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Asymptotic properties of solutions of nonlinear differential equations of the higher order

1. Consider a differential equation

$$y^{(n)}+a_1 y^{(n-1)}+\ldots+a_{n-1}y'+f(y)=p(t) \tag{1}$$

in which $a_1, \ldots, a_{n-1}$ are positive constants and the functions $f(y)$ and $p(t)$ are continuous in their respective arguments.

It will be assumed in the sequel without any further explicit mention that all the roots of the polynomial

$$\psi(\lambda)=\lambda^{n-1}+a_1\lambda^{n-2}+\ldots+a_{n-1}$$

have negative real parts ($\psi(\lambda)$ is a Hurwitz polynomial) and that

$$yf(y)>0$$

for all $y\neq 0$.

The solutions of (1) are said to be *globally bounded*, if there is a constant $D>0$ such that for any solution $y(t)$ of (1) we have

$$|y(t)|<D,\ |y'(t)|<D,\ \ldots,\ |y^{(n-1)}(t)|<D$$

for $t\geqslant T$, where T depends only on the solution $y(t)$.

The trivial solution, $y(t)\equiv 0$, of (1) is said to be *globally asymptotically stable*, if for every solution $y(t)$ of (1) we have

$$\lim_{t\to\infty}(|y(t)|+|y'(t)|+\ldots+|y^{(n-1)}(t)|=0\,.$$

The object of this note is to give sufficient conditions of the global boundedness (the global asymptotic stability) of the solutions of the equation (1). The results of the note are obtained by the application of the Liapunov direct (or second) method.

We replace (1) by the equivalent system of the first-order differential equations, considered next as a special case of differential equations of the automatic control [1]. Such an approach admits the use of the Liapunov function without its effective construction; the existence of the Liapunov function with required properties follows from the suitable theorems of the theory of automatic control.

The following notations will be used in the sequel. Capital letters A, G, H, I, L, P, Q, S denote matrices, small letters a, b, c, q, r, x denote the column-vectors. An asterisk denotes the hermitian transposition and thus $a^*b = (a, b)$ is a scalar product of the vectors a and b. $|a|$ denotes the norm of the vector a and is defined as $|a| = \sqrt{(a, a)}$; $A > 0$ denotes that the matrix A is symmetric and that the quadratic form (x, Ax) is positive definite. I denotes the unit matrix. If A is square its determinant is $|A|$.

We will consider the cases of bounded and unbounded $f(y)$ separately. The case of the function $f(y)$ bounded as a simpler one will be considered firstly. The theorems of the subsequent section are the n-dimensional analogue of the results given in [13].

2. Theorem 1. *If the functions $f(y)$, $p(t)$ satisfy the inequalities*

$$(2)\qquad \begin{aligned} &|f(y)| \leqslant M_1 && \text{for} \quad y \in (-\infty, \infty), \\ &|p(t)| \leqslant M_0, \quad \left|\int_0^t p(s)\,ds\right| \leqslant M_0 && \text{for} \quad t \geqslant 0, \end{aligned}$$

then the solutions of (1) *are globally bounded.*

Proof. In place of (1) let us consider the equivalent system

$$(3)\qquad x' = Ax + a[f(\sigma) - p(t)], \qquad \sigma' = (b, x),$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ -a_{n-1} & -a_{n-2} & \cdot & \dots & -a_1 \end{bmatrix}, \quad a = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix},$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_{n-1} \end{bmatrix}, \quad y = \sigma.$$

The proof of the theorem is reduced to showing that all solutions of (3) are bounded.

Let the functions $V(x)$ and $W(x, \sigma, t)$ be defined as follows

$$V(x) = (x, Lx), \qquad W(x, \sigma, t) = (c, x) + (a, c)\int_0^t p(s)\,ds + \gamma\sigma,$$

where L, c, γ satisfy the conditions

$$\begin{aligned} &LA+A^*L = -Q, \quad Q>0, \text{ arbitrary} \\ &A^*c+\gamma b = 0, \quad (a,c) = -1. \end{aligned} \tag{4}$$

Since $|A-\lambda I|$ is a Hurwitz polynomial, there is a positive definite symmetric matrix L satisfying (4) (see [1], pp. 14-19).

