

LEMMA 1. *If for matrix $A = (a_{ij}) \geq 0$ $i, j = 1, 2, \ldots, m$ there exists p such $A^p > 0$ (A^p denotes the p-th iterate of A) and there exists a vector $\alpha \in R^m$, $\alpha > 0$ ($\alpha_i > 0$ $i = 1, 2, \ldots, m$) such that $A\alpha = \alpha$ then*

(i) *there exists exactly one vector $y \in R^m$ such that*

$$A'y = y$$

and

$$\sum_{i=1}^m \alpha_i y_i = \sum_{i=1}^m \alpha_i,$$

(ii) *there exists* $\lim_{p\to\infty} A^p = B>0$ $(B = (b_{ij}))$,

(iii) *vector y can be given by the formula*

$$y_j = \sum_{i=1}^{m} b_{ij}\,, \quad j = 1, 2, \ldots, m\,.$$

The proof of this lemma is given in [1].

It is easy to verify that the following is valid

LEMMA 2. *Let matrices* $B_r = (b^r_{ij})\subset 0$ $i, j = 1, 2, \ldots, k^n$ $r = 1, 2, \ldots, k$ *satisfy one of the two conditions*

$$\text{(g)}\ \begin{cases} b^r_{ij}>0 & \text{if} \quad ((i-1)(\mathrm{mod}\,k^{n-1}))k<j\leqslant(i(\mathrm{mod}\,k^{q-1}))k \ \text{ for } i \text{ such that} \\ & \qquad (r-1)k^{n-1}<i<rk^{n-1}; \\ b^r_{ij}>0 & \text{if} \quad ((i-1)(\mathrm{mod}\,k^{n-1}))k<j\leqslant k^n \quad \text{for} \quad i = rk^{n-1}, \\ b^r_{ij} = 0 & \text{if} \quad i<r-1k^{n-1} \quad \text{or} \quad i\geqslant rk^{n-1}; \end{cases}$$

$$\text{(h)}\ \begin{cases} b^r_{ij}>0 & \text{if} \quad ((i-1)(\mathrm{mod}\,k^{n-1}))k\leqslant k^n-j<(i(\mathrm{mod}\,k^{n-1}))k \ \text{ for } i \ \text{ such that} \\ & \qquad (r-1)k^{n-1}<i<rk^{n-1}, \\ b^r_{ij}>0 & \text{if} \quad ((i-1)(\mathrm{mod}\,k^{n-1}))k\leqslant k^n-j<(i(\mathrm{mod}\,k^{n-1}))k \ \text{ for } \quad i = rk^{n-1}, \\ b^r_{ij} = 0 & \text{if} \quad i<(r-1)k^{n-1} \quad \text{or} \quad i\geqslant rk^{n-1}. \end{cases}$$

If matrix $A = B_1+B_2+\ldots+B_k$ *then there exists p such that* $A^p>0$.

Example. Let $A = B_1+B_2+B_3$, where

$$B_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{24} & a_{25} & a_{26} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{37} & a_{38} & a_{39} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdots & & & & & & & 0 \\ 0 & \cdots & & & & & & & 0 \\ 0 & \cdots & & & & & & & 0 \\ 0 & \cdots & & & & & & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdots & & & & & & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{47} & a_{48} & a_{49} \\ 0 & 0 & 0 & a_{54} & a_{55} & a_{56} & 0 & 0 & 0 \\ a_{61} & a_{62} & a_{63} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdots & & & & & & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{71} & a_{72} & a_{73} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{84} & a_{85} & a_{86} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{97} & a_{98} & a_{99} \end{bmatrix}$$

for $n = 2$. If $B_i \geqslant 0$ for $i = 1, 2, 3$ then $A^p > 0$ for $p = 2$.

We write "$A \subset B$ a.e." if $A, B \subset R$ and $x \in B$ for almost all x in A. We write "$A = B$ a.e." if both $A \subset B$ a.e. and $B \subset A$ a.e. are satisfied.

Let τ: $[0, 1] \to [0, 1]$. We say a set A is invariant under τ if A is a measurable subset of $[0, 1]$ and $\tau(A) = A$ a.e.

Let f be a function on $[0, 1]$. We call the set on which the function f is non-zero, the support of f and denote it supp$f$. Notice that supp$f$ need not be closed in our definition.

THEOREM (Yorke–Li). *If τ: $[0, 1] \to [0, 1]$ is a piecewise C^2 function with $\{x_1, x_2, \dots, x_k\}$, the points of discontinuity of τ and τ' and $\inf_{x \in [0,1]} |\tau'(x)| > 1$ then there exists a finite collection of sets $L_1, L_2, \dots, L_n$ and a set of functions $\{f_1, f_2, \dots, f_k\} \subset L^1$ such that:*

(i) *each L_i $(1 \leqslant i \leqslant n)$ is a finite union of closed intervals and $\tau(L_i) = L_i$,*

(ii) *$L_i \cap L_j$ contains at most a finite number of points when $i \neq j$,*

(iii) *each L_i $(i = 1, 2, \dots, n)$ contains at least one point of discontinuity x_j $(j = 1, 2, \dots$ $\dots, k)$ in its interior, hence $n \leqslant k$,*

(iv) *$f_i(x) = 0$ for $x \notin L_i$ $i = 1, 2, \dots, n$ and $f_i(x) > 0$ for almost all $x \in L_i$,*

(v) *$\int_{L_i} f_i(x)dx = 1$ for $i = 1, 2, \dots, n$,*

(vi) *if g satisfies* (iv), (v) *for some $i = 1, 2, \dots, n$ and $P_\tau g = g$ then $g = f_i$ almost everywhere,*

(vii) *every f such that $P_\tau f = f$ can be written as*

$$f = \sum_{i=1}^{n} a_i f_i$$

with a suitably chosen a_i, and supp$f$ *is invariant under τ.*

The proof of this theorem is given in [12].

