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On a Boundary Value Problems for Systems of Ordinary Differential Equations of Second Order

INTRODUCTION

In the present note we are concerned with a system of ordinary differential equations

(0.1) $$x_i'' = f_i(t, x_1, \dots, x_m, x_1', \dots, x_m') \quad (i = 1, \dots, m),$$

and the boundary value problem

(0.2) $$x_i(0) = x_i(h) = 0 \quad (i = 1, \dots, m).$$

Problem (0.1)—(0.2) can be written shortly in vector notation as

(0.3) $$x'' = f(t, x, x'), \quad x(0) = x(h) = 0,$$

where $x = (x_1, \dots, x_m)$, $f = (f_1, \dots, f_m)$.

By a *solution* of (0.3) we mean any function $x: [0, h] \to R^m$ with absolutely continuous derivative $x'(t)$, satisfying (0.3) almost everywhere on $[0, h]$.

Problem (0.3) has been considered by I. Kovač and I. Savtchenko [3]; they have given a solution to this problem in the case of $f(t, x, x') = f(t, x)$. In the present note, making use of the method of A. Lasota and Z. Opial [4], we give a generalization and an extension of their results.

Section 1 contains some inequalities of Wirtinger's and Opial's type. Section 2 is devoted to some differential inequalities and in Section 3 we formulate and prove the main theorems.

1. INEQUALITIES

For $x \in R^m$, $|x|$ denotes the Euclidean norm of x. In what follows all functions $x: [0, h] \to R^m$ are supposed to be absolutely continuous and such that $x(0) = x(h) = 0$.

Inequality 1. If the derivative $x'(t)$ is absolutely continuous, then

$$\int_0^h |x'(t)|^2 dt \leqslant \int_0^h |x(t)||x''(t)| dt .$$

The equality occurs for $x(t) = (x_1(t), \ldots, x_m(t))$, where $x_i(t) = A \sin \frac{\pi t}{h}$ $(i = 1, \ldots, m)$.

Inequality 2 ([2], p. 184). Under the general assumptions,

$$\int_0^h |x(t)|^2 dt \leqslant \frac{h^2}{\pi^2} \int_0^h |x'(t)|^2 dt .$$

The equality occurs for $x(t) = (x_1(t), \ldots, x_m(t))$, where $x_i(t) = A \sin \frac{\pi t}{h}$ $(i = 1, \ldots, m)$.

Inequality 3 [6]. Under the general assumptions,

$$\int_0^h x(t)||x'(t)| dt \leqslant \frac{h}{4} \int_0^h |x'(t)|^2 dt .$$

The equality occurs for $x(t) = (x_1(t), \ldots, x_m(t))$, where

$$x_i(t) = \begin{cases} \frac{1}{2} t & \text{for} \quad 0 \leqslant t \leqslant \frac{h}{2} \\ \frac{1}{2}(h-t) & \text{for} \quad \frac{h}{2} \leqslant t \leqslant h \end{cases} \qquad (i = 1, \ldots, m).$$

Inequality 4. For any summable function $p: [0, h] \to [0, +\infty)$,

$$\int_0^h p(t)|x(t)|^2 dt < \frac{h}{4} \int_0^h p(t) dt \int_0^h |x'(t)|^2 dt .$$

This inequality is the best possible and we have the equality only if $x(t) \equiv 0$ or $p(t) = 0$ almost everywhere in $[0, h]$.

Inequality 5. For any square summable function $p: [0, h] \to [0, +\infty)$,

$$\int_0^h p(t)|x(t)||x'(t)| dt \leqslant \left(\frac{h}{4} \int_0^h p^2(t) dt\right)^{\frac{1}{2}} \int_0^h |x'(t)|^2 dt .$$

Proof of Inequality 1. For $x(t) = (x_1(t), \ldots, x_m(t))$, $x(0) = x(h) = 0$, integrating by parts yields

$$\int_0^h x_i(t) x_i''(t) dt = - \int_0^h x_i'^2(t) dt$$

and hence

$$\int_0^h x_i'^2(t) dt \leqslant \int_0^h |x_i(t)||x_i''(t)| dt .$$

Using the Cauchy inequality, we obtain

$$\int_0^h |x'(t)|^2 dt = \sum_{i=1}^m \int_0^h x_i'^2(t)\,dt \leqslant \sum_{i=1}^m \int_0^h |x_i(t)||x_i''(t)|dt$$

$$\leqslant \int_0^h \Big(\sum_{i=1}^m x_i^2(t)\Big)^{\frac{1}{2}} \Big(\sum_{i=1}^m x_i''^2(t)\Big)^{\frac{1}{2}} dt = \int_0^h |x(t)||x''(t)|dt.$$

