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Lie algebras of infinitesimal holonomy groups of a Cartan connection

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1. Preliminaries. Throughout this paper, we assume that all differentiable manifolds, fibre bundles, functions, vector fields and differential forms are of class C^∞.

Let M be a manifold. The tangent vector space of M at a point $x \in M$ will be denoted by T_xM and the set of all vector fields on M by $\mathscr{X}(M)$. Let M and N be two manifolds and let f be a mapping of M into N. The differential of f at $x \in M$ will be denoted by $d_x f$. If γ is an r-form on N, then $f^*\gamma$ is the form on M defined as follows:

$$f^*\gamma = \gamma \circ f \text{ i.e. } f^*\gamma(X_1, \ldots, X_r) = \gamma_{f(x)}(d_x f(X_1), \ldots, d_x f(X_r)),$$

where $X_1, \ldots, X_r \in T_xM$.

Let M be a manifold of dimension n, G a Lie group, G' a closed subgroup of G with $\dim G/G' = n$ and $P'(M, G')$ a principal fibre bundle over M with structure group G'. The Lie algebras of G and G' will be denoted by $\mathfrak{g}$ and $\mathfrak{g}'$ respectively.

A Cartan connection in the bundle $P'(M, G')$ is a 1-form ω on P' with values in the Lie algebra $\mathfrak{g}$ satisfying the following conditions:

1. $\omega(A^*) = A$ for every $A \in \mathfrak{g}'$(A^* denotes the fundamental vector field corresponding to A);

2. $R_a^*\omega = ad_{a^{-1}}\omega$ for every element $a \in G'$, where R_a is the transformation of P' induced by $a \in G'$, i.e. $R_a u = ua$ and ad denotes the adjoint representation of G in $\mathfrak{g}$;

3. $\omega(X) \neq 0$ for every non-zero vector X of P'.

Condition 3. means that ω defines a linear isomorphism of the tangent space T_uP' onto the Lie algebra $\mathfrak{g}$ for every $u \in P'$.

The two following propositions are well-known.

Proposition 1. *For a principal fibre bundle $P'(M, G')$ there is a unique (unique up to an isomorphism) principal fibre bundle $P(M, G)$ such that $P'(M, G')$ is a subbundle of $P(M, G)$.*

$P(M, G)$ will be called the *overbundle of* $P'(M, G')$. The injection of $P'(M, G')$ into $P(M, G)$ will be denoted by ι.

Proposition 2. *If $P(M, G)$ is the overbundle of $P'(M, G')$ and ω is a Cartan connection in $P'(M, G')$, then ω can be uniquely extended to the usual connection $\tilde{\omega}$ in $P(M, G)$.*

The *holonomy group* (*restricted, infinitesimal*) of a *Cartan connection* ω will be defined as the holonomy group (restricted, infinitesimal) of the extended connection $\tilde{\omega}$.

By Propositions 1 and 2 this definition is meaningful.

Let Q_u (resp. $\tilde{Q}_u$) denotes the vertical subspace of the tangent space T_uP' (resp. T_uP). Clearly $Q_u \subset \tilde{Q}_u$. The horizontal subspace of T_uP will be denoted by Γ_u. Every vector $X \in T_uP$ can be uniquely written as $X = \tilde{\vartheta}X + hX$, where $\tilde{\vartheta}X \in \tilde{Q}_u$ and $hX \in \Gamma_u$ i.e. $\tilde{\vartheta}X$ is the vertical component of X and hX is the horizontal component of X.

Take a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{m}$ (the direct sum of the vector spaces). We set $\Gamma_u^{\mathfrak{m}} = \omega_u^{-1}(\mathfrak{m})$ for $u \in P'$. The space $\Gamma_u^{\mathfrak{m}}$ will be called the $\mathfrak{m}$-*horizontal space at* u. It is clear that $\Gamma_u^{\mathfrak{m}} \oplus Q_u = T_uP'$. This means that every vector $X \in T_uP'$ can be uniquely written as $X = \vartheta X + mX$, where $\vartheta X \in Q_u$ and $mX \in \Gamma_u^{\mathfrak{m}}$. The vector mX is called the $\mathfrak{m}$-*horizontal component of* X.

Let π (resp. $\tilde{\pi}$) denotes the canonical projection of a bundle $P'(M, G')$ (resp. $P(M, G)$) onto the base space M. Then a differential $d_u\pi\colon T_uP' \to T_{\pi(u)}M$ is an epimorphism. Since $d_u\pi|_{Q_u} \equiv 0$, $d_u\pi|_{\Gamma_u^{\mathfrak{m}}}\colon \Gamma_u^{\mathfrak{m}} \to T_{\pi(u)}M$ is an isomorphism.

Let $V \in \mathfrak{X}(M)$. The vector field V' on P' defined by $V'_u = (d_u\pi|_{\Gamma_u^{\mathfrak{m}}})^{-1}(V_{\pi(u)})$ will be called the $\mathfrak{m}$-*horizontal lift of* V. The horizontal lift (with respect to $\tilde{\omega}$) of V will be denoted by V^*. For any $X \in T_uP'$, $X = mX + \vartheta X$. Since $h\colon T_uP \to \Gamma_u$ is linear, $hX = h(mX) + h(\vartheta X)$ and $h(\vartheta X) = 0$. Consequently, $h(mX) = hX$ for every $X \in T_uP'$, $u \in P'$.

The curvature form of the connection $\tilde{\omega}$ will be denoted by $\tilde{\Omega}$. We set $\Omega = \iota^*\tilde{\Omega}$ and we call Ω the *curvature form of a Cartan connection* ω. Clearly, forms ω and Ω satisfy the equation

$$d\omega = -\tfrac{1}{2}[\omega, \omega] + \Omega .$$

This equation will be called the *basic structure equation.*

2. Structure equations and Bianchi's identities. Let $P'(M, G')$ be a principal fibre bundle with a Cartan connection ω. Assume that there exists a linear decomposition of $\mathfrak{g}$; $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{m}$ such that $[\mathfrak{m}, \mathfrak{m}] = 0$ ($[\ ,\ ]$ denotes the bracket operation in $\mathfrak{g}$). Such a decomposition exists, for instance, for affine, projective and conformal structures, see [2].