LEMMA 3. *If transformation τ:* $[0, 1] \to [0, 1]$ *satisfies the conditions* (e) *and* (f) *then*

(i) *there exists exactly one measure μ which is invariant under τ, absolutely continuous under Lebesgue measure and $\mu([0, 1]) = 1$,*

(ii) *the density g of measure μ under Lebesgue measure is constant on the interval (b^p_{i-1}, b^p_i),*

(iii) *the vector* $y = (y_1, y_2, \ldots, y_{k^p})$ $y_i = g(x)$ *for* $x \in (b_{i-1}^p, b_i^p)$ *is a solution of the linear equation*

$$A'y = y,$$

where A *is given by* (1) *and* (2),

(iv) $g(x) > 0$ *for* $x \in [0, 1]$ *and* $g(x)$ *may be chosen as a function continuous from the left.*

Proof. Let $A^n = (a_{ij}^n)$ $i, j = 1, 2, \ldots, k^p$ denotes the n-th iterate of matrix A. It is easy to see that matrix A satisfies the assumptions of Lemma 2 and we have $A\alpha = \alpha$ for

$$\alpha = (b_1^p - b_2^p, b_2^p - b_3^p, \ldots, b_{k^p}^p - b_{k^p-1}^p).$$

Therefore, from Lemma 1 it follows that there exists $\lim_{n\to\infty} A^n = B$. Computing the Frobenius–Perron operator for τ we obtain

$$P_\tau h = \sum_{l=1}^{k} h(\varphi_l(x))|\varphi_l'(x)|,$$

where $\varphi_l = \tau_l^{-1}$ and τ_l is an extension of $\tau|_{(a_{l-1}, a_l)}$ to a continuous function from $[a_{l-1}, a_l]$ onto $[0, 1]$. By its very definition the operator P_τ is a mapping from L^1 into L^1, but the last formula enables us to consider P_τ as a map from the space of functions defined on $[0, 1]$ into itself.

Let 1_E denotes the characteristic function of the set $E \subset [0, 1] = I$. By simple induction we obtain

$$P_\tau^n 1_I(x) = \sum_{i=1}^{k^p} a_{ij}^n \quad \text{for} \quad x \in (b_{j-1}^p, b_j^p).$$

We know that A^n is convergent to a matrix $B > 0$, therefore $P_\tau^n 1_I$ is uniformly convergent on $[0, 1] \setminus \{b_0^p, b_1^p, \ldots, b_{k^p}^p\}$ to a function

$$g(x) = \sum_{i=1}^{k^p} b_{ij} > 0 \quad \text{for} \quad x \in (b_{j-1}^p, b_j^p). \tag{4}$$

Since $P_\tau g = g$, from (b) and (d) it follows that measure $d\mu = g\,dm$ is invariant under τ and $\|g\|_{L^1} = 1$. From the Yorke–Li Theorem it follows that for any h such that $P_\tau h = h$ we have $g = h$ (if $E = \operatorname{supp} h$ then there exists m_0 such that $\tau^{m_0}(E) = [0, 1]$), therefore transformation τ has exactly one absolutely continuous invariant measure. From Lemma 1 and (4) we obtain (ii) and (iii). Since measure μ does not depend on value function g on the set $E \subset [0, 1]$ such that $m(E) = 0$ we may assume that g is continuous from the left. From (4) we get $g > 0$. This completes the proof.

LEMMA 4. *If transformation* τ: $[0, 1] \to [0, 1]$ *satisfies the assumptions of Theorem* 1 *then for any* $f \in L^1$

$$\lim_{N\to\infty} \|P_{\tau_N} f - P_\tau f\|_{L^1} = 0,$$

where τ_N *are given by formulas* (3) *and* P_τ, P_{τ_N} *denote Frobenius–Perron operators.*

Proof. It is obvious that τ_N is uniformly convergent to τ on $[0, 1]$. As in the proof of Lemma 3 let $\varphi_l = \tau_l^{-1}$ and $\varphi'_{Nl} = \tau_{Nl}^{-1}$ where τ_l and τ_{Nl} are extensions of $\tau|_{(a_{l-1}, a_l)}$ and $\tau_N|_{(a_{l-1}, a_l)}$ to a continuous function from $[a_{l-1}, a_l]$ onto $[0, 1]$. Since

$$|\varphi'_{Nl}(x) - \varphi'_l(x) \leqslant \max_{l=1,\dots,k} \Big(\max_{x,y \in (b^N_{i-1}, b^N_i)} |\varphi'_l(x) - \varphi'_l(y)| \Big) \leqslant Ms^{-N},$$

where

$$M = \max_{l=1,\dots,k} \Big(\max_{x \in [0,1]} |\varphi''_l(x)| \Big)$$

we also have φ'_{Nl} uniformly convergent to φ'_l for $l = 1, 2, \dots, k$ on the set

$$[0, 1] \setminus \bigcup_{n=1}^{\infty} \tau^{-n}(\{a_0, a_1, \dots, a_k\}).$$

For any $f \in L^1$ we have

$$\begin{aligned}\|P_{\tau_N} f - P_\tau f\|_{L^1} &= \int_0^1 |P_{\tau_N} f - P_\tau f|\, dm \\ &= \int_0^1 \Big| \sum_{l=1}^k f(\varphi_{Nl}(x)) |\varphi'_{Nl}(x)| - \sum_{l=1}^k f(\varphi_l(x)) |\varphi'_l(x)| \Big| \\ &\leqslant \int_0^1 \sum_{l=1}^k \big| f(\varphi_{Nl}(x)) |\varphi'_{Nl}(x)| - f(\varphi_l(x)) |\varphi'_l(x)| \big| .\end{aligned}$$

Since φ'_{Nl} and φ_{Nl} are convergent uniformly, from the last inequality we obtain for any continuous f

$$\lim_{N \to \infty} \|P_{\tau_N} f - P_\tau f\|_{L^1} = 0 .$$

Because the set of all continuous functions is dense in L^1 and $\|P_{\tau_N}\| = \|P_\tau\| = 1$ for any N, therefore for any $f \in L^1$

$$\lim_{N \to \infty} \|P_{\tau_N} f - P_\tau f\|_{L^1} = 0 .$$

Thus the lemma is completely proved.

Denote by $\bigvee_a^b h = \bigvee_{[a,b]} h$ the variation of h over the closed interval $[a, b]$.