Define in (x, σ, t)-space the sets

$$\Delta(C_1, C_2) = \{(x, \sigma, t)\colon V(x) \leqslant C_1,\ |W(x,\sigma,t)| \leqslant C_2\},$$

$$\pi(C_1, C_2, t_0) = \{(x,\sigma)\colon (x,\sigma,t) \in \Delta(C_1, C_2),\ t = t_0\},$$

$$\Sigma(C_1, C_2) = \operatorname{Fr}\Delta(C_1, C_2).$$

The sets Δ, π, Σ have the following properties:

(α) For each pair of contants C_1, C_2, there exists a bounded domain $\Omega(C_1, C_2)$ independent of t such that $\pi(C_1, C_2, t) \subset \Omega(C_1, C_2)$, for $t \geqslant 0$. Corresponding to any point (x_0, σ_0) there are $\bar{C}_1, \bar{C}_2$ such that $(x_0, \sigma_0, t) \in \Delta(\bar{C}_1, \bar{C}_2)$ for all t.

(β) There exist constants $\tilde{C}_1, \tilde{C}_2$ such that if a solution of (3) has an initial point belonging to the set $\Sigma(C_1, C_2)$ $(C_1 \geqslant \tilde{C}_1, C_2 \geqslant \tilde{C}_2)$ then it enters the domain $\Delta(C_1, C_2) \setminus \Sigma(C_1, C_2)$ and so it remains in Δ.

(γ) Corresponding to any $\bar{\bar{C}}_1 \geqslant \tilde{C}_1$, $\bar{\bar{C}}_2 \geqslant \tilde{C}_2$ there is an $M(\tilde{C}_1, \tilde{C}_2) > 0$ such that if $(x, \sigma, t) \in \Sigma(C_1, C_2)$, $\tilde{C}_i \leqslant C_i \leqslant \bar{\bar{C}}_i$, then

$$d|W|/dt \leqslant -M(\bar{\bar{C}}_1, \bar{\bar{C}}_2), \qquad dV/dt \leqslant -M(\bar{\bar{C}}_1, \bar{\bar{C}}_2).$$

$d|W|/dt$, dV/dt denote the derivatives of the functions $|W|$, V along the solutions of (3) (the total derivatives of $|W|$ and V).

The property (α) follows from the assumption (2) and the formulas

$$\lim_{|x|\to\infty} V(x) = \infty, \qquad \limsup_{|x|,|\sigma|\to\infty} |W(x,\sigma,t)| = \infty.$$

Since Σ consists of the surfaces given by conditions $V(x) = C_1$, $|W(x,\sigma,t)| = C_2$, so to show (β) it is enough to demonstrate that for sufficiently large C_1, C_2 if $(x,\sigma,t) \in \Sigma(C_1, C_2)$, then derivatives dV/dt, $d|W|/dt$ calculated in this point are both negative. These derivatives by virtue of (4) are

$$\begin{aligned} dV/dt &= -(x, Qx) + 2(x, La)[f(\sigma) - p(t)] \\ &\leqslant -(x,Qx) + 2M_2(M_1+M_0)|x| \quad (|(x, La)| \leqslant M_2|x|), \\ d|W|/dt &= -f(\sigma)\operatorname{sgn} W(x,\sigma,t). \end{aligned} \tag{5}$$

If ϱ_0 is large enough, then for $|x| > \varrho_0$ the first of the formulas (5) implies $dV/dt < 0$. By the second of the formulas (5) the inequality $d|W|/dt < 0$ holds if and only if the points $(x,\sigma,t) \in \Sigma$ satisfying $W(x,\sigma,t) = C_2$ lie in the half-space $\sigma > 0$, and the points satisfying $W(x,\sigma,t) = -C_2$ lie in the half-space $\sigma < 0$. This property of Σ will take place, if corresponding to C_1 we shall choose (what always can be done) a constant $C_2(C_1)$ such that the system of equations

$$|W(x, 0, t)| = C_2(C_1), \quad V(x) = C_1$$

has no real roots.

Taking $\tilde{C}_1$ so large that the sphere $(x, x) = \varrho_0^2$ is contained in the set $V(x) \leqslant \tilde{C}_1$ and putting $\tilde{C}_2 = C_2(\tilde{C}_1)$, we obtain constants $\tilde{C}_1, \tilde{C}_2$ satisfying (β).

The property (γ) follows from the definition of numbers $\tilde{C}_1, \tilde{C}_2$ and the continuity of the functions $d|W|/dt$, dV/dt in the set $\Gamma = \{(x, \sigma, t): (x, \sigma, t) \in \Sigma(C_1, C_2),\ \tilde{C}_i \leqslant C_i \leqslant \bar{\bar{C}}_i,\ i = 1, 2\}$.