Let γ be a differential form on P' of degree r. Then the form

$$D\gamma = d\gamma \circ m, \text{ i.e. } D\gamma(X_1, \ldots, X_{r+1}) = d\gamma(mX_1, \ldots, mX_{r+1})$$

is called the exterior covariant derivative of γ.

PROPOSITION 3. *The exterior covariant derivative of* ω *satisfies the condition*

$$d\omega(X, Y) = -\tfrac{1}{2}[\omega(X), \omega(Y)] + D\omega(X, Y)$$

for $X, Y \in T_uP'$, $u \in P'$.

Proof. We first show the following lemma.

LEMMA. *Let* $u \in P'$. *For any vector* $Y \in T_uP'$, *there exists a right invariant vector field* X *on* $P'|_U$ (U *is a neighbourhood of the point* $x = \pi(u)$) *such that* $X_u = Y$.

Proof of Lemma. Suppose that $\psi = (\pi, \varphi)$ is a local triviality mapping of the bundle $P'(M, G')$. Let $\psi = (\pi, \varphi)$: $P'|_U \to U \times G'$ and $\psi(u) = (x, e)$. We define the following vector field $\tilde{X}$ on $U \times G'$:

$$\tilde{X}_{(y,a)} = (\bar{X}_y, d_e R_a(A)),$$

where $A = d_u \varphi(Y)$ and $\bar{X}$ is a vector field on M such that $\bar{X}_x = d_u \pi(Y)$. R_a denotes here the right translation of G'. We define X by the formula

$$X_v = d\psi^{-1}(\tilde{X}_{\psi(v)})$$

for $V \in P'$. It is easy to check that X is right invariant and $X_u = Y$. This completés the proof of the Lemma.

Since both sides of the equality are bilinear and skew-symmetric in X and Y, it is sufficient to verify the equality in the following three special cases:

1. X and Y are $\mathfrak{m}$-horizontal. Since $[\omega(X), \omega(Y)] = 0$, by the assumption $[\mathfrak{m}, \mathfrak{m}] = 0$, the equality reduces to the definition of D.

2. X and Y are vertical. Let $X = A^*_u$ and $Y = B^*_u$, where $A, B \in \mathfrak{g}'$. Then

$$2d\omega(A^*, B^*) = A^*(\omega(B^*)) - B^*(\omega(A^*)) - \omega([A^*, B^*]) = -[A, B].$$

On the other hand, $D\omega(A^*, B^*) = 0$.

3. X is $\mathfrak{m}$-horizontal and Y is vertical. Let $Y = B^*_u$, where $B \in \mathfrak{g}$. By virtue of the Lemma, we can extend X to a right invariant vector field, which we denote also by X. Since ω is a 1-form, we have

$$2d\omega(X, B^*) = X(\omega(B^*)) - B^*(\omega(X)) - \omega([X, B^*]).$$

Let $b_t = \exp tB$. Then

$$\begin{aligned} B^*(\omega(X)) &= (d_u \omega(X))(B^*_u) \\ &= \lim_{t\to 0} \frac{1}{t} \{\omega_{ub_t}(X_{ub_t}) - \omega_u(X_u)\} \\ &= \lim_{t\to 0} \frac{1}{t} \{\omega_{ub_t}(d_u R_{b_t}(X)) - \omega_u(X_u)\} \\ &= \lim_{t\to 0} \frac{1}{t} \{ad_{b_t^{-1}} \omega_u(X_u) - \omega_u(X_u)\} \\ &= [\omega_u(X_u), B] \end{aligned}$$

$$\text{and } [B^*, X] = \lim_{t\to 0} \frac{1}{t} \{X - dR_{b_t} \circ dR_{b_t^{-1}} \circ X\} = 0.$$

This completes the proof of our proposition.

From Proposition 3 and the basic structure equation we obtain that $\Omega = D\omega$.

Let $\xi \in \mathfrak{m}$. We associate with ξ an $\mathfrak{m}$-horizontal vector field $B(\xi)$ on P' by the formula:

$$B(\xi)_u = \omega_u^{-1}(\xi).$$

$B(\xi)$ will be called the standard $\mathfrak{m}$-horizontal vector field corresponding to ξ.

PROPOSITION 4. *The standard* $\mathfrak{m}$*-horizontal vector fields have the following properties*

1. *If* $\xi \neq 0$, *then* $B(\xi)$ *never vanishes*;
2. $B(\alpha\xi_1+\beta\xi_2) = \alpha B(\xi_1)+\beta B(\xi_2)$ *for* $\xi_1, \xi_2 \in \mathfrak{m}$; $\alpha, \beta \in R$;
3. $\omega_u(B(\xi)_u) = \xi$ *for* $\xi \in \mathfrak{m}$;
4. $d_u R_a(B(\xi)_u) = B((ad_{a^{-1}}\xi)_{\mathfrak{m}})_{ua}+((ad_{a^{-1}}\xi)_{\mathfrak{g}'})^*_{ua}$;

(For an element $A \in \mathfrak{g}$, let $A_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component of A, $A_{\mathfrak{g}'}$ the $\mathfrak{g}'$-component of A);

5. $\omega_u([A^*, B(\xi)]_u) = [A, \xi]$ *for* $A \in \mathfrak{g}'$ *and* $\xi \in \mathfrak{m}$.

Proof. The first three assertions are trivial. Applying ω_{ua} (we know it is an isomorphism) to the equality 4, we obtain $\mathrm{ad}_{a^{-1}}\xi$ on both sides of it.