LEMMA 5. *If transformation* $\tau\colon [0, 1] \to [0, 1]$ *satisfies the assumptions of Theorem* 1, *then*

(i) *for any* $N = 1, 2, \dots$ *there exists exactly one function* $f_N \in L^1$ *such that* $\|f_N\| = 1$, $f_N \geqslant 0$ *and the measure* $d\mu_N = f_N dm$ *is invariant under* τ_N,

(ii) *function* f_N *is constant on the interval* (b^N_{i-1}, b^N_i) $i = 1, 2, \dots, k^N$ *and vector* $y^N = (y^N_1, y^N_2, \dots, y^N_{k^N})$ $y^N_i = f_N(x)$ *for* $x \in (b^N_{i-1}, b^N_i)$ *is a solution of the linear equation*

$$A'_N y = y ,$$

(iii) f_N *may be chosen as a function continuous from the left,*

(iv) *there exists* $f \in L^1$ *such that* $\lim_{N\to\infty} f_N = f$ *in* L^1 *norm,*

(v) *measure* $d\mu = fdm$ *is invariant under* τ,

(vi) *there exist constant* K *such that for any* $x \in [0, 1]$, $N = 1, 2, \ldots$, $n = 1, 2, \ldots$, $f_N(x) \leqslant K$, $P^n_{\tau_N} 1_I(x) \leqslant K$,

$$\bigvee_0^x f_N \leqslant K \quad \text{and} \quad \bigvee_0^x P^n_{\tau_N} 1_I(x) \leqslant K .$$

Proof. (i), (ii) and (iii) follow directly from Lemma 3. Set $P_{\tau_N} = P_N$. We know that

$$P_N 1_I(x) = \sum_{i=1}^{k^N} a_{ij}^{Nn} \quad \text{for } x \in (b_{j-1}^N, b_j^N) ,$$

where (a_{ij}^{Nn}) is the n-th iterate of matrix (a_{ij}^N), and that $P_N^n 1_I$ is convergent uniformly on the set $[0, 1] \setminus \tau^{-N+1}(\{a_0, a_1, \ldots, a_k\})$. Changing, if necessary the values of functions $P_n^N 1_I$ on the set $\{b_0^N, b_1^N, \ldots, b_{k^N}^N\}$ we may assume without loss of generality that for any $n = 1, 2, \ldots$ $P_N^n 1_I$ are functions continuous from the left and continuous in 0. Then, $P_N^n 1_I$ is convergent uniformly on $[0, 1]$ to a function f_N which is continuous from the left on $[0, 1]$ and continuous in 0. Using notations as in the proof of Lemma 4 we set

$$M = \max_{l=1,\ldots,k} \Big(\sup_{x\in[0,1]} |\varphi_l''(x)| \Big) \tag{5}$$

and

$$m = \min_{l=1,\ldots,k} \Big(\inf_{x\in[0,1]} |\varphi_l'(x)| \Big) . \tag{6}$$

Let N_0 be so large that for any $N > N_0$

$$\frac{Ms^{-N}}{m} + s^{-1} \leqslant \beta < 1 .$$

We have

$$\bigvee_0^1 P_N^{n+1} 1_I = \sum_{i=1}^{k^N} \Big| \sum_{l=1}^{k} P_N^n 1_I(\varphi_{Nl}(x_i)) |\varphi_{Nl}'(x_i)| - \sum_{l=1}^{k} P_N^n 1_I(\varphi_{Nl}(x_{i+1})) |\varphi_{Nl}'(x_{i+1})| \Big| ,$$

where $x_i \in (b_{i-1}^N, b_i^N)$ $i = 1, 2, \ldots, k^N$. Therefore

$$\begin{aligned}
\bigvee_0^1 P_N^{n+1} 1_I \leqslant & \sum_{i=1}^{k^N} \sum_{l=1}^{k} \big| P_N^n 1_I(\varphi_{Nl}(x_i)) |\varphi_{Nl}'(x_i)| - \\
& \qquad - P_N^n 1_I(\varphi_{Nl}(x_{i+1})) |\varphi_{Nl}'(x_{i+1})| \big| \\
\leqslant & \sum_{i=1}^{k^N} \sum_{l=1}^{k} |\varphi_{Nl}'(x_i)| \, |P_N^n 1_I(\varphi_{Nl}(x_i)) - P_N^n 1_I(\varphi_{Nl}(x_{i+1}))| + \\
& + \sum_{i=1}^{k^N} \sum_{l=1}^{k} P_N^n 1_I(\varphi_{Nl}(x_{i+1})) \big| |\varphi_{Nl}'(x_i)| - |\varphi_{Nl}'(x_{i+1})| \big|
\end{aligned}$$

and consequently

$$\bigvee_0^1 P_N^{n+1} 1_I \leqslant s^{-1} \bigvee_0^1 P_N^n 1_I + \sum_{i=1}^{k^N} \sum_{l=1}^{k} P_N^n 1_I(\varphi_{Nl}(x_{i+1})) \big| |\varphi_{Nl}'(x_i)| - |\varphi_{Nl}'(x_{i+1})| \big| . \tag{7}$$

From the definition of transformations τ_N and the mean value theorem there exist points $\xi_{Nli} \in (b_{i-1}^N, b_i^N)$ such that

$$\varphi_l'(\xi_{Nli}) = \varphi_{Nl}'(x_i)\,. \tag{8}$$

Since $P_N^n 1_I$ is constant on (b_{i-1}^N, b_i^N) from (5) and (8) we obtain

$$\sum_{i=1}^{k^N}\sum_{l=1}^{k} P_N^n 1_I(\varphi_{Nl}(x_{i+1}))\,\big|\,|\varphi_{Nl}'(x_i)|-|\varphi_{Nl}'(x_{i+1})|\,\big| \leqslant M\sum_{i=1}^{k^N}\sum_{l=1}^{k} P_N^n 1_I(\varphi_{Nl}(\xi_{Nli+1}))\,|\xi_{Nli}-\xi_{Nli+1}|\,. \tag{9}$$

Set $\theta_{Nli} = \varphi_{Nl}(\xi_{Nli})$. From the definitions of transformations τ_N and (6) it follows that there exist real numbers $\zeta_{Nli} \geqslant m$ such that

$$|\theta_{Nli}-\theta_{Nli+1}| = \zeta_{Nli}|\xi_{Nli}-\xi_{Nli+1}|\,.$$

This and (9) imply the inequality

$$\sum_{i=1}^{k^N}\sum_{l=1}^{k} P_N^n 1_I(\varphi_{Nl}(x_{i+1}))\,\big|\,|\varphi_{Nl}'(x_i)|-|\varphi_{Nl}'(x_{i+1})|\,\big|$$

$$\leqslant \frac{M}{m}\sum_{i=1}^{k^N}\sum_{l=1}^{k} P_N^n 1_I(\varphi_{Nl}(\xi_{Nli+1}))\zeta_{Nli}|\xi_{Nli}-\xi_{Nli+1}|$$

$$= \frac{M}{m}\sum_{i=1}^{k^N}\sum_{l=1}^{k} P_N^n 1_I(\theta_{Nli+1})|\theta_{Nli}-\theta_{Nli+1}| \leqslant \frac{M}{m}\|P_N^n 1_I\| +$$

$$+\frac{M}{m}\sum_{i=1}^{k^N}\sum_{l=1}^{k} |P_N^n 1_I(\theta_{Nli+1})-P_N^n 1_I(\theta_{Nli})|\,|\theta_{Nli+1}-\theta_{Nli}|\,.$$