Proof of Inequality 2. From the well-known Wirtinger inequality [2] it follows that

$$\int_0^h |x(t)|^2 dt = \sum_{i=1}^m \int_0^h x_i^2(t)\,dt$$

$$\leqslant \sum_{i=1}^m \frac{h^2}{\pi^2} \int_0^h x_i'^2(t)\,dt = \frac{h^2}{\pi^2} \int_0^h |x'(t)|^2 dt.$$

Proof of Inequality 3. Our proof is a simple modification (for the vector case) of the proof give in [5] (see also [6]) for the one dimensional version of this inequality.

For

$$y(t) = \int_0^t |x'(t)|dt \quad \text{and} \quad z(t) = \int_t^h |x'(t)|dt$$

we have $y'(t) = |x'(t)| = -z'(t)$ and $|x(t)| \leqslant y(t)$, $|x(t)| \leqslant z(t)$ for $t \in [0, h]$. Hence

$$\int_0^{h/2} |x(t)||x'(t)| \leqslant \int_0^{h/2} y(t)y'(t)\,dt = \frac{1}{2} y^2\left(\frac{h}{2}\right),$$

$$\int_{h/2}^h |x(t)||x'(t)|dt \leqslant -\int_{h/2}^h z(t)z'(t)\,dt = \frac{1}{2} z^2\left(\frac{h}{2}\right).$$

Thus

$$\int_0^h |x(t)||x'(t)|dt \leqslant \frac{1}{2}\left[y^2\left(\frac{h}{2}\right) + z^2\left(\frac{h}{2}\right)\right].$$

Furthermore, using the Buniakowski inequality we get

$$y^2\left(\frac{h}{2}\right) \leqslant \frac{h}{2} \int_0^{h/2} |x'(t)|^2 dt, \quad z^2(t) \leqslant \frac{h}{2} \int_{h/2}^h |x'(t)|^2 dt$$

and this proves Inequality 3.

Proof of Inequality 4. We have, for every $t \in [0, h]$,

$$|x(t)| \leqslant \int_0^t x'(t)|dt, \quad |x(t)| \leqslant \int_t^h |x'(t)|dt.$$

Hence

$$|x(t)| \leqslant \frac{1}{2}\int_0^h |x'(t)|dt$$

and therefore

$$\int_0^h p(t)|x(t)|^2 dt \leqslant \frac{1}{4}\int_0^h p(t)dt\Big[\int_0^h |x'(t)|dt\Big]^2 \tag{1.1}$$

Finally, applying the Schwartz inequality

$$\Big[\int_0^h |x'(t)|dt\Big]^2 \leqslant h\int_0^h |x'(t)|^2 dt \tag{1.2}$$

we get Inequality 4. It is easy to see that equalities in (1.1) and (1.2) with assumption $\int_0^h p(t)dt > 0$ imply $x(t) \equiv 0$ in $[0, h]$.

The following example shows that, if we replace $\frac{h}{4}$ by any number $\alpha < \frac{h}{4}$, then Inequality 4 fails to be true. Let

$$x_i(t) = \begin{cases} t & \text{for} \quad 0 \leqslant t \leqslant k \\ k & \text{for} \quad k \leqslant t \leqslant h-k \\ -t+k & \text{for} \quad h-k \leqslant t \leqslant h, \end{cases}$$

$$p(t) = \begin{cases} 0 & \text{for} \quad 0 \leqslant t \leqslant k \\ 1 & \text{for} \quad k < t < h-k \\ 0 & \text{for} \quad h-k \leqslant t \leqslant h, \end{cases}$$

where $2\alpha < k < \frac{h}{2}$, $i = 1, \dots, m$. We have

$$\int_0^h p(t)|x(t)|^2 dt = mk^2(h-2k)$$

$$\int_0^h p(t)dt = h-2k, \qquad \int_0^h |x'(t)|^2 dt = 2km.$$

Hence $mk^2(h-2k) \leqslant 2\alpha mk(h-2k)$, in contradiction with $2\alpha < k$.