Let $a_t = \exp tA$. We have

$$\begin{aligned}
&\omega_u([A^*, B(\xi)]_u)\\
&= \omega_u\left\{\lim_{t\to 0}\frac{1}{t}\,(B(\xi)_u - dR_{a_t}(B(\xi))_{ua_t^{-1}}\right\}\\
&= \omega_u\left\{\lim_{t\to 0}\frac{1}{t}\,\left(B(\xi)_u - B((ad_{a_t^{-1}}\xi)_{\mathfrak{m}})_u - ((ad_{a_t^{-1}}\xi)_{\mathfrak{g}'})^*_u\right)\right\}\\
&= \lim_{t\to 0}\frac{1}{t}\,(\xi - ad_{a_t^{-1}}\xi) = [A, \xi]\,,
\end{aligned}$$

which completes the proof of our assertion.

Let $\omega = \omega'+\theta$ be the decomposition of ω corresponding to the decomposition of $\mathfrak{g}$; $\mathfrak{g} = \mathfrak{g}'\oplus\mathfrak{m}$. We set $\Omega' = D\omega'$, $\Theta = D\theta$. The form Θ will be called the *torsion form of a Cartan connection* ω.

From the basic structure equation, we have

$$\begin{aligned}
(1)\quad d\omega'(X, Y) &= -\tfrac{1}{2}([\omega'(X), \omega'(Y)]+[\omega'(X), \theta(Y)]_{\mathfrak{g}'}+[\theta(X), \omega'(Y)]_{\mathfrak{g}'})+\Omega'(X, Y)\,,\\
d\theta(X, Y) &= -\tfrac{1}{2}([\omega'(X), \theta(Y)]_{\mathfrak{m}}+[\theta(X), \omega'(Y)]_{\mathfrak{m}})+\Theta(X, Y)\,,
\end{aligned}$$

where $X, Y \in T_u P'$, $u \in P'$.

In the following, the cyclic sum with respect to X, Y and Z will be denoted by σ.

PROPOSITION 5. *Let* Ω *and* Θ *be the curvature form and the torsion form of a Cartan connection* ω. *Then*

$$3\,D\Omega(X, Y, Z) = \sigma[\Omega(X, Y), \theta(Z)]$$

for $X, Y, Z \in T_u P'$, $u \in P'$.

Proof. It is sufficient to deal with the case where X, Y, Z are $\mathfrak{m}$-horizontal. Assume that $X = B(\xi)$ and $Y = B(\eta)$. Since Ω is a form of degree 2 and by the basic structure equation, we have

$$\begin{aligned}
3D\Omega(X, Y, Z) &= 3d\Omega(X, Y, Z) = \sigma\{X\Omega(Y, Z)-\Omega([X, Y], Z)\}\\
&= \sigma\{Xd\omega(Y, Z)-d\omega([X, Y], Z)\}-\tfrac{1}{2}\sigma[\omega([X, Y]), \omega(Z)]\,.
\end{aligned}$$

We have also

$$\Omega(X, Y) = -\tfrac{1}{2}\big(X(\omega(B(\eta)))-Y(\omega(B(\xi)))-\omega([X, Y])\big) = -\tfrac{1}{2}\omega([X, Y])$$

hence

$$3D\Omega(X, Y, Z) = dd\omega(X, Y, Z)+\sigma[\Omega(X, Y), \omega(Z)] = \sigma[\Omega(X, Y), \theta(Z)]\,.$$

From this proposition and from the equality $D\Omega = D\Omega'+D\Theta$, we obtain the following identities

$$\text{(2)}\qquad \begin{aligned} 3D\Omega'(X, Y, Z) &= \sigma[\Omega'(X, Y), \theta(Z)]_{\mathfrak{g}'},\\ 3D\Theta(X, Y, Z) &= \sigma[\Omega'(X, Y), \theta(Z)]_{\mathfrak{m}},\end{aligned}$$

where $X, Y, Z \in T_uP'$, $u \in P'$.

In the case when ω is an affine connection and ω' is a linear connection induced by ω, equations (1) and identities (2) are known structure equations and Bianchi's identities for a linear connection.

Now consider a special situation. Assume that $\mathfrak{g}$ is a graded Lie algebra:

$$\mathfrak{g} = \mathfrak{m}_{-1}+\mathfrak{m}_0+\mathfrak{m}_1$$

such that $\mathfrak{m}_0+\mathfrak{m}_1 = \mathfrak{g}'$ and $\mathfrak{m}_{-1} = \mathfrak{m}$. Take, for instance, the Lie algebra of the projective group and the Lie algebra of the Möbius group. They have the above property (see [2], 132—135).

Let $\omega = \theta+\omega_0+\omega_1$ be the decomposition of ω corresponding to the decomposition of $\mathfrak{g}$; $\mathfrak{g} = \mathfrak{m}_{-1}+\mathfrak{m}_0+\mathfrak{m}_1$. We set $\Omega_0 = D\omega_0$, $\Omega_1 = D\omega_1$. By equations (1) and identities (2), we obtain equations:

$$\text{(3)}\qquad \begin{aligned} d\theta(X, Y) &= -\tfrac{1}{2}([\omega_0(X), \theta(Y)]-[\omega_0(Y), \theta(X)])+\Theta(X, Y)\,,\\ d\omega_0(X, Y) &= -\tfrac{1}{2}([\omega_0(X), \omega_0(Y)]+[\omega_1(X), \theta(Y)]+[\theta(X), \omega_1(Y)])+\Omega_0(X, Y)\,,\\ d\omega_1(X, Y) &= -\tfrac{1}{2}([\omega_1(X), \omega_0(Y)]+[\omega_0(X), \omega_1(Y)])+\Omega_1(X, Y)\end{aligned}$$

and identities:

$$\text{(4)}\qquad \begin{aligned} 3D\Theta(X, Y, Z) &= \sigma[\Omega_0(X, Y), \theta(Z)]\,,\\ 3D\Omega_0(X, Y, Z) &= \sigma[\Omega_1(X, Y), \theta(Z)]\,,\\ D\Omega_1 &\equiv 0\,,\end{aligned}$$

where $X, Y, Z \in T_uP'$, $u \in P'$.