Since $|\theta_{Nli+1}-\theta_{Nli}| \leqslant s^{-N}$ from the last inequality we obtain

$$\sum_{i=1}^{k^N}\sum_{l=1}^{k} P_N^n 1_I(\varphi_{Nl}(x_{i+i}))\,\big|\,|\varphi_{Nl}'(x_i)|-|\varphi_{Nl}'(x_{i+1})|\,\big| \leqslant \frac{M}{m}\|P_N^n 1_I\| + \frac{M}{m}s^{-N}\bigvee_0^1 P_N^n 1_I\,. \tag{10}$$

From (7) and (10) it follows

$$\bigvee_0^1 P_N^{n+1} 1_I \leqslant \frac{M}{m}\|P_N^n 1_I\| + \beta\bigvee_0^1 P_N^n 1_I$$

for $N > N_0$ and consequently

$$\bigvee_0^1 P_N^n 1_I \leqslant \frac{M}{m}\sum_{p=0}^{\infty}\beta^p = \frac{M}{m}\frac{1}{1-\beta}$$

for any n and $N>N_0$. Letting $n\to\infty$ we obtain

$$\bigvee_0^1 f_N \leqslant \frac{M}{m}\frac{1}{1-\beta}.$$

Thus we have proved (vi). Since $\|f_N\| = 1$ for any N from Helly's theorem it follows that the sequence $\{f_N\}_{N=0}^{\infty}$ is relatively compact in L^1. Therefore there exists a subsequence f_{N_j} which is convergent in L^1 norm to a function $f\in L^1$. We show that f is the density of the invariant measure under τ. With this aim we show that for any $\varepsilon>0$

$$\|P_\tau f-f\|_{L^1}\leqslant\varepsilon . \tag{11}$$

Since

$$\|P_\tau f-f\|\leqslant\|P_\tau f-P_{N_j}f\|+\|P_{N_j}f-P_{N_j}f_{N_j}\|+\|P_{N_j}f_{N_j}-f\|\leqslant\|P_\tau f-P_{N_j}f\|+2\|f-f_{N_j}\|$$

we obtain (11) by Lemma 4. From the Yorke–Li Theorem it follows that transformation τ have only one absolutely continuous invariant measure (if $P_\tau h = h$ and $E = \operatorname{supp} h$ then there exists such m_0 that $\tau^{m_0}(E) = [0, 1]$). Since $\{f_N\}_{N=0}^{\infty}$ is relatively compact in L^1 the above implies the convergence of f_N to f in L^1 norm. Thus the lemma is proved.

Let function h: $[a, b]\to R$ be monotonic and continuous and let h satisfy the condition

$$h(x) = \frac{h(a_i)-h(a_{i-1})}{a_i-a_{i-1}}(x-a_{i-1})+h(a_{i-1}) \tag{k}$$

for $x\in[a_{i-1}, a_i]$ where $a = a_0<a_1<\ldots<a_k = b$ is a partition of interval $[a, b]$. Set

$$Dh(x) = \begin{cases}[h'_-(x), h'_+(x)] & \text{if } h'_-(x) < h'_+(x)\,,\\ [h'_+(x), h'_-(x)] & \text{if } h'_+(x) < h'_-(x)\,,\\ h'_-(x) & \text{if } h'_-(x) = h'_+(x)\,,\end{cases}$$

where $h'_-(x)$, $h'_+(x)$ denote the left and right derivatives of h respectively.

By $h'(x)$ we shall denote any real number from $Dh(x)$.

Let $A\subset R$ and $B\subset R$ (R is the set of real numbers). Denote

$$AB = \{ab\colon a\in A, b\in B\}.$$

LEMMA 6. *If functions f_1: $[\alpha, \beta]\to[\gamma, \delta]$ and f_2: $[\gamma, \delta]\to R$ are monotonic and continuous, and f_1 satisfies condition* (k) *for a partition $\alpha = a_0<a_1<\ldots<a_{k_1} = \beta$, f_2 satisfies condition* (k) *for a partition $\gamma = b_0<b_1<\ldots<b_{k_2} = \delta$ then for any $x\in[\alpha, \beta]$*

$$D(f_2\circ f_1)(x)\subset(Df_2)(f_1(x))\,Df_1(x).$$

Proof. From the definition of derivative from the left and derivative from the right we have

$$(f_2\circ f_1)'_-(x) = (f_2)'_-(f_1(x))(f_1)'_-(x)$$

or

$$(f_2\circ f_1)'_-(x) = (f_2)'_+(f_1(x))(f_1)'_-(x)$$

and

$$(f_2\circ f_1)'_+(x) = (f_2)'_+(f_1(x))(f_1)'_+(x)$$

or

$$(f_2 \circ f_1)'_+(x) = (f_2)'_-(f_1(x))(f_1)'_+(x) .$$

From these identities it follows that

$$(f_1 \circ f_2)'_-(x) \in (Df_2)(f_1(x))\, Df_1(x)$$

and

$$(f_1 \circ f_2)'_+(x) \in (Df_2)(f_1(x))\, Df_1(x) .$$

Since $(Df_2)(f_1(x))\, Df_1(x)$ is an interval, therefore

$$D(f_2 \circ f_1) \subset (Df_2)(f_1(x))\, Df_1(x) .$$

This finishes the proof of lemma.

It is easy to verify

LEMMA 7. *If function* $h\colon [a, b] \to R$ *is continuous and satisfies condition* (k) *for a certain partition* $a = a_0 < a_1 < \ldots a < a_k = b$ *of interval* $[a, b]$ *then there exists a point* $\xi \in (a. b)$ *and there exists* $h'(\xi) \in Df(x)$ *such that*

$$\frac{h(a) - h(b)}{a - b} = h'(\xi) .$$

LEMMA 8. *If* $h\colon [a, b] \to R$ *is* C^1 *function and* $|h'(x)| > 0$ *then for any* $c \in (a, b)$ *and any number* $d \in D$ *there exists* $\xi \in (a, b)$ *such that* $h'(\xi) = d$, *where*

$$D = \begin{cases} \left[\dfrac{h(c)-h(a)}{c-a}, \dfrac{h(c)-h(b)}{c-b}\right] & \text{if} \quad \dfrac{h(c)-h(a)}{c-a} < \dfrac{h(c)-h(b)}{c-b}, \\[2ex] \left[\dfrac{h(c)-h(b)}{c-b}, \dfrac{h(c)-h(a)}{c-a}\right] & \text{if} \quad \dfrac{h(c)-h(b)}{c-b} < \dfrac{h(c)-h(a)}{c-a}. \end{cases}$$

Proof. From the mean value theorem there exist $\xi_1 \in (a, b)$ and $\xi_2 \in (a, b)$ such that

$$h'(\xi_1) = \frac{h(c) - h(a)}{c - a} \quad \text{and} \quad h'(\xi_2) = \frac{h(c) - h(b)}{c - b}.$$

Since $h'(x)$ has the Darboux property, therefore there exists $\xi \in (a, b)$ such that $h'(\xi) = d$.