Proof of Inequality 5 is an immediate consequence of Inequality 4 and the Schwartz inequality.

$$\int_0^h p(t)|x(t)||x'(t)|dt \leqslant \Big(\int_0^h p^2(t)|x(t)|^2 dt\Big)^{\frac{1}{2}}\Big(\int_0^h |x'(t)|^2 dt\Big)^{\frac{1}{2}}$$

$$\leqslant \Big(\frac{h}{4}\int_0^h p^2(t)dt\int_0^h |x'(t)|^2 dt\Big)^{\frac{1}{2}}\Big(\int_0^h |x'(t)|^2 dt\Big)^{\frac{1}{2}} \leqslant \Big(\frac{h}{4}\int_0^h p^2(t)dt\Big)^{\frac{1}{2}}\int_0^h |x'(t)|^2 dt.$$

2. DIFFERENTIAL INEQUALITIES

Throughout this Section the function $x:[0,h]\to R$ is assumed to have absolutely continuous first derivative x' and to vanish at both ends of the interval $[0,h]$, i.e., $x(0)=x(h)=0$.

Lemma 1. *If*

$$|x''(t)|\leqslant N|x(t)| \tag{2.1}$$

almost everywhere on $[0,h]$ *and*

$$h<\frac{\pi}{\sqrt{N}}, \tag{2.2}$$

then $x(t)\equiv 0$ *in* $[0,h]$.
Condition (2.2) *is the best possible.*

Proof. Inequality 1 and (2.1) imply that

$$\int_0^h |x'(t)|^2dt\leqslant\int_0^h |x''(t)||x(t)|\,dt\leqslant N\int_0^h |x(t)|^2dt.$$

Hence, by Inequality 2,

$$\int_0^h |x'(t)|^2dt\leqslant N\frac{h^2}{\pi^2}\int_0^h |x'(t)|^2dt.$$

If $x(t)\not\equiv 0$ in $[0,h]$, then $\int_0^h |x'(t)|^2dt>0$ and therefore $1\leqslant N\frac{h^2}{\pi^2}$. Thus (2.2) implies that $x(t)\equiv 0$. For $h=\frac{\pi}{\sqrt{N}}$ the function $x(t)=\big(x_1(t),\dots,x_m(t)\big)$, where $x_i(t)=A\sin\sqrt{N}\,t$ $(i=1,\dots,m)$, satisfies (2.1).

Lemma 2. *If*

$$|x''(t)|\leqslant N|x(t)|+M|x'(t)| \tag{2.3}$$

almost everywhere on $[0,h]$ *and*

$$N\frac{h^2}{\pi^2}+M\frac{h}{4}<1, \tag{2.4}$$

then $x(t)\equiv 0$ *in* $[0,h]$.

Proof. By Inequalities 1, 2, 3 and (2.3) we have

$$\int_0^h |x'(t)|^2dt\leqslant\int_0^h |x''(t)||x(t)|dt\leqslant N\int_0^h |x(t)|^2dt+M\int_0^h |x(t)||x'(t)|dt$$

$$\leqslant N\frac{h^2}{\pi^2}\int_0^h |x'(t)|^2dt+M\frac{h}{4}\int_0^h |x'(t)|^2dt.$$

Therefore, if $x(t) \not\equiv 0$, we would have $1 \leqslant N\frac{h^2}{\pi^2}+M\frac{h}{4}$, in contradiction with (2.4).

