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**HADAMARD'S FUNDAMENTAL SOLUTION AND
 MULTIPLE-CHARACTERISTIC PROBLEMS [†]**

Abstract. In this work, Hadamard's construction of fundamental solutions for linear holomorphic PDE is generalized to a rather broad class of linear holomorphic PDE. In the case of simple-characteristics, solutions with polynomial singularity along the characteristic conoid are constructed which are very close to Hadamard's fundamental solution. In the case of multi-characteristics, solutions with exponential singularity are obtained. It is clear that Hadamard's procedure is just a kind of asymptotic method; using this method and singular solutions mentioned above, some new results for the propagation of singularities are also proved which are essentially a generalization of Huygens' Principle.

1. Introduction

Among Hadamard's contributions to the theory of linear PDE, the construction of fundamental solution is very prominent and of basic importance. In a certain sense, this contribution of his summed vast progress in this area to his time, and was very clearly presented in his classical treatise: "Le Problème de Cauchy" [1] and also in his posthumous monograph: "La Théorie des Équations aux Dérivées Partielles" [2], which was published in China (1964) in accordance with his will, hence little known in the West and is now a literature rarity even in China. Hadamard defined the fundamental solution (solution élémentaire) to be solutions with certain singularity and tried to find them in the form of an asymptotic series

$$(1) \quad \Gamma(x, y) = \sum_{h=0}^{\infty} U_h(x, y) k^{p+h} / \Gamma(p + h + 1)$$

where $k = 0$ is the equation for the characteristic conoid, and $k(x, y)$ satisfies an important first relation:

$$(2) \quad A\left(x, \frac{\partial k}{\partial x}\right) = 4k,$$

where $A(x, \xi)$ is the principal symbol of the 2nd order linear holomorphic partial differential operator

$$(3) \quad Lu = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial u}{\partial x_i} + c(x)u,$$

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$$a_{ij}(x) = a_{ji}(x), \quad \det(a_{ij}(x)) \neq 0.$$

Hadamard's method is actually just the asymptotic method widely used among mathematical physicists. (2) is just the eiconal equation. Hadamard then proceeded and obtained the transport equations for U_h and the convergence of the series (1) was proved by standard majoration methods.

Hadamard's argument is variational: characteristic conoid is composed of bicharacteristics issuing from the vertex, while the latter are geodesics in a certain metric defined through the principal symbol which is quadratic. This approach restricted him to partial differential operators of 2nd order only. But as early as H. Poincaré and E. Cartan, it was known that Hamiltonian variational principles are equivalent to the principle of integral invariants [3]; the latter principle is symplectic in nature and thus can be used in a much broader field. J. Leray is aware of this, in the late 50's and early 60's, he published a series of important papers under the general title "Problème de Cauchy" [4], which contains a generalization of (2) as

$$(4) \quad g(x, k_x) = \frac{k}{m-1}.$$

Also, it should be mentioned, the 6th or the last of these series, i.e., L. Gårding, T. Kotake and J. Leray [5] developed systematically the asymptotic approach to the Cauchy problem.

But there is another approach to the theory of fundamental solutions. The year 1930's saw I. G. Petrowsky's work on general PDE which started an algebro-geometrical approach to the theory of linear PDE with constant coefficients. Fourier-Laplace transform is the main tool, and this is the beginning of the period when the theory of distributions "dominated" this field. Fundamental solutions are now defined to be the solutions of the equation $L(u) = \delta$. Ehrenpreis, Hörmander and Malgrange proved the existence of the fundamental solutions for general linear PDE with constant coefficients. For general linear PDE with variable coefficients, Lewy's famous example showed the deep difference between the analytic and C^∞ frames. For the latter, we have now the micro- local analysis, with the help of the PsDO and the FIO, we can construct parametrices, which are approximate fundamental solutions. Thus there arises a problem: can we also construct distributional fundamental solutions for general linear PDE with variable coefficients and what are the relations between these two kinds of fundamental solutions? Another problem is: Hadamard established his theory for normal elliptic and hyperbolic equations, which are all of principal type. Then, what can we say about the multiple-characteristic problems? As will be seen in what follows, Hadamard's theory is actually a Fuchsian theory, while PDE's with multiple-characteristics can be considered as an analogy of the ODE's with irregular singularities. For the latter, we have the arsenal of such things as the asymptotic expansions in a sector, the Borel-Laplace transforms etc. All these come from the exponential growth of the solutions. Thus the Gevrey classes and their dual, the ultra- distributions, offer a natural frame for the multiple-characteristic problems. For a very clear treatment, see [14], also see [15] for an up-to-date survey with comprehensive literature. But can we also consider Hadamard's theory from a distributional point of view? It is the author's aim to give partial answers to these problems. We can prove that Hadamard's fundamental solution is only one from a broad category of solutions with definite singularity, and both approaches are closely related and can be unified. But in the present paper, we must restrict ourselves to first extend Hadamard's approach to a class of linear holomorphic PDE of higher order with simple characteristics, and next construct for a class of linear holomorphic PDE's with multiple- characteristics a solution with exponential singularity following Hadamard's procedure, hence we call it the Hadamard fundamental solution, although its relation with the distributional fundamental solution is not clear yet and will be treated later.

