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On the Boundary Value Problems for the Ordinary Differential Equation of Second Order

1. We shall consider a second order differential equation

$$x'' = f(t, x, x') \qquad a \leqslant t \leqslant b \tag{1}$$

and boundary conditions

$$L_i x = r_i \qquad i = 1, 2 \tag{2}$$

where L_i is a continuous linear functional defined on the space $C^1_{\langle a,b\rangle}$ of all real-valued functions continuously differentiable in closed interval $\langle a, b\rangle$.

We shall say that the function $f(t, u, v)$ defined in the set D:

$$a \leqslant t \leqslant b\,, \qquad -\infty < u, v < +\infty$$

satisfies the condition (C) (Carathéodory's condition) if

(i) for every fixed $t, f(t, u, v)$ is continuous in the variables u, v,

(ii) for fixed $u, v, f(t, u, v)$ is measurable with respect to t.

The general theorems included in papers [3], [4], [5] applied to the equation (1) and conditions (2) allow us to formulate the following

Theorem A. *If the right-hand member of equation* (1) *satisfies in the set D condition* (C) *and inequality*

$$|f(t, u, v)| \leqslant M(t) + p_0(t)\,|u| + q_0(t)\,|v| \tag{3}$$

where $M(t)$, $p_0(t)$, $q_0(t)$ are non-negative summable functions in the interval $\langle a, b\rangle$ and if for every pair of measurable functions $p(t)$, $q(t)$ satisfying conditions

$$|p(t)| \leqslant p_0(t)\,, \qquad |q(t)| \leqslant q_0(t) \tag{4}$$

the homogeneous linear equation

$$x'' + p(t)x' + q(t)x = 0 \tag{5}$$

has unique solution $x(t) \equiv 0$ satisfying homogeneous conditions

$$L_i x = 0 \qquad i = 1, 2\,, \tag{6}$$

then there exists at least one solution in sense of Carathéodory of the problem (1), (2).

Of great interest here are conditions which functions $p_0(t)$, $q_0(t)$ have to satisfy in order that the linear homogeneous problem (5), (6) has unique solution $x(t)\equiv 0$. The best possible evaluations in the norm $\|g\|=\max\limits_{\langle a,b\rangle}|g(t)|$ have been received by Ch. de la Vallée Poussin [8] and later, in a simpler form, by Z. Opial [7] and Ph. Hartman, A. Wintner [2].

The purpose of this paper is to establish conditions in the integral norm

$$\|g\|=\int_a^b |g(t)|\,dt$$

for the functions $p_0(t)$, $q_0(t)$ so that the linear homogeneous problem (5), (6) has exclusively a trivial solution. The conditions obtained (the best possible in the integral norm) make it possible for us to formulate two theorems on the existence of solutions of equation (1) which satisfy the boundary conditions (2) of certain type.

2. We shall employ the following lemma analogous to the well known inequality of Beurling (see [1], p. 124).

Lemma 1. *If the function $x(t)$ is a non-trivial solution of equation*

$$x''+h(t)x=0\,, \tag{7}$$

where $h(t)$ denotes a summable function in the interval $\langle a,b\rangle$ and if $x(t)$ satisfies the condition

$$x(a)=x'(k)=0,\qquad a<k<b \tag{8}$$

then the following inequality

$$\int_a^k |h(t)|\,dt>\frac{1}{k-a} \tag{9}$$

holds true.

Proof. Put

$$M=\int_a^k |x''(t)|\,dt$$

then by (7) we obtain the evaluation

$$M\leqslant \max_{\langle a,k\rangle}|x(t)|\int_a^k |h(t)|\,dt\,.$$

On the other hand we have

$$\max_{\langle a,k\rangle}|x(t)|\leqslant (k-a)\max_{\langle a,k\rangle}|x'(t)|=(k-a)x'(t_1)\,,$$

where $t_1\in(a,k)$. It is easy to see that equality is only possible when $x'(t)=\text{const.}$ in the interval $\langle a,k\rangle$ which in the present case is excluded.