Lemma 3. *If the function* $p:[0,h]\to[0,+\infty)$ *is summable, if*

$$(2.5) \qquad |x''(t)| \leqslant p(t)|x(t)|$$

almost everywhere on $[0,h]$ *and*

$$(2.6) \qquad \int_0^h p(t)\,dt \leqslant \frac{4}{h},$$

then $x(t)\equiv 0$ *in* $[0,h]$.
Condition (2.6) *is the best possible.*

Proof. The assumption $x(t)\not\equiv 0$ implies that $\int_0^h |x'(t)|^2dt>0$. Since, by Inequalities 1,4 and (2.5), we have

$$\int_0^h |x'(t)|^2dt \leqslant \int_0^h |x''(t)||x(t)|dt \leqslant \int_0^h p(t)|x(t)|^2dt$$

$$<\frac{h}{4}\int_0^h p(t)\,dt\int_0^h |x'(t)|^2dt,$$

it follows that

$$1<\frac{h}{4}\int_0^h p(t)\,dt.$$

A simple modification of the example of Denkowski [1] shows that the number $\frac{4}{h}$ cannot be replaced by any greater.

Lemma 4. *If the function* $p:[0,h]\to[0,+\infty)$ *is summable and the function* $q:[0,h]\to[0,+\infty)$ *is square summable, if the inequality*

$$(2.7) \qquad |x''(t)| \leqslant p(t)|x(t)|+q(t)|x'(t)|$$

is fulfiled almost everywhere on $[0,h]$ *and*

$$(2.8) \qquad \frac{h}{4}\int_0^h p(t)\,dt+\sqrt{\frac{h}{4}\int_0^h q^2(t)\,dt}<1,$$

then $x(t)\equiv 0$ *in* $[0,h]$.

Proof. Using Inequalities 1, 4, 5 and assumption (2.7), we have

$$\int_0^h |x'(t)|^2dt \leqslant \int_0^h |x''(t)||x(t)|dt$$

$$\leqslant \int_0^h p(t)|x(t)|^2 dt + \int_0^h q(t)|x(t)||x'(t)|dt$$

$$\leqslant \frac{h}{4}\int_0^h p(t)dt \int_0^h |x'(t)|^2 dt + \sqrt{\frac{h}{4}\int_0^h q^2(t)dt \int_0^h |x'(t)|^2 dt}.$$

Hence, for $x(t) \not\equiv 0$, it follows that

$$1 \leqslant \frac{h}{4}\int_0^h p(t)dt + \sqrt{\frac{h}{4}\int_0^h q^2(t)dt}.$$

in contradiction with (2.8).

3. APPLICATION TO DIFFERENTIAL EQUATIONS

Now we state and prove two theorems which are particular cases of the general theory of Lasota and Opial [4].

We shall say that the function $f\colon [0, h]\times R^n \to R^m$ satisfies *condition* (C) (*Carathéodory's condition*), if for every fixed $t \in [0, h]$, $f(t, p)$ is continuous in p and for every fixed $p \in R^n$ $f(t, p)$ is measurable with respect to t.

Theorem 1. *Suppose that in* (0, 3) *the function* $f\colon [0, h]\times R^m\times R^m \to R^m$ *satisfies condition* (C) *and the inequality*

(3.1) $$|f(t, x, y)| \leqslant p(t)|x| + q(t)|y| + g(t, x, y)$$

for $t \in [0, h]$, $x, y \in R^m$, *where* $p(t)$ *and* $q(t)$ *are summable on* $[0, h]$ *and the function* $g\colon [0, h]\times R^m\times R^m \to R$ *satisfies condition* (C) *and the following assumption*

(3.2) $$\lim_{k\to\infty}\frac{1}{k}\int_0^h \sup_{|x|+|y|\leqslant k} g(t, x, y)dt = 0.$$

Under these assumptions, if $x(t) \equiv 0$ *is the unique solution on* $[0, h]$ *of the unequality*

(3.3) $$|x''(t)| \leqslant p(t)|x(t)| + q(t)|x'(t)|$$

satisfying the boundary condition $x(0) = x(h) = 0$, *then problem* (0. 3) *has at least one solution.*

Theorem 2. *Suppose that in* (0. 3) *the function* $f\colon [0, h]\times R^m\times R^m \to R^m$ *satisfies condition* (C) *and the Lipchltz inequality*

(3.4) $$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \leqslant p(t)|u-\bar{u}| + q(t)|v-\bar{v}|$$

for each $u, \bar{u}, v, \bar{v} \in R^m$, *where the functions* $p(t)$, $q(t)$ *are summable on* $[0, h]$ *and* $f(t, 0, 0)$ *is summable over* $[0, h]$. *Under these assumptions, if* $x(t) \equiv 0$ *is the unique* *solution on* $[0, h]$ *of inequality* (3.3) *satisfying the boundary condition* $x(0) = x(h) = 0$, *then problem* (0. 3) *has exactly one solution.*

By a *solution* of inequality (3.3) we mean any function $x: [0, h] \to R^m$ with absolutely continuous derivative $x'(t)$ satisfying this inequality almost everywhere on $[0, h]$.