LEMMA 9. *If transformation* $\tau\colon [0, 1] \to [0, 1]$ *satisfies the assumptions of Theorem* 1 *then there exists a constant* $L > 0$ *such that for any* n $(n = 1, 2, \ldots)$, *for any* N $(N = 1, 2, \ldots)$ *and for any measurable set* $A \subset [0, 1]$

$$m\big(\tau_N^{-n}(A)\big) \leqslant L m(A) .$$

Proof. Let $0 = b_0^{Nn} < b_1^{Nn} < \ldots < b_{k^n}^{Nn} = 1$ be the partition of interval $[0, 1]$ such that

$$\tau_N^{-n+1}(\{a_0, a_1, \ldots, a_k\}) = \{b_0^{Nn}, b_1^{Nn}, \ldots, b_{k^n}^{Nn}\} .$$

It is easy to see that transformations $\tau_N^j\colon (b_{i-1}^{Nn}, b_i^{Nn})\to(0,1)$ are injections for $j\leqslant n$ and $i=1,2,\dots,k^n$, $\tau_N^n\colon (b_{i-1}^{Nn}, b_i^{Nn})\to(0,1)$ are bijections for $i=1,2,\dots,k^n$, for $i=1,2,\dots,k^n$ there exists l such that

$$\tau_N^{n-1}\big((b_{i-1}^{Nn}, b_i^{Nn})\big) = (a_{l-1}, a_l)$$

and for $i=1,2,\dots,k^n$, $j<n$ there exists l such that

$$\tau_N^j\big((b_{i-1}^{Nn}, b_i^{Nn})\big)\subset(a_{l-1}, a_l)\,.$$

From this and the definition of number s for any $x\in(b_{i-1}^{Nn}, b_i^{Nn})$, $y\in(b_{i-1}^{Nn}, b_i^{Nn})$ and $j<n$ we have

$$|\tau_N^{j+1}(x)-\tau_N^{j+1}(y)|\geqslant|\tau_N^j(x)-\tau_N^j(y)|\,s$$

and consequently

$$|\tau_N^j(x)-\tau_N^j(y)|\leqslant\frac{|\tau_N^n(x)-\tau_N^n(y)|}{s^{n-j}}\leqslant\frac{1}{s^{n-j}} \tag{12}$$

for $j\leqslant n$, $x\in(b_{i-1}^{Nn}, b_n^{Nn})$ and $y\in(b_{i-1}^{Nn}, b_i^{Nn})$.

From Lemma 6 it follows that for $x\in(b_{i-1}^{Nn}, b_i^{Nn})$ and $y\in(b_{i-1}^{Nn}, b_i^{Nn})$ there are such derivatives $\tau_N'(\tau_N^j(x))\in(D\tau_N)(\tau_N^j(x))$ and $\tau_N'(\tau_N^j(y))\in(D\tau_N)(\tau_N^j(y))$ $j=1,2,\dots,n-1$ that

$$\left|\frac{(\tau_N^n(x))'}{(\tau_N^n(y))'}\right| = \prod_{j=1}^{n-1}\left|\frac{\tau_N'((\tau_N^j)(x))}{\tau_N'((\tau_N^j)(y))}\right|. \tag{13}$$

It is easy to see that if $j\leqslant n-N$, $x\in(b_{i-1}^{Nn}, b_1^{Nn})$, $y\in(b_{i-1}^{Nn}, b_i^{Nn})$ then there exists q such that $\tau_N^j(x)\in(b_{q-1}^N, b_q^N)$, $\tau_N^j(y)\in(b_{q-1}^N, b_q^N)$ and consequently

$$\tau_N'(\tau_N^j(x))-\tau_N'(\tau_N^j(y)) = 0\,. \tag{14}$$

If $n-1\geqslant j\geqslant n-N$ $x, y\in(b_{i-1}^{Nn}, b_i^{Nn})$ then there exist p_1, p_2 and l such that

$$\tau_N^j(x)\in[b_{p_1-1}^N, b_{p_1}^N]\subset[a_{l-1}, a_l] \quad\text{and}\quad \tau_N^j(y)\in[b_{p_2-1}^N, b_{p_2}^N]\subset[a_{l-1}, a_l]\,. \tag{15}$$

From Lemma 8 and the definition of τ_N it follows that there exists ξ_{xj} and ξ_{yj} such that

$$|\tau_N^j(x)-\xi_{xj}|\leqslant\max_{i=1,2,\dots,k^N}|b_i^N-b_{i-1}^N|\leqslant s^{-N},$$

$$|\tau_N^j(y)-\xi_{yj}|<s^{-N}$$

and

$$\tau_N'(\tau_N^j(x)) = \tau'(\xi_{xj}),\qquad \tau_N'(\tau_N^j(y)) = \tau'(\xi_{yj})\,.$$