3. Main results. Let $P(M, G)$ be a principal fibre bundle with a connection $\tilde{\omega}$. We define a series of $\mathfrak{g}$-valued functions on P in the following way:

$$(\mathrm{I}_0)\ f^0 = \tilde{\Omega}(X, Y)\,,$$

$$\vdots$$

$$(\mathrm{I}_k)\ f^k = V_k \dots V_1(\tilde{\Omega}(X, Y))\,,$$

where X, Y, $V_1, \ldots, V_k$ are arbitrary horizontal vector fields on P. Let $\mathfrak{m}_0(u)$ be the subspace of $\mathfrak{g}$ spanned by all elements of the form $\tilde{\Omega}_u(X, Y)$, where X and Y are horizontal vectors at u. Let $\mathfrak{m}_k(u)$ be the subspace of $\mathfrak{g}$ spanned by $\mathfrak{m}_{k-1}(u)$ and by the values at u of all functions $\bar{f}^k$ of the form (I_k). The union of all $\mathfrak{m}_k(u)$, $k = 0, 1, 2, \ldots$ will be denoted by $\mathfrak{h}'(u)$.

The subspace $\mathfrak{h}'(u)$ of $\mathfrak{g}$ is a subalgebra of the Lie algebra $\mathfrak{g}$. The connected Lie subgroup $\Phi'(u)$ of G generated by $\mathfrak{h}'(u)$ is called the *infinitesimal holonomy group at u*.

Consider a $\mathfrak{g}$-valued function $\bar{f}^k$ on P of the form

$$(\mathrm{II}_k)\ \bar{f}^k = V_k^* \ldots V_1^* \tilde{\Omega}(X^*, Y^*),$$

where $X, Y, V_1, \ldots, V_k \in \mathscr{X}(M)$. It is easy to verify that for each k, $k = 1, 2, \ldots, \mathfrak{m}_k(u)$ is spanned by $\mathfrak{m}_{k-1}(u)$ and by the values at u of all functions of the form (II_k).

The infinitesimal holonomy groups have the following property

$$\Phi'(ua) = Ad_{a^{-1}}\Phi'(u),$$

where $Ad_a x = axa^{-1}$ for $x \in G$ and $a \in G$. Consequently, $\dim \Phi'(u)$ is constant on $\tilde{\pi}^{-1}(\tilde{\pi}(u))$.

Later on, we shall use the two following theorems.

THEOREM 1. *If* $\dim \Phi'(u)$ *is constant on* P, *then* $\Phi'(u) = \Phi^0(u)$ *for every* $u \in P$, *where* $\Phi^0(u)$ *denotes the restricted holonomy group of* $\tilde{\omega}$ *with the reference point* u.

THEOREM 2. *For a real analytic fibre bundle* $P(M, G)$ *with a real analytic connection* $\tilde{\omega}$, *we have* $\Phi'(u) = \Phi^0(u)$ *for every* $u \in P$.

For more details on this point, see [1]; Chapter II, § 10.

Assume now that $P'(M, G')$ is a principal fibre bundle with a Cartan connection ω and $P(M, G)$ is the overbundle of $P'(M, G')$ with the extended connection $\tilde{\omega}$. Let a linear decomposition of $\mathfrak{g}$, $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{m}$, be fixed.

LEMMA 1. *Let* f *be a* $\mathfrak{g}$*-valued function on* P *of type* ad_G. *Then, for any vector field* V *on* M, *the function* V^*f *is also of type* ad_G *and its restriction to* P' *is equal to the function*

$$V'f' + [\omega'(V'), f'],$$

where f' *denotes the restriction of* f *to* P'.

Proof. Since V^* is right invariant on P, V^*f is of type ad_G. At any point u of P', we have

$$V_u^* = hV_u^* = hV_u'.$$

Hence the vector $\tilde{V}_u = V_u' - V_u^*$ is vertical in P.

From the fact that f is of type ad_G, it follows

$$\tilde{V}_u f = -[\tilde{\omega}_u(\tilde{V}_u), f(u)]$$

We have also

$$\tilde{\omega}_u(\tilde{V}_u) = \tilde{\omega}_u(V_u') = \omega_u(V_u')$$

Therefore, at each point u of P',

$$V_u^* f = V_u' f - \tilde{V}_u f = V_u' f + [\tilde{\omega}_u(\tilde{V}_u), f(u)] = V_u' f' + [\omega_u(V_u'), f'(u)].$$

THEOREM 3. (Ozeki [3]) *The Lie algebra $\mathfrak{h}'(u)$ of the infinitesimal holonomy group of a Cartan connection ω at a point $u \in P'$ is spanned by the values at u of all functions f^k on P' obtained in the following way: For any vector fields $X, Y, V_1, \ldots, V_k \ldots$ on M, we define $\mathfrak{g}$-valued functions f^k on P' successively by:*

$$(\mathrm{II}'_0)\ f^0 = \Omega(X', Y'),$$
$$(\mathrm{II}'_1)\ f^1 = V'_1 f^0 + [\omega(V'_1), f^0],$$
$$\vdots$$
$$(\mathrm{II}'_{k+1})\ f^{k+1} = V'_{k+1} f^k + [\omega(V'_{k+1}), f^k],$$
$$\vdots$$

Proof. Let $u \in P'$. Then $\Omega_u(X', Y') = \tilde{\Omega}_u(X^*, Y^*)$, because $hX'_u = hX^*_u$ and $hY'_u = hY^*_u$. This shows that $f^0 = \Omega(X', Y')$ is just the restriction of the function $\tilde{f}^0 = \tilde{\Omega}(X^*, Y^*)$ to P'. Applying Lemma 1. to the function $\tilde{f}^0 = \tilde{\Omega}(X^*, Y^*)$ we see that f^1 is also the restriction of $\tilde{f}^1 = V^*_1 \tilde{\Omega}(X^*, Y^*)$ to P'. In the same way, we see that f^k is the restriction to P' of $V^*_k \ldots V^*_1 \tilde{\Omega}(X^*, Y^*)$ which is a $\mathfrak{g}$-valued function on P of type ad_G. This completes the proof.