Denote now by $x(t), \sigma(t)$ a solution of the system (3) through the point (x_0, σ_0, t_0). By (α) there exist $\bar{C}_1, \bar{C}_2$ such that $(x_0, \sigma_0, t_0) \in \Sigma(\bar{C}_1, \bar{C}_2)$, by (β) we have $\big(x(t), \sigma(t)\big) \in \Delta(\bar{C}_1, \bar{C}_2)$ for $t \geqslant t_0$ and from (γ) it follows that $\big(x(t), \sigma(t)\big)$ reaches the set $\Delta(\tilde{C}_1, \tilde{C}_2)$ in a finite time. As $\pi(\tilde{C}_1, \tilde{C}_2, t)$ is bounded, so all solutions of (3) are bounded and the assertion of Theorem 1 holds.

Theorem 2. *If the functions $f(y), p(t)$ satisfy the conditions*

$$(6) \qquad \begin{aligned} &0 < M_2 \leqslant f(y)\,\mathrm{sgn}\,y \leqslant M_1, && \textit{for} \quad |y| > y_0, \\ &|p(t)| \leqslant M_0 < M_2, && \textit{for} \quad t \geqslant 0, \end{aligned}$$

then the solutions of (1) *are globally bounded.*

Proof. As previously we will demonstrate the boundedness of solutions of the equivalent system (3).

Define the functions $V_1(x)$, $W_1(x, \sigma)$ by

$$V_1(x) = (x, Lx), \quad W_1(x, \sigma) = (c, x) + \gamma\sigma,$$

where L, c, γ satisfy (4) and let Δ_1, Σ_1 be the sets

$$\Delta_1(C_1, C_2) = \{(x, \sigma): V_1(x) \leqslant C_1,\ |W_1(x, \sigma)| \leqslant C_2\},$$
$$\Sigma_1(C_1, C_2) = \mathrm{Fr}\,\Delta_1(C_1, C_2).$$

The sets Δ_1, Σ_1 have the properties (α), (β) (γ) formulated in the proof of Theorem 1. We shall prove (β), the remaining properties are demonstrated as previously.

The total derivatives of V_1 and $|W_1|$ are

$$dV_1/dt = -(x, Qx) + 2(x, La)[f(\sigma) - p(t)],$$
$$d|W_1|/dt = -[f(\sigma) - p(t)]\,\mathrm{sgn}\,W_1(x, \sigma).$$

So $dV_1/dt < 0$ in the points of Σ_1 satisfying the condition $V_1(x) \geqslant \tilde{C}_1$ with sufficiently large $\tilde{C}_1$. Let $C_2(\tilde{C}_1)$ be chosen so large that the system

$$W_1(x, y_0) = \tilde{C}_2, \quad V_1(x) = \tilde{C}_1$$

or the system

$$W_1(x, -y_0) = -\tilde{C}_2, \quad V_1(x) = \tilde{C}_1$$

has no real roots.

Then in the points (x, σ) satisfying $|W_1(x, \sigma)| = C_2$ $(C_2 \geq \tilde{C}_2)$ we have $d|W_1|/dt$, what shows the property (β).

From (α), (β), (γ) in the way just described the direct proof of Theorem 2 follows at once.

Corollary. *If the function $p(t)$ is periodic with the period ω and the assumptions of Theorem 1 or 2 hold, then the equation (1) has at least one periodic solution with the period ω.*

Proof (see [12]). Let $x(t; x_0, \sigma_0, t_0)$, $\sigma(t; x_0, \sigma_0, t_0)$ be a solution of (3) through the point (x_0, σ_0, t_0). We define the transformation τ_ω of (x, σ)-space:

$$\tau_\omega\colon (x_0, \sigma_0) \to \{x(t_0+\omega; x_0, \sigma_0, t_0), \sigma(t_0+\omega; x_0, \sigma_0, t_0)\}.$$

The sets Δ_1 and $\pi(C_1, C_2, t_0)$ (t_0 fixed) are homeomorphic with a closed n-cell and are mapped by τ_ω onto itself. So by the Brouwer fixed-point theorem the mapping τ_ω has a fixed point and this proves the corollary.

3. In the subsequent sections the case of unbounded function $f(y)$ will be considered. We will be concerned with equations of more general form (including (1) as a special case)

$$y^{(n)} + \ldots + a_{k-1} y^{(n-k+1)} + f_k(y^{(n-k)}) + a_{k+1} y^{(n-k-1)} + \ldots + a_n y = p(t), \tag{7}$$

where a_j $(j = 1, \ldots, n,\ j \neq k)$ are positive constants.