From this and (15) we obtain

$$|\tau_N'(\tau_N^j(x))-\tau_N'(\tau_N^j(y))| = |\tau'(\xi_{xj}|-\tau'(\xi_{yj})|\leqslant M|\xi_{xj}-\xi_{yj}|\leqslant M(|\tau_N^j(x)-\tau_N^j(y)|+2s^{-N})\,, \tag{16}$$

where $M = \sup_{x\in[0,1]} |\tau'(x)|$. Inequalities (12), (16) and identities (13) and (14) imply

$$\left|\frac{(\tau_N^n(x))'}{(\tau_N^n(y))'}\right| \leqslant \prod_{j=1}^{n-1}\left(1+\frac{|\tau_N'(\tau(x_N^j))-\tau_N'(\tau_N^j(y))|}{\tau_N'(\tau_N^j(y))}\right)$$

$$\leqslant \exp\left(\sum_{j=n-N+1}^{n-1} \frac{M(|\tau_N^j(x)-\tau_N^j(y)|+2s^{-N})}{s}\right)$$

$$\leqslant \exp\left(2NMs^{-N}+\frac{M}{s}\sum_{j=n-N+1}^{n-1} |\tau_N^j(x)-\tau_N^j(x)-\tau_N^j(y)|\right)$$

$$\leqslant \exp\left(2NMs^{-N}+\frac{M}{s}\sum_{j=n-N+1}^{n-1}\frac{1}{s^{n-j}}\right)\leqslant \exp\left(2NMs^{-N}+\frac{M}{s}\frac{1}{1-\frac{1}{s}}\right)$$

for $x, y\in(b_{i-1}^{Nn}, b_i^{Nn})$. This inequality implies that there exists a constant L such that for $N = 1, 2, \ldots, n = 1, 2, \ldots$ and $x, y \in (b_{i-1}^{Nn}, b_i^{Nn})$

$$\left|\frac{(\tau_N^n(x))'}{(\tau_N^n(y))'}\right| \leqslant L .$$

Since $|\tau'(x)| \geqslant s > 0$ for any $x\in[0, 1]$ therefore

(17) $$\left|\frac{(\tau_N^n(x))'}{(\tau_N^n(y))'}\right| \geqslant \frac{1}{L}$$

for $x, y\in(b_{i-1}^{Nn}, b_i^{Nn})$ $n = 1, 2, 3, \ldots$ and $N = 1, 2, 3, \ldots$

Since τ_N^n: $(b_{i-1}^{Nn}, b_i^{Nn})\to(0, 1)$ is a bijection, from Lemma 7 it follows that for $N = 1, 2, 3, \ldots, n = 1, 2, 3, \ldots$ and $i = 1, 2, \ldots, k^n$ there exist points $\xi_{Nni}\in(b_{i-1}^{Nn}, b_i^{Nn})$ and derivatives $(\tau_N^n(\xi_{Nni}))'\in D\tau_N^n(\xi_{Nni})$ such that

$$(b_i^{Nn}-b_{i-1}^{Nn})|(\tau_N^n(\xi_{Nni}))'| = 1 .$$

Because

$$\sum_{i=1}^{k^n}(b_i^{Nn}-b_{i-1}^{Nn}) = 1$$

from the last identity we have

(18) $$\sum_{i=1}^{k^n}\frac{1}{|(\tau_N^n(\xi_{Nni}))'|} = 1 .$$

Finally, from (17) and (18) we obtain

$$\sum_{i=1}^{k^n} \frac{1}{\inf\limits_{x\in(b_{i-1}^{Nn},\, b_i^{Nn})} |(\tau_N^n(x))'|} \leqslant L \tag{19}$$

for $n = 1, 2, 3, \ldots$ and $N = 1, 2, 3, \ldots$

Let $E \subset [0, 1]$ be any measurable set. Set

$$E_i^{Nn} = \tau_N^{-n}(E) \cap (b_{i-1}^{Nn}, b_i^{Nn}) .$$

It is obvious that

$$\tau_N^{-n}(E) = \bigcup_{i=1}^{k^n} E_i^{Nn} . \tag{20}$$

Since $\tau_N^n(E_i^{Nn}) = E$ a.e. therefore

$$m(E) \geqslant m(E_i^{Nn}) \inf_{x\in[b_{i-1}^{Nn},\, b_i^{Nn}]} |(\tau_N^n(x))'| . \tag{21}$$

From (19), (20) and (21) we obtain

$$m(\tau_N^{-n}(E)) = \sum_{i=1}^{k^n} m(E_i^{Nn}) \leqslant \sum_{i=1}^{k^n} \frac{m(E)}{\inf\limits_{x\in[b_{i-1}^{Nn},\, b_i^{Nn}]} |(\tau_N^n(x))'|} \leqslant m(E)L$$

for $n = 1, 2, 3, \ldots$ and $N = 1, 2, 3, \ldots$ This finishes the proof of the lemma.

A sequence of functions $\{h_n\}_{n=1}^{\infty}$ h_n: $[0, 1] \to R$ is said to be quasi-equicontinuous on $[0, 1]$ if for every $\varepsilon > 0$ there exists n_0 and $\delta > 0$ such that

$$|h_n(x) - h_n(y)| < \varepsilon$$

whenever $|x-y| < \delta$, $x, y \in [0, 1]$, and $n > n_0$.

For a proof of Theorem 1 we shall need the following generalization of Arzela theorem.

THEOREM (Arzela). *If sequence $\{h_n\}_{n=1}^{\infty}$ h_n: $[0, 1] \to R$ is uniformly bounded on $[0, 1]$ and quasi-equicontinuous on $[0, 1]$ then*

(i) *$\{h_n\}_{n=1}^{\infty}$ contains a uniformly convergent subsequence $\{h_{n_j}\}$,*

(ii) *$\lim\limits_{j\to\infty} h_{n_j}$ is a continuous function.*

The proof of this Theorem is identical with that of the well known Arzela theorem.

Let $E = \bigcup\limits_{p=1}^{m} [c_p, d_p] \subset [0, 1]$ be such that $[c_p, d_p] \cap [c_q, d_q] = \phi$ for $p \neq q$ and let f: $[0, 1] \to R$. We define the variation of f over the set E by the formula

$$\bigvee_E f = \sum_{p=1}^{n} \bigvee_{c_p}^{d_p} f .$$

Proof of Theorem 1. (i) and (ii) follows directly from Lemma 5. To prove (iii) we show first that there exists N_0 such that the sequence of functions $\{f_N\}_{N=N_0}^{\infty}$ is uniformly bounded on $[0, 1]$ and quasi-equicontinuous. Using notations as in the proof of Lemma 4 we denote

$$m = \inf_{l=1,2,\dots,k} \left(\inf_{x\in[1,0]} |\varphi_l'(x)|\right)$$

and

$$M = \sup_{l=1,2,\dots,k} \left(\sup_{x\in[0,1]} |\varphi_l''(x)|\right).$$

Let N_0 be so large that

$$\frac{M}{m} s^{-N} + s^{-1} < \beta < 1$$

for $N > N_0$ and h: $[0, 1] \to [0, \infty]$ be a function continuous in 0, continuous from the left on $[0, 1]$ and constant on the intervals (b_{i-1}^N, b_i^N). Furthermore, let

$$E = \bigcup_{p=1}^{r} [c_p, d_p] \subset [0, 1]$$

be such that $[c_p, d_p] \cap [c_q, d_q] = \phi$ for $p \neq q$.