Introduce now the following functions

$$(\mathrm{I}'_0)\ f^0 = \Omega(X, Y),$$
$$(\mathrm{I}'_1)\ f^1 = V_1 f^0 + [\omega(V_1), f^0],$$
$$\vdots$$
$$(\mathrm{I}'_{k+1})\ f^{k+1} = V_{k+1} f^k + [\omega(V_{k+1}), f^k],$$

where $X, Y, V_1, \ldots, V_{k+1}$ are arbitrary $\mathfrak{m}$-horizontal vector fields on P'.

It is evident that $\mathfrak{m}_k(u)$, $k = 1, 2, \ldots$, is spanned by $\mathfrak{m}_{k-1}(u)$ and by the values at u of all functions f^k of the form (I'_k).

Observe that a function f^k (resp. $\tilde{f}^k$) of the form (I'_k) or (II'_k) (resp. (I_k) or (II_k)) may be defined also by means of local vector fields. In the following, a function f^k (resp. $\tilde{f}^k$) will be called a *function of the form* (I'_k) or (II'_k) (resp. (I_k) or (II_k), both in the case when the suitable vector fields are global as well as when they are local.

A diffeomorphism f of P' (f may be local) onto itself is called a *transformation of* ω if it preserves the form ω, i.e. $f^*\omega = \omega$. If f is a transformation of ω, then $d_u f(\Gamma^{\mathfrak{m}}_u) = \Gamma^{\mathfrak{m}}_{(u)}$ for every $u \in P'$. If a transformation f of ω is a bundle automorphism, i.e. commutes with the right transformation $R_a (a \in G')$, then f is called a *transformation of* (P', ω).

A vector field X on P' is called an *infinitesimal transformation of* ω (resp. (P', ω)) if the local 1-parameter group of local transformations generated by X in a neighbourhood of each point of P' consists of transformations of ω (resp. (P', ω)).

The set $\check{k}$ of vector fields on P' is called

a) *horizontally transitive at* $u \in P'$ if for any $V \in \Gamma_u$, there is $X \in \check{k}$ such that $h(X_u) = V$;
b) *horizontally transitive on* P' if it is horizontally transitive at each point $u \in P'$.

The set $\check{k}$ of vector fields on P' is horizontally transitive at $u \in P'$ if for any linear decomposition $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{m}$ and for any $V \in \Gamma^{\mathfrak{m}}_u$ there is $X \in \check{k}$ such that $m(X_u) = V$. This follows from the fact that $h(mX) = hX$ for $X \in T_u P'$.

PROPOSITION 6. *Let f be a* $\mathfrak{g}$*-valued function on P' and let $f(u)\in\mathfrak{m}_{k-1}(u)$ for every $u\in P'$. Then*

$$V'_u f+[\omega_u(V'_u),f(u)]\in\mathfrak{m}_k(u)$$

for every $u\in P'$ and $V\in\mathscr{X}(M)$.

Proof. From our assumption, we have that f can be expressed as follows:

$$f=\alpha_0 f_0+\ldots+\alpha_r f_r\,,$$

where $\alpha_0,\ldots,\tilde{a}_r$ are R-valued functions of class C^∞ and f_i, $0\leqslant i\leqslant r$, is a function of the form (II'_s) for some s, $0\leqslant s\leqslant k-1$. Then

$$V'f=V'(\sum_{i=0}^{r}\alpha_i f_i)=\sum_{i=0}^{r}((V'\alpha_i)f_i+\alpha_i(V'f_i))\,.$$

Therefore

$$V'_u f+[\omega_u(V'_u),f(u)]=\sum_{i=0}^{r}(V'\alpha_i)(u)f_i(u)+\sum_{i=0}^{r}\alpha_i(u)((V'f_i)(u)+[\omega_u(V'_u),f_i(u)])\,.$$

It is clear that $\sum_{i=0}^{r}(V'\alpha_i)(u)f_i(u)\in\mathfrak{m}_{k-1}(u)$ and

$$\sum_{i=0}^{r}\alpha_i(u)((V'f_i)(u)+[\omega_u(V'_u),f_i(u)])\in\mathfrak{m}_k(u)\,,$$

which proves the proposition.

POPOSITION 7. *If f is a* $\mathfrak{g}$*-valued function of type $\mathrm{ad}_{G'}$ on P', then for any vector field X on P', we have*

$$(\vartheta X)_u f=-[\omega_u(\vartheta X),f(u)]\,.$$

Proof. Let $\omega_u(\vartheta X_u)=A$. A belongs to $\mathfrak{g}'$. Let $a_t=\exp tA$. Then

$$(\vartheta X)_u f=A^*_u f=\lim_{t\to 0}\frac{1}{t}\{f(ua_t)-f(u)\}$$

$$=\lim_{t\to 0}\frac{1}{t}\{ad_{a_t^{-1}}f(u)-f(u)\}=-[A,f(u)]=-[\omega_u(\vartheta X)_u,f(u)]$$

and this completes the proof.

Let u_0 be an arbitrary point of P' which we choose as a reference point. We define a linear mapping $\Lambda\colon\mathscr{X}(P')\to\mathfrak{g}$ by

$$\Lambda(X)=-\omega_{u_0}(X)\,.$$

PROPOSITION 8. *If X and Y are infinitesimal transformations of ω, then the curvature form Ω satisfies the following condition:*

$$2\Omega_{u_0}(X,Y)=[\Lambda(X),\Lambda(Y)]-\Lambda([X,Y])\,.$$

Proof. Since ω and Ω satisfy the basic structure equation and ω is a 1-form, we have

$$2\Omega(X, Y) = 2d\omega(X, Y)+[\omega(X), \omega(Y)]$$
$$= X(\omega(Y))-Y(\omega(X))-\omega([X, Y])+[\omega(X), \omega(Y)]\,.$$

X and Y are infinitesimal transformations of ω, hence

$$0 = (L_X\omega)(Y) = X(\omega(Y))-\omega([X, Y])\,,$$
$$0 = (L_Y\omega)(X) = Y(\omega(X))-\omega([Y, X])\,.$$

Finally, we obtain

$$2\Omega_{u_0}(X, Y) = [\omega_{u_0}(X), \omega_{u_0}(Y)]-\Lambda([X, Y]) = [\Lambda(X), \Lambda(Y)]-\Lambda([X, Y])\,.$$