Similarly, we have

$$|x'(t_1)| \leqslant \int_a^k |x''(t)|\,dt = M\,.$$

The inequalities obtained so far and the remark just made immediately imply inequality

$$M < (k-a)M\int_a^k |h(t)|\,dt$$

which together with the assumption that $x(t) \not\equiv 0$ results in inequality (9) and our lemma is proved.

From the above Lemma 1 it easily follows:

Lemma 2. *If the function* $x(t)$ *is a non-trivial solution of differential equation* (5) *which satisfies the condition* (8) *then the following inequality*

$$\int_a^k |q(t)|\,dt\, e^{\int_a^k |p(t)|dt} > \frac{1}{k-a} \tag{10}$$

holds true.

Proof. Let us put

$$P(t) = \int p(t)\,dt$$

and introduce independent variable $\tau = \chi(t)$, where

$$\chi(t) = \int_a^t e^{-P(s)}\,ds$$

then the equation (5) and condition (8) have the form

$$\frac{d^2\bar{x}}{d\tau^2} + \bar{q}(\tau)e^{2\bar{P}(\tau)}\bar{x} = 0\,, \tag{11}$$

where $\bar{x}(\tau) = x(\chi^{-1}(\tau))$, $\bar{q}(\tau) = q(\chi^{-1}(\tau))$, $\bar{P}(\tau) = P(\chi^{-1}(\tau))$ and

$$\bar{x}(\chi(a)) = 0\,, \qquad \frac{d\bar{x}}{d\tau}\Big|_{\tau=\chi(k)} = \frac{dx}{dt}\Big|_{t=k}\frac{dt}{d\tau}\Big|_{\tau=\chi(k)} = 0\,. \tag{12}$$

Now a straightforward application of Lemma 1 to equation (11) and conditions (12) gives the inequality

$$\int_{\chi(a)}^{\chi(k)} |\bar{q}(\tau)|\,e^{2\bar{P}(\tau)}\,d\tau > \frac{1}{\chi(k)-\chi(a)}\,.$$

Hence, we have

$$\int_a^k |q(t)|\,e^{P(t)}\,dt > \frac{1}{\int_a^k e^{-P(t)}dt}\,.$$

Applying the mean-value theorem we have the following inequality

$$e^{P(t_1)}\int\limits_a^k |q(t)|\,dt > \frac{1}{(k-a)e^{-P(t_2)}}\,, \quad \text{where} \quad a < t_1, t_2 < k\,.$$

Multiplying the above inequality by $e^{-P(t_2)}$ and applying the obvious inequality

$$e^{P(t_1)-P(t_2)} \leqslant e^{\int\limits_a^k |p(t)|\,dt}$$

we obtain the required inequality (10) and Lemma 2 is proved.

The Lemma 2 permits us to obtain the inequality of Levine published, without proof, in [6]. In particular we can formulate the following

Lemma 3. *If the function* $x(t)$ *is a non-trivial solution of differential equation* (5) *which satisfies the condition*

$$x(a) = x(b) = 0 \tag{13}$$

then the following inequality

$$\int\limits_a^b |q(t)|\,dt\, e^{\frac{1}{2}\int\limits_a^b |p(t)|dt} > \frac{4}{b-a} \tag{14}$$

holds true.

Proof. We note that from assumptions of our lemma it follows that there exists in interval (a, b) a number k such that $x'(k) = 0$. Now a straightforward application of Lemma 2 to the intervals $\langle a, k\rangle$ and $\langle k, b\rangle$ gives respectively the inequalities

$$\int\limits_a^k |q(t)|\,dt\, e^{\int\limits_a^k |p(t)|dt} > \frac{1}{k-a}\,, \qquad \int\limits_k^b |q(t)|\,dt\, e^{\int\limits_k^b |p(t)|dt} > \frac{1}{b-k}\,.$$

Hence, we obtain

$$\Big(\int\limits_a^k |q(t)|\,dt\Big)^{\frac{1}{2}} \cdot \Big(\int\limits_k^b |q(t)|\,dt\Big)^{\frac{1}{2}} e^{\frac{1}{2}\int\limits_a^b |p(t)|dt} > \frac{1}{\sqrt{(k-a)(b-k)}}\,.$$

The above inequality and elementary inequalities

$$\alpha\cdot\beta \leqslant \tfrac{1}{2}(\alpha^2+\beta^2)\,, \text{ where } \alpha = \Big(\int\limits_a^k |q(t)|\,dt\Big)^{\frac{1}{2}}, \ \beta = \Big(\int\limits_k^b |q(t)|\,dt\Big)^{\frac{1}{2}}$$

and

$$\frac{1}{\sqrt{(k-a)(b-k)}} \geqslant \frac{2}{b-a}$$

immediately yield inequality (14) and our lemma is proved.