As in the proof of Lemma 5, changing if necessary value of functions $P_N^n h$ on the set $\{b_0^N, b_1^N, \dots, b_{k^N}^N\}$ we may assume without loss of generality that $P_N^n h$ are functions continuous from the left and continuous in 0 for $n = 1, 2, 3, \dots$ Furthermore, since

$$\bigvee_a^b P_N^n h = 0$$

for any n if $[a, b] \subset (b_{i-1}^N, b_i^N]$ for a certain i we may assume that

$$[c_p, d_p] \cap \{b_0^N, b_1^N, \dots, b_{k^N}^N\} \neq \phi \quad \text{for } p = 1, 2, \dots, r.$$

We have

$$\bigvee_{c_p}^{d_p} P_N h = \sum_{i=1}^{k^N} \Big| \sum_{l=1}^{k} h(\varphi_{Nl}(x_i)) |\varphi_{Nl}'(x_i)| - \sum_{l=1}^{k} h(\varphi_{Nl}(x_{i+1})) |\varphi_{Nl}'(x_{i+1})|$$

$$\leqslant \sum_{l=1}^{k} \sum_{i=1}^{k^N} \big| h(\varphi_{Nl}(x_i)) |\varphi_{Nl}'(x_i)| - h(\varphi_{Nl}(x_{i+1})) |\varphi_{Nl}'(x_{i+1})| \big|$$

$$\leqslant \sum_{l=1}^{k} \sum_{i=1}^{k^N} |\varphi_{Nl}'(x_i)|\, |h(\varphi_{Nl}(x_i)) - h(\varphi_{Nl}(x_{i+1}))| +$$

$$+ \sum_{l=1}^{k} \sum_{i=1}^{k^N} h(\varphi_{Nl}(x_{i+1}))|\, \big||\varphi_{Nl}'(x_i)| - |\varphi_{Nl}(x_{i+1})|\big|,$$

where $x_i \in [c_p, d_p] \cap (b_{i-1}^N, b_i^N]$ and $\varphi_{Nl}'(x_i) = \lim_{x \to x_i} \varphi_{Nl}'(x)$. Since $|\varphi_{Nl}'(x_i)| \leqslant s^{-1}$, from the last inequality we obtain

$$\bigvee_{c_p}^{d_p} P_N h \leqslant s^{-1} \bigvee_{\tau_N^{-1}([c_p, d_p])} h + \sum_{l=1}^{k} \sum_{i=1}^{k^N} h(\varphi_{Nl}(x_{i+1}))|\, \big||\varphi_{Nl}'(x_i)| - |\varphi_{Nl}(x_{i+1})|\big|. \tag{22}$$

As in the proof of Lemma 5, for any x_i and $l = 1, 2, \ldots, k$ there exists $\xi_{Nli} \in (b^N_{i-1}, b^N_i)$ such that $\varphi'_{Nl}(x_i) = \varphi'_l(\xi_{Nli})$. Therefore

$$\sum_{l=1}^{k}\sum_{i=1}^{k^N} h(\varphi_{Nl}(x_{i+1}))||\varphi'_{Nl}(x_i)|-|\varphi'_{Nl}(x_{i+1})||$$
$$\leqslant \sum_{l=1}^{k}\sum_{\zeta_{Nli},\zeta_{Nli+1}\in[c_p,d_p]} h(\varphi_{Nl}(\xi_{Nli+1}))\big||\varphi'_l(\xi_{Nli})|-|\varphi'_l(\xi_{Nli+1})|\big|+(\sup h)B_{Ep}$$

where $B_{Ep} = \sum \big||\varphi'_l(\xi_{Nli})|-|\varphi'_l(\xi_{Nli+1})|\big|$ is a sum of that component of the sum $\sum_{i=1}^{k^N}\big||\varphi'_l(\xi_{Nli})|-|\varphi'_l(\xi_{Nli+1})|\big|$ for which $\xi_{Nli} \notin [c_p, d_p]$ or $\xi_{Nli+1} \notin [c_p, d_p]$. Set $\theta_{Nli} = \varphi_{Nl}(\xi_{Nli})$. As in the proof of Lemma 5 there exist numbers $\zeta_{Nli} \geqslant m$ such that

$$|\theta_{Nli}-\theta_{Nli+1}| = \zeta_{Nli}|\xi_{Nli}-\xi_{Nli+1}|\,.$$

This and the last inequality imply

$$\sum_{l=1}^{k}\sum_{i=1}^{k^N} h(\varphi_{Nl}(x_{i+1}))\big||\varphi'_{Nl}(x_i)|-|\varphi'_{Nl}(x_{i+1})|\big|$$
$$\leqslant \frac{M}{m}\sum_{l=1}^{k}\sum_{\zeta_{Nli},\zeta_{Nli+1}\in[c_p,d_p]} h(\varphi_{Nl}(\xi_{Nli+1}))\zeta_{Nli}|\xi_{Nli}-\xi_{Nli+1}|+(\sup h)B_{Ep}$$
$$= \frac{M}{m}\sum_{l=1}^{k}\sum_{\zeta_{Nli},\zeta_{Nli+1}\in[c_p,d_p]} h(\theta_{Nli})|\theta_{Nli+1}-\theta_{Nli}|+(\sup h)B_{Ep} \leqslant \frac{M}{m}\sum_{l=1}^{k}\int_{\tau_l^{-1}([c_p,d_p])} h(s)\,ds +$$
$$+\frac{M}{m}\sum_{l=1}^{k}\sum_{\zeta_{Nli},\zeta_{Nli+1}\in[c_p,d_p]} |h(\theta_{Nli})-h(\theta_{Nli+1})|\,|\theta_{Nli}-\theta_{Nli+1}|+$$
$$+\sup h\, B_{Ep} \leqslant \frac{M}{m}\int_{\tau_N^{-1}([c_p,d_p])} h(s)\,ds+\frac{M}{m}s^{-1}\bigvee_{\tau_N^{-1}([c_p,d_p])} h+(\sup h)B_{Ep}\,.$$

From (22) and the last inequality we obtain

$$\bigvee_{c_p}^{d_p} P_N h \leqslant \left(s^{-1}+\frac{M}{m}s^{-N}\right)\bigvee_{c_p}^{d_p} h+\frac{M}{m}\int_{\tau_N^{-1}([c_p,d_p])} h(s)\,ds+(\sup h)B_{Ep}$$
$$\leqslant \beta\bigvee_{c_p}^{d_p} h+\frac{M}{m}\int_{\tau_N^{-1}([c_p,d_p])} h(s)\,ds+(\sup h)B_{Ep}$$

and finally

$$\bigvee_E P_N h \leqslant \beta \bigvee_E h + \frac{M}{m} \int_{\tau_N^{-1}(E)} h(s)\,ds + (\sup h) B_E , \tag{23}$$

where $B_E = \sum_{p=1}^{r} B_{Ep}$.