Assume that, corresponding to numbers a_j, there is a positive constant h_k such that

$$\psi(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \ldots + h_k \lambda^{n-k} + \ldots + a_n$$

is a Hurwitz polynomial.

For $k = n$ this assumption is reduced to the one given in Section 1.

The third-order equations of the form considered have been investigated by E. A. Barbasin [2], J. O. C. Ezeilo [4], [5], A. J. Ogurtsov [9], V. A. Pliss [10], [11], M. Cartwright [3] and J. O. C. Ezeilo [6].

Our hypotheses on $f(y)$ are weaker than the corresponding conditions in the quoted papers, except [11]. Instead of assumption of the class C^1 for $f(y)$ and hypotheses concerning the derivative of $f(y)$, we consider more general assumption (10). But conditions $\lim\limits_{|y|\to\infty} |f(y)| = \infty$, $\lim\limits_{|y|\to\infty} |ky - f(y)| = \infty$ assumed in [12] are replaced by slight more restrictive (13), (21). It allows to shorten and simplify the proof of Theorem 6.

We replace (7) by the equivalent system

$$x' = Ax + a\varphi(\sigma) + cp(t), \qquad \sigma = (b, x) \tag{8}$$

putting

$$A = \begin{bmatrix} 0 & 1. & 0 & 0 & \ldots & 0 \\ \cdot & \ldots & \cdot & \cdot & \ldots & \cdot \\ -a_n & \ldots & -a_{k+1} & -e_k & \ldots & -a_1 \end{bmatrix}, \quad a = \begin{bmatrix} 0 \\ \cdot \\ 0 \\ -1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ \cdot \\ 1 \\ 0 \end{bmatrix} \text{ (the unity on the } n+1-k\text{-th place)},$$

$$x^* = (x_1, x_2, \ldots, x_n), \qquad c = -a, \qquad x_1 = y$$

and $\varphi(\sigma) = f(\sigma) - e_k\sigma$, for $k < n$; $\varphi(\sigma) = e_n\sigma - f(\sigma)$, $a^* = (0, \ldots, 0, 1)$, $a = c$; for $k = n$.

Positive numbers e_k, e_n are so chosen that eigenvalues of A lie in the half plane $\operatorname{Re} z \leqslant 0$.

We call the autonomous system of differential equations

$$x' = Ax + a\varphi(\sigma), \qquad \sigma = (b, x) \tag{9}$$

absolutely stable, provided the trivial solution of (9) is globally asymptotically stable for every continuous function $\varphi(\sigma)$ satisfying

$$0 < \varphi(\sigma)\sigma < k\sigma^2. \tag{10}$$

We will investigate the boundedness of solutions of (8) by use of the Liapunov function of the form

$$V(x) = (x, Lx) + \vartheta \int_0^\sigma \varphi(u)\,du, \tag{11}$$

which assures the absolute stability of (9).

Denote $\chi(\lambda) = \big(b, (A - \lambda I)^{-1}a\big)$ and assume that A has exactly two characteristic roots on an imaginary axis, $\pm\omega_0 i$ say. Then $\chi(\lambda)$ is of the form

$$\chi(\lambda) = \chi_1(\lambda) + (\alpha\lambda + \beta)/(\lambda^2 + \omega_0^2),$$

where $\chi_1(\lambda)$ is holomorphic on an imaginary axis.

The following result of V. A. Yakubovich [8] gives the sufficient condition for the existence of the Liapunov function in this case:

If for every real ω

$$\pi(\omega) = k^{-1} + \beta/\omega_0^2 + \operatorname{Re}\big[\big(1 + i\beta\omega/(\alpha\omega_0^2)\big)\chi_1(\omega i)\big] > 0, \tag{12}$$

then there exists a Liapunov function of the form (11) *which guarantees the absolute stability of* (9) *in the class of functions* $\varphi(\sigma)$ *satisfying* (10).

Using this condition, we shall prove the global asymptotic stability of the trivial solution of (7) for $n = 3$.

4. Theorem 3. *Let* $a_1 > 0$, $a_2 > 0$, $f_3(y)$ *be continuous*, $f_3(0) = 0$ *and*

$$0 < \varepsilon \leqslant f_3(y)/y < a_1 a_2, \quad \text{if} \quad y \neq 0, \tag{13}$$

then the trivial solution of the differential equation

$$y''' + a_1 y'' + a_2 y' + f_3(y) = 0 \tag{14}$$

is globally asymptotically stable

Theorem 4. *If the function* $f_2(y)$ *is continuous and satisfies the conditions*

$$f_2(0) = 0, \qquad a_3/a_1 < f_2(u)/u < K < \infty \quad \text{for} \quad u \neq 0; \tag{15}$$

where a_1, a_3 *are positive, then the trivial solution of the equation*

$$y''' + a_1 y'' + f_2(y') + a_3 y = 0 \tag{16}$$

is globally asymptotically stable.