3. Lemma 1 and Lemma 2 allow us to prove the following two theorems. Theorem 1. *Let the right-hand side of equation* (1) *satisfy the condition* (C) *and inequality* (3) *in the set* D. *Under the above assumptions if the inequality*

$$\int_a^b q_0(t)\,dt\, e^{\int_a^b p_0(t)dt} \leqslant \frac{1}{b-a} \tag{15}$$

is fulfilled, then for every two pairs of numbers (t_1, r_1), (t_2, r_2), *where* $a \leqslant t_1 < t_2 \leqslant b$, *there exists at least one solution in the sense of Carathéodory of equation* (1) *satisfying the boundary conditions* $x(t_1) = r_1$, $x'(t_2) = r_2$.

Theorem 2. *Under the assumptions of Theorem* 1 *if inequality* (15) *is replaced by the following inequality*

$$\int_a^b q_0(t)\,dt\, e^{\frac{1}{2}\int_a^b p_0(t)dt} \leqslant \frac{4}{b-a} \tag{16}$$

then for every pairs of numbers (t_1, r_1), (t_2, r_2), *where* $a \leqslant t_1 < t_2 \leqslant b$, *there exists at least one solution in the sense of Carathéodory of equation* (1) *satisfying the boundary conditions* $x(t_1) = r_1$, $x(t_2) = r_2$.

In fact let us define the functionals L_i in the following form

$$L_1 x = x(t_1)\,, \qquad L_2 x = x'(t_2)$$

in the case of Theorem 1 and

$$L_1 x = x(t_1)\,, \qquad L_2 x = x(t_2)$$

in the case of Theorem 2. Then these theorems are immediate conclusions from Theorem A and from Lemma 1 and Lemma 2 respectively.

4. The example quoted below shows that if we replace the number $1/(k-a)$ in the inequality (9) by number α, where $\alpha > 1/(k-a)$, then there exist non--trivial solutions of problem (7), (8). Let α be a fixed positive number. Without loss of generality we can assume that $a = 0$. Next suppose that n is an integer sufficiently large so that inequality $0 < \sqrt{\alpha/n} < \pi/2$ is fulfilled.

Let us put $k = k_n$, where

$$k_n = \frac{1}{\alpha}\left(\frac{\alpha}{n} + \frac{\sqrt{\frac{\alpha}{n}}}{\operatorname{tg}\sqrt{\frac{\alpha}{n}}}\right)$$

and

$$h_n(t) = \begin{cases} 0\,, & \text{for } 0 \leqslant t \leqslant t_n \\ n_\alpha\,, & \text{for } t_n < t \leqslant k_n\,, \end{cases}$$

where $t_n = k - \frac{1}{n}$. It is easy to verify that the function

$$x_n(t) = \begin{cases} tA\sqrt{na}\sin\sqrt{\frac{a}{n}} & 0 \leqslant t \leqslant t_n \\ A\cos\left[\sqrt{na}(t-k)\right] & t_n < t \leqslant k_n', \end{cases}$$

where A is an arbitrary constant other than 0, is a non-trivial solution of equation $x'' + h(t)x = 0$ satisfying the condition $x(0) = x'(k) = 0$.

Since

$$\int_0^{k_n} h_n(t)\,dt = a \quad \text{and} \quad \lim_{n\to\infty} k_n = \frac{1}{a} \quad \left(k_n > \frac{1}{a}\right)$$

it is easy to see that in inequality (9) the number $1/(k-a)$ cannot be replaced by any greater number.

In a similar way one can show that the evaluations in inequalities (10) and (14) and also in (15) and (16), are best possible.

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