The plan of this paper is as follows. In part I, we consider the simple- characteristic case

where (4) is valid. In part II, the simplest case of multiple-characteristic problem is considered where (4) is not valid.

2. Part I. Simple-characteristic Problems

2.1. Notations

Let X be a domain in an analytic complex manifold with complex dimension n , with local coordinates for its element $x = (x_1, \dots, x_n)$. The complex projective space $\Xi^* = \mathbf{C}^{n+1} \setminus 0 / \sim$ (\sim denotes dilation) is just the space of complex affine functions defined on X , and we denote

$$(5) \quad \langle \xi, x \rangle = \xi_0 + \xi \cdot x = \xi_0 + \sum_{i=1}^n \xi_i \cdot x_i .$$

Let $P(x, \partial_x)$ be a linear holomorphic partial differential operator of order m :

$$(6) \quad P(x, \partial_x) = \sum_{|\alpha| \leq m} a_\alpha(x) \partial^\alpha ,$$

with principal symbol

$$(7) \quad P_m(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha .$$

2.2. Characteristic conoid

Bicharacteristics strips for $P(x, \partial_x)$ are defined as orbits of the Hamiltonian system

$$(8) \quad \begin{aligned} \frac{dx}{dt} &= \frac{\partial P_m(x, \xi)}{\partial \xi} x|_{t=0} = y , \\ \frac{d\xi}{dt} &= -\frac{\partial P_m(x, \xi)}{\partial x} \xi|_{t=0} = \eta , \\ P_m(y, \eta) &= 0 , \end{aligned}$$

which give extremal curves for the action integral

$$W = \int \sum \xi_i dx_i - H dt .$$

In analytic dynamics, the Hamiltonian H is the total energy, i.e., the sum of kinetic energy (quadratic in ξ) and potential energy, and in our case, H should be replaced by $P_m(x, \xi)$ which is homogeneous of degree m in ξ now. H. Poincaré and E. Cartan realized that Hamilton's principle of least action is equivalent to the principle of invariance of energy-momentum form. Thus in our case, we should add another equation for ξ_0 to (8)

$$(9) \quad \begin{aligned} \frac{d\xi_0}{dt} &= \sum_i x_i \frac{\partial}{\partial x_i} P_m(x, \xi) - P_m(x, \xi) , \\ \xi_0|_{t=0} &= \eta_0 , \end{aligned}$$

and for initial values we should assume

$$\begin{aligned}\langle \eta, y \rangle &= \eta_0 + \sum_{i=1}^n \eta_i y_i = 0, \\ P_m(y, \eta) &= 0.\end{aligned}$$

It is readily seen, the system (8), (9) has first integrals

$$P_m(x, \xi), \langle \xi, x \rangle + (1-m)t P_m(x, \xi)$$

and also an invariant differential form

$$(10) \quad \omega = \langle d\xi, x \rangle + P_m(x, \xi) dt.$$

Actually, set $\omega = 0$ and $d\omega = 0$ as differential forms in dx , $d\xi$ and dt , we could recover (8) and (9), meaning that this system is the characteristic system for ω and hence ω is invariant form, of (8) and (9) [3] (Chap. 5).

x -components of the system (8), i.e., the bicharacteristic curves in X , through a fixed point y with $P_m(y, \eta) = 0$, $\eta \neq 0$, form a surface with y as a conic point which is called the characteristic conoid with vertex at y . We have

THEOREM 1 (J. LERAY). *The characteristic conoid can be written as $k(x, y) = 0$ where $k(x, y)$ is holomorphic in x and y when $|x - y|$ is small enough, and*

$$(11) \quad P_m(x, k_x) = \frac{k}{m-1}$$

under the assumption

$$(12) \quad \text{Hess}_\eta P_m(y, \eta) \neq 0,$$

when $\eta \neq 0$.

Proof. The following proof is reproduced from Leray [4] (paper I) where η is assumed to be complex and (12) holds when $P_m(y, \eta) = 0$. The solution to (8) when $P_m(y, \eta)$ is arbitrary gives a mapping

$$\begin{aligned}x &= y + P_{m\xi}(x, \xi)t + o(t) \\ &= y + P_\eta m(y, \eta)t + o(t) \\ (\xi &= \eta - P_{mx}(x, \xi)t + o(t))\end{aligned}$$

when $|t|$ is small enough. Thus we may write

$$(13) \quad \begin{aligned}x &= x(t, y, \eta), \\ \xi &= \xi(t, y, \eta),\end{aligned}$$

and prove immediately that

$$\frac{D(x_1(t, y, \eta), \dots, x_n)}{D(\eta_1, \dots, \eta_n)} = t \text{Hess}_\eta P_m(y, \eta) + o(t),$$

which does not vanish for $t \neq 0$ when $\text{Hess}_\eta P_m(y, \eta) \neq 0$ and $\eta \neq 0$. Thus we may replace (t, x, y) by (t, η, y) and vice versa. The invariance of ω (see (10)) gives

$$\langle d\xi, x \rangle = -P_m(x, \xi) dt + \langle d\eta, y \rangle$$

from which follows the relation:

$$d\langle \xi, x \rangle = -P_m(x, \xi) dt + \sum_{i=1}^n \xi_i dx_i - \sum_{i=1}^n \eta_i dy_i,$$