THEOREM 4. *Let $P'(M, G')$ be a principal fibre bundle with a Cartan connection ω. Let $\check{k}$ be a horizontally transitive at $u_0 \in P'$ set of infinitesimal transformations of ω. Then the Lie algebra $\mathfrak{h}'(u_0)$ of the infinitesimal holonomy group $\Phi'(u_0)$ of the connection ω is given by*

$$\mathfrak{m}_0+[\Lambda(\check{k}), \mathfrak{m}_0]+[\Lambda(\check{k}), [\Lambda(\check{k}), \mathfrak{m}_0]]+\dots,$$

where $\mathfrak{m}_0$ is the subspace of $\mathfrak{g}$ spanned by

$$\{[\Lambda(X), \Lambda(Y)]-\Lambda([X, Y]);\ X, Y\in \check{k}\}$$

Proof. We define the following series of subspaces $\mathfrak{m}_k$:

$$\mathfrak{m}_1 = \mathfrak{m}_0+[\Lambda(\check{k}), \mathfrak{m}_0]\,,$$
$$\vdots$$
$$\mathfrak{m}_k = \mathfrak{m}_{k-1}+[\Lambda(\check{k}), \mathfrak{m}_{k-1}]\,.$$

To prove the theorem, it is sufficient to show that $\mathfrak{m}_k = \mathfrak{m}_k(u_0)$ for $k = 0, 1, 2, \dots$

We first prove that $\mathfrak{m}_0 = \mathfrak{m}_0(u_0)$. Using Proposition 8, we have

$$\mathfrak{m}_0 = \text{the subspace of } \mathfrak{g} \text{ spanned by } \{\Omega_{u_0}(X, Y);\ X, Y\in \check{k}\}\,.$$

The set $\check{k}$ is horizontally transitive at u_0 and

$$\Omega_{u_0}(X, Y) = \Omega_{u_0}(mX, mY), \text{ thus } \mathfrak{m}_0 = \mathfrak{m}_0(u_0)\,.$$

Let f^k be a function of the form (II_k'). By virtue of Lemma 1, we know that f^k is of type $\mathrm{ad}_{G'}$. Applying Proposition 7 to f^k, we have

$$(mX)_{u_0}f^k+[\omega_{u_0}(mX)_{u_0}, f^k(u_0)]$$
$$= -(\vartheta X)_{u_0}f^k+X_{u_0}f^k+[\omega_{u_0}(mX)_{u_0}, f^k(u_0)]$$
$$= [\omega_{u_0}(\vartheta X)_{u_0}, f^k(u_0)]+X_{u_0}f^k+[\omega_{u_0}(mX)_{u_0}, f^k(u_0)]$$
$$= [\omega_{u_0}(X_{u_0}), f^k(u_0)]+X_{u_0}f^k$$

for any $X\in \mathfrak{X}(P')$. Hence we have

$$(mX)_{u_0}f^k+[\omega_{u_0}(mX)_{u_0}, f^k(u_0)] = -[\Lambda(X), f^k(u_0)]+X_{u_0}f^k \tag{5}$$

for every f^k of the form (II_k') and $X\in \mathfrak{X}(P')$.

We now prove the following two lemmas.

LEMMA 2. *If Y is an $\mathfrak{m}$-horizontal vector field on P' and $X \in \check{k}$, then $[X, Y]$ is $\mathfrak{m}$-horizontal.*

Proof of Lemma 2. ω is a 1-form on P', thereby

$$X(\omega(Y)) = L_X\omega(Y) + \omega([X, Y]) .$$

Since Y is $\mathfrak{m}$-horizontal, $\omega(Y)$ is an $\mathfrak{m}$-valued function on P' so that $X(\omega(Y))$ is also an $\mathfrak{m}$-valued function on P'. On the other hand, $L_X\omega = 0$. Thus $\omega_u([X, Y]_u)$ belongs to $\mathfrak{m}$ for every $u \in P'$.

LEMMA 3. *Let f^k be a function of the form (II'_k) and $X \in \check{k}$. Then $X_u f^k \in \mathfrak{m}_k(u)$ for every $u \in P'$, $k = 0, 1, \ldots$*

Proof of Lemma 3. This lemma will be proved by induction. At first, we show that $X_u\Omega(W', Z') \in \mathfrak{m}_0(u)$ for $W, Z \in \mathscr{X}(M)$, $X \in \check{k}$. Since Ω is a 2-form on P', we have

$$X_u\Omega(W', Z') = (L_X\Omega)(W', Z') + \Omega_u([X, W'], Z') + \Omega_u(W', [X, Z']) .$$

By Lemma 2, $\Omega_u([X, W'], Z') \in \mathfrak{m}_0(u)$ and $\Omega_u(W', [X, Z']) \in \mathfrak{m}_0(u)$. On the other hand, $L_X\Omega = 0$ so that $X_u f^0 \in \mathfrak{m}_0(u)$ for any function f^0 of the form (II'_0) and for every $u \in P'$.

Suppose now that $X_u f^{k-1} \in \mathfrak{m}_{k-1}(u)$ for every function f^{k-1} of the form (II'_{k-1}) and for every $u \in P'$. Let f^k be a function of the form (II'_k). Let

$$f^k = V'f^{k-1} + [\omega(V'), f^{k-1}] ,$$

where f^{k-1} is of the form (II'_{k-1}). Observe that

$$\begin{aligned} X_u f^k &= X_u V'f^{k-1} + X_u[\omega(V'), f^{k-1}] \\ &\equiv V'_u Xf^{k-1} + X_u[\omega(V'), f^{k-1}] - [\omega_u([X, V']_u), f^{k-1}(u)] \mod \mathfrak{m}_k(u) , \end{aligned}$$

because

$$[X, V']_u f^{k-1} + [\omega_u([X, V']_u), f^{k-1}(u)] \in \mathfrak{m}_k(u) .$$