Theorem 5. *Let* a_2, a_3 *be positive constants. Let* $f_1(v)$ *be continuous,* $f_1(0) = 0$, *and*

$$a_3/a_2 < f_1(v)/v \leqslant (a_3^2/a_3) + (a_3/a_2) - \varepsilon \tag{17}$$

with arbitrary small positive ε.

Then the solution $y = 0$ *of the equation*

$$y''' + f_1(y'') + a_2 y' + a_3 y = 0 \tag{18}$$

is globally asymptotically stable.

Proof of Theorems 3, 4, 5. Consider instead of (14), (16), (18) the equivalent system (9) (in every case the parameters A, a, b of this system will be different of course). Let a matrix A of (9) have in any case exactly two pure imaginary eigenvalues. For this sake let us put $e_3 = a_1 a_2$, $e_2 = + a_3/a_1$, $e_1 = a_3/a_2$ respectively in equivalent systems corresponding to (14), (16), (18).

By (13), (15), (17), functions $\varphi_3(\sigma) = a_1 a_2 \sigma - f_3(\sigma)$, $\varphi_2(\sigma) = f_2(\sigma) - \sigma a_3/a_1$ $\varphi_1(\sigma) = f_1(\sigma) - \sigma a_3/a_2$ satisfy the inequalities

$$0 < \varphi_i(\sigma)/\sigma < k_i \quad (i = 1, 2, 3),$$

in which $k_1 = a_2^2/a_3 - \varepsilon$, $k_2 = K - a_3/a_1$, $k_3 = a_1 a_2 - \varepsilon$.

To prove the theorems it is enough to examine that (12) in every case holds.

Let us denote by λ_0, $\omega_0 i$, $-\omega_0 i$ the characteristic roots of A and let $\psi(\lambda) = -|A - \lambda I|$ be its characteristic polynomial.

For (14) we have

$$\chi(\lambda) = [\psi(\lambda)]^{-1} \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ -a_1 a_2 & -a_2 & -a_1 - \lambda & 1 \\ 1 & 0 & 0 & 0 \end{vmatrix} = -1/\psi(\lambda) = \gamma/(\lambda - \lambda_0) + (\alpha\lambda + \beta)/(\lambda^2 + \omega_0^2),$$

where $\gamma = -1/p$, $\alpha = 1/p$, $\beta = \lambda_0/p$, $p = \lambda_0^2 + \omega_0^2$. (12) takes the form

$$k^{-1} + \lambda_0/(p\omega_0^2) + \mathrm{Re}[(1 + i\omega\lambda_0/\omega_0^2)(p\lambda_0 - p\omega i)^{-1}] > 0 .$$

The latter inequality holds, if

$$\omega^2 p \omega_0^2 + p\lambda_0(k + \omega_0^2 \lambda_0) > 0$$

i.e. if $k < -\omega_0^2 \lambda_0 = a_1 a_2$. But this inequality is satisfied by virtue of the definition of k_3.

For the equation (16) we have analogously

$$\chi(\lambda) = \lambda/\psi(\lambda) = \gamma/(\lambda - \lambda_0) + (\alpha\lambda + \beta)/(\lambda^2 + \omega_0^2),$$

where $\gamma = \lambda_0/p$, $\alpha = -\lambda_0/p$, $\beta = \omega_0^2/p$.

Substituting these constants in (12) after simple calculations we obtain

$$\pi(\omega) = 1/k_2 .$$

The condition of Yakubovich will hold if $0 < k_2 < \infty$, what proves Theorem 4.

The characteristic $\chi(\lambda)$ in the case of Theorem 5 is

$$\chi(\lambda) = \lambda^2/\psi(\lambda) .$$

The coefficients of a partial fraction decomposition are: $\alpha = \omega_0^2/p$, $\gamma = \lambda_0^2/p$, $\beta = \lambda_0 \omega_0^2/p$, $p = \lambda_0^2 + \omega_0^2$. The simple calculation gives

$$\pi(\omega) = [\omega^2 p(k_1 \lambda_0 + \omega_0^2) + \lambda_0^2 \omega_0^2 p][k_1 \omega_0^2(\omega_0^2 + \lambda_0^2)]^{-1} > 0 .$$

$\pi(\omega) > 0$ provided $k_1 < -\omega_0^2/\lambda_0 = a_2^2/a_3$. By (17) $k_1 < a_2^2/a_3$, thus Theorem 5 is proved.