Proof of Theorem 1. In order to prove Theorem 1, it is sufficient to verify the assumptions of Theorem 2.1 in [4]. For $t \in [0, h]$, $x, y \in R^m$ we set:

$$\tilde{f}(t, x, y) = (y, f(t, x, y)),$$
$$\tilde{F}(t, x, y) = \{(u, v) \in R^m \times R^m;\ u = y,\ |v| \leqslant p(t)|x| + q(t)|y|\},$$
$$\tilde{L}(x, y) = (x(0), x(h)).$$

From the definition it is clear that $\tilde{F}(t, x, y) \in \mathrm{cf}(R^{m^2})$ is homogeneous and continuous in (x, y) for each $t \in [0, h]$. If $A \subset R^{m^2}$ is closed, then for every fixed (x, y) we have

$$\{t \in [0, h];\ \tilde{F}(t, x, y) \cap A \neq \emptyset\}$$
$$= \{t \in [0, h];\ p(t)|x| + q(t)|y| \geqslant \delta(0, A \cap \{(y, v);\ v \in R^m\})\}.$$

Hence $\tilde{F}(t, x, y)$ is Lebesgue measurable in t. Setting $\varphi(t) = p(t) + q(t)$ we have

$$|(u, v)| \leqslant \varphi(t) \quad \text{for} \quad (u, v) \in \bigcup_{|(x, y)| = 1} \tilde{F}(t, x, y)$$

and $\varphi(t)$ is summable on $[0, h]$.

Condition (C) follows from the assumptions of Theorem 1. Note that $\delta(\tilde{f}(t, x, y), \tilde{F}(t, x, y)) \leqslant g(t, x, y)$ so (3.2) implies assumption (2.1) in Theorem 2.1 of [4].

It follows from the definition that $\tilde{L}(x, y)$ is continuous and homogeneous. Thus conditions 1°, 2°, 3° of Theorem 2.1 in [4] are fulfiled. Finally, the uniqueness of solution of the homogeneous boundary value problem

$$(x'(t), y'(t)) \in \tilde{F}(t, x(t), y(t)), \qquad \tilde{L}(x, y) = (0, 0)$$

is assumed in our theorem.

Now, a straightforward application of Theorem 2.1 of [4] completes the proof.

Proof of Theorem 2. By condition (3.4), the integrability of $f(t, 0, 0)$ and Theorem 1, the existence of a solution of problem (0. 3) is evident.

For to prove its uniqueness, suppose that $x_1(t)$, $x_2(t)$ are two solutions of (0. 3); then $x(t) = x_1(t) - x_2(t)$ satisfies (3.3) and the boundary condition $x(t) = x(h) = 0$, so that from the assumptions of Theorem 2 we have $x(t) \equiv 0$, which proves our theorem.

The following theorems and corollary are simple conclusions from Theorems 1 and 2 and Lemmas 1—4, respectively.

Theorem 3. *Assume that the function* $f: [0, h] \times R^m \to R^m$ *satisfies condition* (*C*) *and the inequality*

$$|f(t, x)| \leqslant N|x| + g(t, x) \quad \textit{for} \quad t \in [0, h],\ x \in R^m$$

where the function $g: [0, h]\times R^m \to R$ *satisfies condition* (C) *and the following assumption*

$$\lim_{k\to\infty} \frac{1}{k}\int_0^h \sup_{|x|\leqslant k} g(t, x,)dt = 0. \tag{3.5}$$

Under these hypotheses, if condition (2.2) *holds true, then there exists at least one solution of problem* (0. 3).

Corollary 1. *If function* $f(t, x)$ *satisfies condition* (C) *and the inequality*

$$|f(t, u)-f(t, v)| \leqslant N|u-v| \quad \textit{for every} \quad u, v \in R^m,$$

with $f(t, 0)$ *summable on* $[0, h]$ *and, if condition* (2.2) *holds true, then there exists exactly one solution of problem* (0. 3).