This follows from the fact that $[X, V']$ is $\mathfrak{m}$-horizontal (by Lemma 2) and from Proposition 6. Denote the right hand side of this congruence by $\mathscr{P}$. We have

$$\begin{aligned} \mathscr{P} = V'_u(Xf^{k-1}) + [\omega_u(V'_u), X_u f^{k-1}] + X_u[\omega(V'), f^{k-1}] - \\ - [\omega_u([X, V']), f^{k-1}(u)] - [\omega_u(V'_u), X_u f^{k-1}] . \end{aligned}$$

By the inductive assumption and by Proposition 6,

$$V'_u(Xf^{k-1}) + [\omega_u(V'_u), X_u f^{k-1}] \in \mathfrak{m}_k(u) .$$

Since the bracket operation in $\mathfrak{g}$ is bilinear,

$$X_u[\omega(V'), f^{k-1}] = [X_u(\omega(V')), f^{k-1}(u)] + [\omega_u(V'_u), X_u f^{k-1}] .$$

We have also

$$0 = (L_X\omega)(V') = X(\omega(V')) - \omega([X, V'])$$

and consequently

$$X_u[\omega(V'), f^{k-1}] - [\omega_u([X, V']), f^{k-1}(u)] - [\omega_u(V'_u), X_u f^{k-1}] = 0\,.$$

This completes the proof of our lemma.

Going back to the proof of the theorem, we see

$$(6)\qquad (mX)_{u_0} f^k + [\omega_{u_0}(mX)_{u_0}, f^k(u_0)] \equiv -[\Lambda(X), f^k(u_0)] \operatorname{mod} \mathfrak{m}_k(u_0)$$

by the equality (5) and by Lemma 3.

Assume now that $\mathfrak{m}_r = \mathfrak{m}_r(u_0)$ for all $r<s$. Since $\check{k}$ is horizontally transitive at u_0, every $\mathfrak{m}$-horizontal vector at u_0 is of the form $(\mathfrak{m}X)_{u_0}$ for some $X \in \check{k}$. Therefore, $\mathfrak{m}_s(u_0)$ is spanned by $\mathfrak{m}_{s-1}(u_0)$ and by the set of all

$$(mX)_{u_0} f^{s-1} + [\omega_{u_0}(mX)_{u_0}, f^{s-1}(u_0)]\,,$$

where $X \in \check{k}$ and f^{s-1} is a function of the form (II'_{s-1}). From the inductive assumption and from the congruence (6) it follows that $\mathfrak{m}_s(u_0)$ is spanned by $\mathfrak{m}_{s-1}$ and by $[\Lambda(\check{k}), \mathfrak{m}_{s-1}]$. On the other hand, $\mathfrak{m}_s$ is spanned just by $\mathfrak{m}_{s-1}$ and by $[\Lambda(\check{k}), \mathfrak{m}_{s-1}]$. Hence we have completed the proof of Theorem 4.

THEOREM 5. *Assume in Theorem 4 that $\check{k}$ is a subalgebra of the Lie algebra of all vector fields on P'. Then*

$$\mathfrak{h}'(u_0) \subset \mathfrak{p} \subset \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}'(u_0))\,,$$

where $\mathfrak{p}$ is the subalgebra of $\mathfrak{g}$ spanned by the set $\Lambda(\check{k})$ and $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}'(u_0))$ denotes the normalizor of the Lie algebra $\mathfrak{h}'(u_0)$ in the Lie algebra $\mathfrak{g}$.

Proof. From Theorem 4, we know that

$$\mathfrak{h}'(u_0) = \mathfrak{m}_0 + [\Lambda(\check{k}), \mathfrak{m}_0] + [\Lambda(\check{k}), [\Lambda(\check{k}), \mathfrak{m}_0]] + \ldots,$$

where $\mathfrak{m}_0$ is the subspace in $\mathfrak{g}$ spanned by

$$\{[\Lambda(X), \Lambda(Y)] - \Lambda([X, Y]);\ X, Y \in \check{k}\}\,.$$

Hence the inclusion $\mathfrak{h}'(u_0) \subset \mathfrak{p}$ is evident.

To verify the inclusion $\mathfrak{p} \subset \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}'(u_0))$, it is sufficient to show that

$$[\Lambda(\check{k}), \mathfrak{h}'(u_0)] \subset \mathfrak{h}'(u_0)\,.$$

But this is obvious, because $[\Lambda(\check{k}), \ldots [\Lambda(\check{k}), \mathfrak{m}_0] \ldots] \subset \mathfrak{h}'(u_0)$.

THEOREM 6. *Let $P'(M, G')$ be a principal fibre bundle with a Cartan connection ω. Assume that there exists a set $\mathscr{L}$ of infinitesimal transformations of (P', ω) horizontally transitive on P'. Let $\check{k}$ be a set of infinitesimal transformations of ω horizontally transitive at u_0. Then the Lie algebra $\mathfrak{h}^0(u_0)$ of the restricted holonomy group $\Phi^0(u_0)$ is equal to the sum in $\mathfrak{g}$*

$$\mathfrak{m}_0 + [\Lambda(\check{k}), \mathfrak{m}_0] + [\Lambda(\check{k}).\ [\Lambda(\check{k}), \mathfrak{m}_0]] + \ldots,$$

where $\Lambda(X) = -\omega_{u_0}(X_{u_0})$ and $\mathfrak{m}_0$ is the subspace of $\mathfrak{g}$ spanned by

$$\{[\Lambda(X), \Lambda(Y)] - \Lambda([X, Y]);\ X, Y \in \check{k}\}\,.$$

Proof. We may assume that M is connected. In view of Theorem 1 and Theorem 4, it is sufficient to show that $\dim\Phi'(u)$ is constant on P'.