It is easy to see that

$$B_E \leqslant 4kMrs^{-N} \quad \text{and} \quad B_E \leqslant 1 . \tag{24}$$

Now set $E = [x_0, x_1]$. From (23) and Lemma 5 we have

$$\bigvee_{x_0}^{x_1} P_N^n 1_I \leqslant \beta \bigvee_{\tau_N^{-1}(E)} P_N^{n-1} 1_I + \frac{M}{m} \int_{\tau_N^{-1}(E)} P_N^{n-1} 1_I + KB_E$$

for $n = 1, 2, 3, \ldots$ and $N > N_0$. Therefore

$$\bigvee_{x_0}^{x_1} P_N^n 1_I \leqslant \beta^N \bigvee_{\tau_N^{-N}(E)} P_N^{n-N} 1_I + \frac{M}{m}\left(\sum_{p=1}^{N} \beta^{p-1} \int_{\tau_N^{-p}(E)} P_N^{n-N} 1_I\,dm\right) + K \sum_{p=1}^{N} \beta^{p-1} B_{E_p} , \tag{25}$$

where $E_p = \tau_N^{-p}([x_0, x_1])$, $n = 1, 2, 3, \ldots$ and $N > N_0$. Obviously $B_{E_p} \leqslant 1$ and $B_{E_p} \leqslant 4k^p Ms^{-N}$. Applying Lemma 9 and Lemma 5 to (25) we obtain

$$\bigvee_{x_0}^{x_1} P_N^n 1_I \leqslant K\beta^N + \frac{M}{m} KL(x_1 - x_0) \sum_{p=1}^{N} \beta^{p-1} + K \sum_{p=1}^{N} \beta^{p-1} B_{E_p}$$

for $n = 1, 2, 3, \ldots$ and $N > N_0$. Letting $n \to \infty$ we have

$$\bigvee_{x_0}^{x_1} f_N \leqslant K\beta^N + \frac{M}{m} KL(x_1 - x_0) \sum_{p=1}^{N} \beta^{p-1} + K \sum_{p=1}^{N} \beta^{p-1} B_{E_p} \tag{26}$$

where $B_{E_p} \leqslant 1$ and $B_{E_p} \leqslant 4k^p Ms^{-N}$.

Let q be such that $k^q s^{-N} \leqslant 1$ and $k^{q+1} s^{-N} > 1$. Since $B_{E_p} \leqslant 1$ and $B_{E_p} \leqslant 4k^p Ms^{-N}$ we have

$$\begin{aligned}
\sum_{p=1}^{N} \beta^{p-1} B_{E_p} &= \sum_{p=1}^{q} \beta^{p-1} B_{E_p} + \sum_{p=q+1}^{N} \beta^{p-1} B_{E_p} \leqslant \frac{4Ms^{-N}}{\beta} \sum_{p=1}^{q} \beta^p k^p + \sum_{p=q+1}^{N} \beta^{p-1} \\
&= 4Ms^{-Nk} \frac{1-(\beta k)^q}{1-\beta k} + \beta^q \frac{1-\beta^{N-q-1}}{1-\beta} \\
&= 4Ms^{-Nk} \frac{1}{1-\beta k} + \frac{4Ms^{-N} \beta^q k^q}{1-\beta k} + \beta^q \frac{1-\beta^{N-q-1}}{1-\beta} \\
&\leqslant 4Ms^{-N} \frac{1}{1-\beta k} + \frac{4M\beta^q}{1-k} + \beta^p \frac{1}{1-\beta} .
\end{aligned}$$

Since $q \to \infty$ as $N \to \infty$ therefore from the last inequality it follows that

$$\sum_{p=1}^{N} \beta^{p-1} B_{E_p} \to 0 \quad \text{as } N \to \infty . \tag{27}$$

From (26), (27) and Lemma 5 it follows that the sequence of functions $\{\bigvee_0^x f_N\}_{N=N_0}^{\infty}$ is quasi-equicontinuous and uniformly bounded. This implies that the sequence of functions $\{f_N\}_{N=N_0}^{\infty}$ is quasi-equicontinuous and uniformly bounded because

$$|h(x_1) - h(x_2)| \leqslant \bigvee_{x_1}^{x_2} h$$ for any h. From Lemma 5 and the

Arzela theorem it follows that f_N is uniformly convergent to a continuous function f and the measure $d\mu = f dm$ is invariant under τ. This completes the proof.

References

[1] R. Bellman, *Introduction to a matrix analysis*, New York–Toronto–London 1960.

[2] N. Dunford and J. T. Schwartz, *Linear operators. I. General Theory*, Pure Appl. Math. 7, Interscience, New York 1958.

[3] A. O. Gel'fond, *A common property of number systems*, Izv. Akad. Nauk SSSR Ser. Mat. 23 (1959), 809–814.

[4] K. Krzyżewski and W. Szlenk, *On invariant measures for expanding differentiable mappings*, Studia Math. 33 (1969), 83–92.

[5] A. Lasota, *Invariant measures and functional equations*, Aequations Math. 9 (1973), 193–200.

[6] — *On the existence of invariant measures for Markov processes*, Ann. Polon. Math. 28 (1973), 207–211.

[7] — and J. A. Yorke, *On the existence of invariant measures for piecewise monotonic transformations*, Trans. Amer. Math. Soc. 186 (1973), 481–488.

[8] W. Parry, *On the β-expansion of real numbers*, Acta Math. Acad. Sci. Hungar. 11 (1960), 401–416.

[9] A. Rényi, *Representation for real numbers and their ergodic properties*, ibidem 8 (1957), 474–493.

[10] V. A. Rohlin, *Exact endomorphism of Lebesgue spaces*, Izv. Akad. Nauk SSSR Ser. Mat. 25 (1961), 499–530.

[11] S. M. Ulam, *A collection of mathemathical problems*, Interscience Tracts in Pure Appl. Math. 8, Interscience, New York 1960.

[12] Tien-Yien Li and J. A. Yorke, *Ergodic transformations from an interval into itself* (preprint).

[13] M. Halfant, *Analytic properties of Rényi's invariant density*, Israel J. Math. 1 (1977), 1–20.

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