The proof of Theorem 3 (Corollary 1) follows immediately from Theorem 1 (Theorem 2, respectively) and Lemma 1.

Theorem 4. *Suppose that the function* $f: [0, h]\times R^m\times R^m \to R^m$ *satisfies condition* (C) *and the inequality*

$$|f(t, x, y)| \leqslant N|x|+M|y|+g(t, x, y) \quad \textit{for} \quad t \in [0, h],\ x, y \in R^m$$

where the function $g: [0, h]\times R^m\times R^m \to R$ *satisfies condition* (C) *and the following hypothese*

$$\lim_{k\to\infty} \frac{1}{k}\int_0^h \sup_{|x|+|y|\leqslant k} g(t, x, y)dt = 0. \tag{3.6}$$

Under these assumptions, if condition (2, 4) *holds true, then there exists at least one solution of problem* (0, 3).

Corollary 2. *If function* $f(t, x, y)$ *satisfies condition* (C) *and the inequality*

$$|f(t, u, v)-f(t, \bar{u}, \bar{v})| \leqslant N|u-\bar{u}|+M|v-\bar{v}|$$

for every $t \in [0, h]$, $u, v, \bar{u}, \bar{v} \in R^m$ *with* $f(t, 0, 0)$ *summable on* $[0, h]$ *and, if condition* (2.4) *holds true, then there exists exactly one solution of problem* (0. 3).

The proof of Theorem 4 (Corollary 2) is an immediate conclusion from Theorem 1 (Theorem 2, respectively) and Lemma 2.

Theorem 5. *Assume that the function* $f: [0, h]\times R^m \to R^m$ *satisfies condition* (C) *and the inequality*

$$|f(t, x)| \leqslant p(t)|x|+g(t, x) \quad \textit{for} \quad t \in [0, h],\ x \in R^m,$$

where the function $g: [0, h]\times R^m \to R$ *satisfies condition* (C) *and the assumption* (3.5) *and the function* $p: [0, h]\to [0, +\infty)$ *is summable on* $[0, h]$.

Under these hypotheses, if condition (2.6) *holds true, then there exists at least one solution of problem* (0. 3).

Corollary 3. *If function* $f(t, x)$ *satisfies condition* (C) *and the inequality*

$$|f(t, u)-f(t, v)| \leqslant p(t)|u-v| \quad \textit{for every} \quad u, v \in R^m,$$

with $f(t, 0)$ and $p(t)$ summable on $[0, h]$ and, if condition (2.6) holds true, then there exists exactly one solution of problem (0. 3).

The proof of Theorem 5 (Corollary 3) is an immediate conclusion from Theorem 1 (Theorem 2, respectively) and Lemma 3.

Theorem 6. *Suppose that the function $f\colon [0, h]\times R^m\times R^m\to R^m$ satisfies condition (C) and the inequality*

$$|f(t, x, y)| \leqslant p(t)|x|+q(t)|y|+g(t, x, y) \quad \text{for} \quad t\in[0, h],\ x, y\in R^m,$$

where the function $g\colon [0, h]\times R^m\times R^m\to R$ satisfies condition (C) and the assumption (3.6). Suppose that the function $p\colon [0, h]\to[0, +\infty)$ is summable and the function $q\colon [0, h]\to[0, +\infty)$ is square summable.

Under these hypotheses, if condition (2.8) holds true, then there exists at least one solution of problem (0. 3).

Corollary 4. *If function $f(t, x, y)$ satisfies condition (C) and the inequality*

$$|f(t, u, v)-f(t, \bar{u}, \bar{v})| \leqslant p(t)|u-\bar{u}|+q(t)|v-\bar{v}|$$

for every $u, \bar{u}, v, \bar{v}\in R^m$ with $f(t, 0, 0)$ and $p(t)$ and $q^2(t)$ summable on $[0, h]$ and, if condition (2.8) holds true, then there exists exactly one solution of problem (0. 3).

The proof of Theorem 6 (Corollary 4) follows immediately from Theorem 1 (Theorem 2, respectively) and Lemma 4.

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