We define the following mapping S:

$$S\colon \mathscr{L}\to\mathscr{X}(M);\ S(X) = \bar{X};\ \bar{X}_x = d_u\pi(X_u)\,,$$

where u is taken freely from the fibre over x. $\bar{X}_u$ is independent of the choice of u, which follows from the fact that an automorphism of a principal fibre bundle is right invariant. Every point x of M has a neighbourhood U and a cross-section $\sigma\colon U\to P'$. It is clear that $\bar{X}|_U = d\pi\circ X\circ\sigma$. Consequently, the vector field $\bar{X}$ is of class C^∞. Local 1-parameter groups of local transformations generated by $\bar{X}$ will be denoted by $(\bar{\varphi}_t)$. From the formula $\bar{X}|_U = d\pi\circ X\circ\sigma$, we obtain that if (φ_t) is the local 1-parameter group of local transformations generated by X in a neighbourhood $\pi^{-1}(U)$ of u, then $(\bar{\varphi}_t)\colon \bar{\varphi}_t = \pi\circ\varphi_t\circ\sigma$ is the local 1-parameter group of local transformations generated by $\bar{X}$ in a neighbourhood U of $\pi(u)$.

Take a linear decomposition of $\mathfrak{g}$: $\mathfrak{g} = \mathfrak{g}'\oplus\mathfrak{m}$. Let $x\in M$ and Y be any tangent vector at x. Let $V\in\mathscr{X}(M)$ be such that $V_x = Y$ and let u be any point $u\in P'$ with $\pi(u) = x$. Then $V'_u\in\Gamma^{\mathfrak{m}}_u$. The set $\mathscr{L}$ is horizontally transitive, thereby there is $X\in\mathscr{L}$ such that $mX_u = V'_u$. Hence

$$\bar{X}_x = d_u\pi(X_u) = d_u\pi(mX_u) = d_u\pi(V'_u) = Y\,.$$

Consequently, we see that $S(\mathscr{L})$ is a transitive on M set of vector fields. This fact implies that for every $x\in M$ there exists a neighbourhood U of x such that for every $y\in U$ there is a local transformation $\bar{\varphi}_t$ satisfying condition $\bar{\varphi}_t(x) = y$. This means that φ_t sends the fibre over x into the fibre over y. Since $\varphi_t^*\omega = \omega$, accordingly $\varphi_t^*\Omega = \Omega$. Consider a function f^0 of the form (I'_0), i.e. $f^0 = \Omega(W, Y)$, where W, Y are $\mathfrak{m}$-horizontal vector fields on P'. Vector fields φ_t^*W and φ_t^*Y are $\mathfrak{m}$-horizontal, hence $\check{f}^0 = \Omega(\varphi_t^*W, \varphi_t^*Y)$ is of the form (I'_0). Furthermore $\check{f}^0\circ\varphi_t = f^0$. Suppose now that for every $r<k$ and for any function f^r of the form (I'_r) there exists a function $\check{f}^r$ of the form (I'_r) such that $\check{f}^r\circ\varphi_t = f^r$.

A function f^k of the form (I'_k) can be written as $f^k = Vf^{k-1}+[\omega(V), f^{k-1}]$, where f^{k-1} is of the form (I'_{k-1}). We define $\check{f}^k$ by the formula

$$\check{f}^k = (\varphi_t^*V)\check{f}^{k-1}+[\omega(\varphi_t^*V), \check{f}^{k-1}]\,,$$

where $\check{f}^{k-1}$ is of the form (I'_{k-1}) and $\check{f}^{k-1}\circ\varphi_t = f^{k-1}$. We have

$$\begin{aligned}\check{f}^k(\varphi_t(u)) &= (d\varphi_t\circ V)_u\check{f}^{k-1}+[\omega_{\varphi_t(u)}(d_u\varphi_t(V_u)), \check{f}^{k-1}(\varphi_t(u))]\\ &= d_u(\check{f}^{k-1}\circ\varphi_t)(V_u)+[\omega_u(V_u), f^{k-1}(u)]\\ &= V_uf^{k-1}+[\omega_u(V_u), f^{k-1}(u)] = f^k(u)\,.\end{aligned}$$

Hence, for every $k = 0, 1, \ldots$ and for any function f^k of the form (I'_k) there exists a function $\check{f}^k$ of the form (I'_k) such that $\check{f}^k\circ\varphi_t = f^k$. This means that $\mathfrak{m}_k(u)\subset\mathfrak{m}_k(\varphi_t(u))$. Using this inclusion for the transformation φ_{-t} and for the point $\varphi_t(u)$, we obtain the inverse inclusion. Therefore, $\mathfrak{m}_k(u) = \mathfrak{m}_k(\varphi_t(u))$ and consequently $\Phi'(u) = \Phi'(\varphi(u))$. Finally, we obtain that $\dim\Phi'(u)$ is constant on P', because it is locally constant and M is connected. Hence our theorem is proved.

As an immediate consequence of Theorems 2 and 4, we obtain

COROLLARY 1. *Let $P'(M, G')$ be a principal fibre bundle with a Cartan connection ω. Assume that the overbundle $P(M, G)$ of $P'(M, G')$ is real analytic and the extended connection $\tilde{\omega}$ in P is also real analytic. If $\check{k}$ is a set of infinitesimal transformations of ω horizontally transitive at u_0, then the Lie algebra $\mathfrak{h}^0(u_0)$ of the restricted holonomy group $\Phi^0(u_0)$ is given by*

$$\mathfrak{m}_0+[\Lambda(\check{k}), \mathfrak{m}_0]+[\Lambda(\check{k}), [\Lambda(\check{k}), \mathfrak{m}_0]]+\ldots,$$

where $\Lambda(X) = -\omega_{u_0}(X_{u_0})$ *and* $\mathfrak{m}_0$ *is the subspace of* $\mathfrak{g}$ *spanned by*

$$\{[\Lambda(X), \Lambda(Y)]-\Lambda([X, Y]);\ X, Y\in \check{k}\}.$$

By arguments similar to the proof of Theorem 5, we obtain

COROLLARY 2. *With the same assumptions as in Theorem* 6 *or in Corollary* 1 *and with the further assumption that $\check{k}$ is a subalgebra of the Lie algebra $\mathfrak{X}(P')$ the following inclusions are true*

$$\mathfrak{h}^0(u_0)\subset \mathfrak{p}\subset \mathfrak{n}_g(\mathfrak{h}^0(u_0)),$$

where the notation is analogous to the notation in Theorem 5.

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