ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
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Jan Bochenek

On an oscillation problem for selfadjoined elliptic equation

1. Let D be a simply connected domain in (x, y)-plane and let the boundary $F(D)$ of D be a piecewise regular curve. It is well known that the first eigenvalue of the equation

$$u''_{xx} + u''_{yy} + \lambda u = 0$$

with the boundary condition $u = 0$ on $F(D)$ for domains D with the same area is smallest if D is a circle. This may be also formulated in the form of the following inequality important for applications

$$\lambda_1 |D| \geqslant \pi j_0^2, \tag{1}$$

where $|D|$ is the area of D and j_0 is the smallest positive zero of Bessel's function $J_0(x)$.

The inequality (1) has been derived for a more general equation by Z. Nehari [2]. He namely proved that the first eigenvalue λ_1 of the equation

$$u''_{xx} + u''_{yy} + \lambda \varrho(x, y) u = 0$$

with boundary condition $u = 0$ on $F(D)$, where $\varrho(x, y) > 0$ and $\log \varrho(x, y)$ is subharmonic function in the closure $\overline{D}$ of D, satisfies the inequality

$$\lambda_1 \iint_D \varrho(x, y) \, dx\, dy \geqslant \pi j_0^2 .$$

The purpose of this paper is to extend the result of Z. Nehari to the equation

$$\big(p(x, y) u'_x\big)'_x + \big(p(x, y) u'_y\big)'_y - q(x, y) u + \lambda \varrho(x, y) u = 0 \tag{2}$$

with boundary condition $u = 0$ on $F(D)$ and by means of this generalization to estimate the number of nodal domains of the n-th eigenfunction of (2).

2. Theorem 1. *If* $p(x, y) > 0$ *is a function of the class* C^2, $q(x, y) \geqslant 0$, $\varrho(x, y) > 0$ *in the closure* $\bar{D}$ *of* D *and if* $\log p(x, y)$ *and* $\log \varrho(x, y)/p(x, y)$ *are functions subharmonic in* $\bar{D}$, *then the first eigenvalue* λ_1 *of the equation* (2) *with the boundary condition* $u = 0$ *on* $F(D)$ *satisfies the inequality*

$$\lambda_1 \iint_D \frac{\varrho(x, y)}{p(x, y)}\, dx\, dy \geqslant \pi j_0^2 . \tag{3}$$

Proof. Put

$$v(x, y) = \sqrt{p(x, y)}\, u(x, y) . \tag{4}$$

Then (2) takes the form

$$v''_{xx} + v''_{yy} - Q(x, y) v + \lambda \frac{\varrho(x, y)}{p(x, y)} v = 0 \tag{5}$$

where

$$Q(x, y) = \frac{1}{2} \Delta[\log p(x, y)] + \frac{1}{4p^2} \operatorname{grad}^2 p(x, y) + \frac{q(x, y)}{p(x,y)} . \tag{6}$$

Observe that due to (4) the first eigenvalue of the equation (2) with boundary condition $u = 0$ on $F(D)$ is the first eigenvalue of the equation (5) with boundary condition $v = 0$ on $F(D)$ and vice versa.

Take the equation

$$w''_{xx} + w''_{yy} + \mu \frac{\varrho(x, y)}{p(x, y)} w = 0 \tag{7}$$

with boundary condition $w = 0$ on $F(D)$ and observe that by the assumptions of Theorem 1 the function Q given by (6) is nonnegative in the closure $\bar{D}$ of D. Therefore (see [1], p. 412)

$$\lambda_1 \geqslant \mu_1 \tag{8}$$

where λ_1 and μ_1 are the first eigenvalues of the equations (5) and (7) with the boundary conditions $v = 0$ and $w = 0$ on $F(D)$ respectively. Since due to our assumptions the equation (7) is the equation considered by Z. Nehari in [2], so

$$\mu_1 \iint_D \frac{\varrho(x, y)}{p(x,y)}\, dx\, dy \geqslant \pi j_0^2 . \tag{9}$$

The inequalities (8) and (9) imply the inequality (3).

3. Theorem 2. *If the functions* $p(x, y)$ *and* $\varrho(x, y)$ *satisfy the assumptions of Theorem* 1 *and if* $N(n)$ *denotes the number of nodal domains of the* n*-th eigenfunction of the equation* (2) *with boundary condition* $u = 0$ *on* $F(D)$, *then*

$$N(n) \leqslant 0{,}7 n \tag{10}$$

for n *sufficiently large.*

Proof. Let $D_1, \ldots, D_{N(n)}$ be nodal domains of the n-th eigenfunction $u_n(x, y)$ of the equation (2) with boundary condition $u = 0$ on $F(D)$. Due to [1] (p. 451), the function $u_n(x, y)$ and the eigenvalue λ_n are the first eigenfunction and the first eigenvalue respectively, for every domain D_i ($i = 1, \ldots, N(n)$) and for the equation (2) with the boundary condition $u = 0$ on $F(D_i)$. Therefore due to Theorem 1

$$\lambda_n \iint\limits_{D_i} \frac{\varrho}{p}\, dx\, dy \geqslant \pi j_0^2 \qquad i = 1, \ldots, N(n),$$

whence

$$\iint\limits_{D} \frac{\varrho}{p}\, dx\, dy \geqslant \frac{N(n)}{\lambda_n} \pi j_0^2 = \frac{n}{\lambda_n} \cdot \frac{N(n)}{n} \cdot \pi j_0^2 . \tag{11}$$

Since (see [1], p. 436)

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = \frac{1}{4\pi} \iint\limits_{D} \frac{\varrho}{p}\, dx\, dy ,$$

we get from (11)

$$\iint\limits_{D} \frac{\varrho}{p}\, dx\, dy \geqslant \frac{j_0^2}{4} \iint\limits_{D} \frac{\varrho}{p}\, dx\, dy \cdot \liminf_{n\to\infty} \frac{N(n)}{n} ,$$

whence

$$\liminf_{n\to\infty} \frac{N(n)}{n} \leqslant \left(\frac{2}{j_0}\right)^2 .$$

Since $j_0 = 2{,}408$, the last inequality implies (10) for all n sufficiently large.

Remark. The inequality (10) sharpens the well known Courant's inequality $N(n) \leqslant n$ (see [1], p. 454) and generalizes the inequality proved by Pleijel [3] in the case $p(x, y) \equiv \varrho(x, y) \equiv 1$, $q(x, y) \equiv 0$.

REFERENCES

[1] R. Courant, D. Hilbert, *Methods of mathematical physics*, New York 1953.

[2] Z. Nehari, *On the principal frequency of a membrane*. Pacific Journ. Math. 8 (1958), 285-293.

[3] A. Pleijel, *Remarks on Courant nodal line theorem*, Comm. Pure Appl. Math. 9 (1956), 543-550.