Remark 1. The stability of solutions of the equation (18) is shown only for functions $f_1(v)$ satisfying

$$0 < k_1 < f_1(v)/v < k_2 \tag{19}$$

although the linear equation $y''' + ky'' + a_2 y' + a_3 y = 0$ has asymptotically stable solutions for all positive k. But V. A. Pliss [10] has constructed the equation with non-linear function $f_1(v)$ not belonging to the class (19), which has the periodic solution.

Remark 2. Generally in the case of differential equations of the order higher than three, it is impossible to find a Liapunov function of the form (11) assuring the global stability of the trivial solution of (9) (see [7]). It follows from the fact that some eigenvalues of A may lay too close to the imaginary axis, and it makes impossible to construct the function V belonging to the class given by (11).

5. In this section we deal with the equation

$$y''' + a_1 y'' + a_2 y' + f(y) = p(t) \tag{20}$$

in which $a_1 > 0$, $a_2 > 0$, $p(t)$ is continuous and for $t \geqslant 0$

$$|p(t)| \leqslant M_0 .$$

Theorem 6. *If $f(y)$ is continuous, $f(0) = 0$ and for $y \neq 0$*

$$0 < \varepsilon < f(y)/y < a_1 a_2 - \varepsilon \tag{21}$$

with arbitrary small ε, then solutions of (20) are globally bounded.

Proof. As mentioned in Section 3 we will study behaviour of solutions of the equivalent system

$$x_1' = x_2, \quad x_2' = x_3, \quad x_3' = -a_1 a_2 x_1 - a_2 x_2 - a_1 x_3 + \varphi(\sigma) + p(t), \quad \sigma = x_1 , \tag{22}$$

where $\varphi(\sigma) = a_1 a_2 \sigma - f(\sigma)$.

By Theorem 3 there exists a function $V(x)$ defined by

(23) $$V(x) = (x, Hx) + \vartheta \int_0^\sigma \varphi(u)\,du \qquad x^* = (x_1, x_2, x_3),$$

which is a Liapunov function for the system

(24) $$x_1' = x_2, \qquad x_2' = x_3, \qquad x_3' = -a_1 a_2 x_1 - a_2 x_2 - a_1 x_3 + \varphi(\sigma), \qquad \sigma = x_1.$$

Using the function $V(x)$ we will construct the Liapunov function, that will be applied in the proof of the theorem. Our procedure is based on the following lemmas concerning the properties of $V(x)$.

Lemma 1. *The derivative of $V(x)$ along the solutions of (24) satisfies the inequality*

(25) $$dV/dt \leqslant -M_1[(a_2 x_1 + x_3)^2 + \varphi^2(\sigma)], \qquad M_1 > 0.$$

Proof. By means of nonsingular transformation $x = S\tilde{x}$,

$$S = \begin{bmatrix} 1 & 0 & 0 \\ -a_1 & 0 & \sqrt{a_2} \\ a_1^2 & -a_2 & 0 \end{bmatrix}$$

we transform (24) into the system

$$\tilde{x}_1' = P_1\tilde{x}_1 + q_1\varphi(\sigma), \qquad \tilde{x}_2' = P_2\tilde{x}_2 + q_2\varphi(\sigma), \qquad \sigma = (\tilde{x}_1, r_1) + (\tilde{x}_2, r_2),$$

where

$$\tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}, \quad \tilde{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \quad \tilde{r} = \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}, \quad P = \begin{bmatrix} \lambda_0 & 0 & 0 \\ 0 & 0 & \omega_0 \\ 0 & -\omega_0 & 0 \end{bmatrix} = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} = S^{-1}AS,$$

$\tilde{x} = S^{-1}x$, $\tilde{q} = Sa$, $\tilde{r} = S^* b$ and $V(x)$ into the form

$$\tilde{V}(\tilde{x}) = (\tilde{x}, \tilde{H}\tilde{x}) + \vartheta \int_0^\sigma \varphi(u)\,du,$$

where $\tilde{H} = S^* H S$, $\tilde{H}\tilde{P} + \tilde{P}^*\tilde{H} = -G = -\begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix}$, $G_1 > 0$.

It may be proved (see [8]) that the total derivative of $\tilde{V}$ is nonpositive and, when (12) holds, it satisfies the inequality

$$d\tilde{V}/dt \leqslant -(\tilde{x}_1, G_1\tilde{x}_1) + \left|\left(\tilde{x}_1, g_1 - \frac{\tau}{2}r_1\right)\right|^2 \frac{k}{\tau} - \frac{k}{\tau}\left[\left(\tilde{x}_1, g_1 - \frac{\tau}{2}r_1\right) + \frac{\tau}{k}\varphi(\sigma)\right]^2$$

($\tau > 0$ is a suitable constant, $-g_1 = H_1 q_1 + \frac{\vartheta}{2}P_1^* r_1$), from which it follows

(26) $$dV/dt \leqslant -M_2[(\tilde{x}_1, \tilde{x}_1) + \varphi^2(\sigma)]$$

for the sufficiently small positive constant M_2. (26) and the formula $\tilde{x}_1 = -a_2(a_2 x_1 + x_3)/|S|$ gives us (25).

Lemma 2. *If $\eta > 0$ is sufficiently small, then the function $W(x) = V(x) + \eta x_2 x_3$ is a Liapunov function for (24). The derivative of $W(x)$ along the solutions of (24) is negative.*

Proof. Since $W(x)$ is continuous (uniformly over the sphere $(x, x) = 1$) in η, the assumption (21) and the inequality $V(x) > 0$ imply $W(x) > 0$ if η is small enough. From the formula $\lim_{|x|\to\infty} V(x) = \infty$ it follows that $W(x) \to \infty$ as $|x| \to \infty$.

By virtue of Lemma 1 the total derivative of W satisfy the inequality

$$dW/dt - M_1[(a_2x_1 + x_3)^2 + \varphi^2(\sigma)] + \eta[x_3^2 - a_1x_2x_3 - a_2x_2^2 - a_1a_2x_1x_2 + \varphi(\sigma)x_2]$$

from which in view of formulas

$$\varepsilon < \varphi(\sigma)/\sigma < a_1a_2, \qquad -a_1a_2x_1x_2 + \varphi(\sigma)x_2 = -x_1x_2(a_1a_2 - \varphi(\sigma)/\sigma)$$

we have

$$dW/dt \leqslant -M_1\{(a_2x_1 + x_3)^2 + \varepsilon^2x_1^2\} + \eta\{x_3^2 - a_1x_2x_3 - a_2x_2^2\} - \eta x_1x_2(a_1a_2 - \varphi(\sigma)/\sigma)$$

The right-hand side of this inequality is negative, if there is $\eta > 0$ such that the quadratic form

$$M_1\{(a_2x_1 + x_3)^2 + \varepsilon^2x_1^2\} - \eta\{x_3^2 - a_1x_2x_3 - a_2x_2^2\} + \eta x_1x_2(a_1a_2 - \varepsilon)$$

is positive-definite.

The matrix $A(\eta)$ of this form is

$$A(\eta) = \begin{bmatrix} M_1(a_2^2 + \varepsilon^2) & 1/2\eta(a_1a_2 - \varepsilon) & M_1a_2 \\ 1/2\eta(a_1a_2 - \varepsilon) & \eta a_2 & 1/2\eta a_1 \\ M_1a_2 & 1/2\eta a_1 & M_1 - \varepsilon \end{bmatrix}.$$

It is easy to see that for sufficiently small positive η_0, $A(\eta_0)$ is positive-definite and corresponding to η_0 there is an $M_3(\eta_0) > 0$ such that

$$dW/dt \leqslant -M_3(\eta_0)(x, x) < 0\,. \tag{27}$$

The function $W(x)$ considered above will be used in the proof of Theorem 6.

The derivative of $W(x)$ along the solution of (22) may be written down as a sum

$$dW/dt = (dW/dt)_{(24)} + l(x; p(t))',$$

where $(dW/dt)_{(24)}$ denotes the derivative of $W(x)$ along the solutions of (24) and $l(x; p(t))$ is a linear form whose coefficients depend on $p(t)$ and are bounded for $t \geqslant 0$. Let $|l(x; p(t))| \leqslant M_4|x|$. By (27) we have

$$dW/dt \leqslant -M_3(x, x) + M_4\sqrt{(x, x)}\,.$$

For sufficiently large $|x|$ we have $dW/dt < 0$, what ends the proof of Theorem 6.

Corollary. *If the function $p(t)$ is periodic with the period ω and the assumptions of Theorem 6 hold, then the equation (20) has at least one periodic solution with the period ω.*

Proof. Let $\Omega(C) = \{x\colon W(x) \leqslant C\}$, where $W(x)$ is the Liapunov function considered above. We shall show that for sufficiently large C $\Omega(C)$ is starlike.

For this sake consider the equation

$$W(s\tau) = C, \tag{28}$$

where s is an arbitrary vector, $|s| = 1$ and τ is real positive number. Notice that (28) has arbitrary large roots if C is large enough.

For sufficiently large τ $dW(s\tau)/d\tau > 0$. In fact, we have $W(x) = (x, Kx) + \vartheta \int_0^\sigma \varphi(u)du$, where the form (x, Kx) is positive-definite, so

$$dW(s\tau)/d\tau = 2\tau(s, Ks) + (b, s)\vartheta\varphi\big(\tau(b, s)\big) = 2/\tau[(s\tau, Ks\tau) + \vartheta\mu(b, s\tau)^2/2],$$
$$\mu = \varphi(\sigma)/\sigma$$

and by the assumptions involving $\varphi(\sigma)$ and $V(x)$ is positive for large τ. Thus for C large enough (28) has exactly one root and so $\Omega(C)$ is starlike. From this it follows that $\Omega(C)$ is homeomorphic to the "solid" sphere.

From the proof of Theorem 6 it is easy to see that for large C all trajectories of (22) starting from the points belonging to $\mathrm{Fr}\,\Omega(C)$, enters the domain $\bar{\Omega}(C)$.

Repeating the reasoning used in the end of Section 2, we obtain the assertion of Corollary.

6. As it was mentioned in Remark 2, for $n > 3$ it is impossible to obtain any stability conditions by use of the Liapunov function of the form (11). But if we make the additional assumption concerning the eigenvalues of A, we obtain the following

Theorem 7. *Let coefficients a_1, a_2, a_3 of the differential equation*

$$y^{(4)} + a_1 y''' + a_2 y'' + a_3 y' + f(y) = 0 \tag{29}$$

satisfy the assertion of Section 1. Let a_4 be such positive constant that

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = (\lambda^2 + \omega_0^2)(\lambda - \lambda_1)(\lambda - \lambda_2),$$

where $\mathrm{Re}\,\lambda_1 < 0$.

If the function $f(y)$ is continuous and

$$\varepsilon y^2 < f(y)y < a_4 y^2, \tag{30}$$

the root λ_1 satisfies the condition

$$|\mathrm{Im}\,\lambda_1| < |\mathrm{Re}\,\lambda_1|, \tag{31}$$

then the trivial solution of (29) *is globally asymptotically stable.*

Proof. We will proceed as in the proof of Theorems 3, 4, 5. The matrix of the system (9) equivalent to (29) has two pure imaginary eigenvalues. By (30) the function $\varphi(\sigma) = a_4\sigma - f(\sigma)$ satisfy

$$0 < \varphi(\sigma)\sigma < k\sigma^2, \qquad k = a_4 - \varepsilon. \tag{32}$$

Using the notations of Section 3, we have

$$\chi(\lambda) = \left(b, (a-\lambda I)^{-1}a\right) = 1/\psi(\lambda) = (\gamma\lambda+\delta)/(\lambda-\lambda_1)(\lambda-\lambda_2)+(\alpha\lambda+\beta)/(\lambda^2+\omega_0^2),$$

where

$$\gamma = p/M, \quad \delta = -(p^2+q)/M, \quad \alpha = -p/M, \quad \beta = q/M,$$
$$p = \lambda_1+\lambda_2, \quad q = \omega_0^2-\lambda_1\lambda_2^2, \quad M = (\omega_0^2+\lambda_1^2)(\omega_0^2+\lambda_2^2).$$

Applying these formulas to (12) after straightforward calculation we obtain

$$\pi(\omega) = \frac{\omega^4 M\omega_0^2+\omega^2\omega_0^2 M(\lambda_1^2+\lambda_2^2)+M\lambda_1\lambda_2(\omega_0^2\lambda_1\lambda_2-k)}{kM\omega_0^2(\omega^2+\lambda_1^2)(\omega^2+\lambda_2^2)} > 0$$

this inequality holds for all real ω, if

(33) $$\lambda_1^2+\lambda_2^2 > 0, \quad \omega_0^2\lambda_1\lambda_2-k > 0.$$

(31) implies the first of inequalities (33), the other follows from (32) and the formula $a_4 = \omega_0^2\lambda_1\lambda_2$.

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