(note that $\langle y, \eta \rangle = 0$). Taking (t, x, y) as independent variables, we have

$$(14) \quad \begin{aligned} \frac{\partial}{\partial t} \langle \xi, x \rangle &= -P_m(x, \xi), \\ \frac{\partial}{\partial x_i} \langle \xi, x \rangle &= \xi_i, \\ \frac{\partial}{\partial y_i} \langle \xi, x \rangle &= -\eta_i. \end{aligned}$$

Since $P_m(x, \xi)$ and $\langle \xi, x \rangle + (1-m)t P_m(x, \xi)$ are first integrals,

$$(15) \quad \begin{aligned} P_m(x, \xi) &= P_m(y, \eta), \\ \langle \xi, x \rangle &= (m-1)t P_m(y, \eta). \end{aligned}$$

Substituting (15) into (14), we have

$$\frac{\partial}{\partial t} \langle \xi, x \rangle = \langle \xi, x \rangle / (1-m)t.$$

Thus there is a function $k(x, y)$ such that

$$(16) \quad \langle \xi(t, \eta, y), x(t, \eta, y) \rangle = t^{\frac{1}{1-m}} k(x, y).$$

$k(x, y)$ should be holomorphic in x and y for x, y close enough. $x = y$ corresponds to $t = 0$ which is singular for $\langle \xi, x \rangle$ as seen from (16).

From the second equation of (14), we have

$$(17) \quad \xi_i = \frac{\partial}{\partial x_i} \langle \xi, x \rangle = t^{\frac{1}{1-m}} k_x(x, y),$$

substituting again into the first equation of (14), we have

$$-\frac{\partial}{\partial t} \langle \xi, x \rangle = \frac{\langle \xi, x \rangle}{(m-1)t} = \frac{t^{\frac{m}{m-1}}}{m-1} k(x, y),$$

but

$$P_m(x, \xi) = P_m\left(x, t^{\frac{1}{1-m}} k_x\right) = t^{\frac{m}{1-m}} P_m(x, k_x),$$

hence (11).

(11) shows that $k(x, y) = 0$ is characteristic. Now take into consideration that $P_m(y, \eta) = 0$. If $x = x(t, \eta, y)$ is a bicharacteristic curve through $y : x|_{t=0} = y$, then, since $P_m(x, \xi)$ is a first integral,

$$\begin{aligned} k[x(t, \eta, y), y] &= (m-1)P_m[x(t, \eta, y), k_x(x(t, \eta, y), y)] \\ &= (m-1)t^{\frac{m}{m-1}} P_m[x(t, \eta, y), \xi(t, \eta, y)] \\ &= (m-1)t^{\frac{m}{m-1}} P_m(y, \eta) = 0. \end{aligned}$$

It means that the whole bicharacteristic curve lies on the surface $k(x, y) = 0$. Thus, it is easily seen, $k(x, y) = 0$ is an equation for the characteristic conoid.

□

In the proof above, the condition $\text{Hess}_\eta P_m(y, \eta) \neq 0$ is very important. It excludes multiple-characteristic problems from consideration. Actually, we have the following

LEMMA 1. *If $a(x, \xi)$ and $b(x, \xi)$ both vanish at a point (x_0, ξ_0) and $\text{grad}_\xi a(x_0, \xi_0)$ is parallel to $\text{grad}_\xi b(x_0, \xi_0)$, then $\text{Hess}_\xi [a(x_0, \xi_0)b(x_0, \xi_0)] = 0$, here $x \in \mathbb{C}^n$, $0 \neq \xi \in \mathbb{C}^n$.*

Proof. Denote differentiation in ξ by sub-indices. Since

$$\begin{aligned} [a(x, \xi)b(x, \xi)]_{ij} &= a(x, \xi)b_{ij}(x, \xi) + a_i(x, \xi)b_j(x, \xi) \\ &\quad + a_j(x, \xi)b_i(x, \xi) + a_{ij}(x, \xi)b(x, \xi), \end{aligned}$$

$\text{Hess}_\xi [a(x, \xi)b(x, \xi)]$ can be written as a sum of 2^n determinants and we may arrange them such that every row is of either the following forms:

$$(a_i b_1, a_i b_2, \dots, a_i b_n) = a_i(b_1, b_2, \dots, b_n)$$

or

$$(a_1 b_i, a_2 b_i, \dots, a_n b_i) = b_i(a_1, a_2, \dots, a_n)$$

Since these vectors are parallel, each determinant contain 2 linearly dependent rows making this determinant vanishing. Thus the lemma is proved. \square

Because of this lemma, arguments in this part are usually invalid for partial differential operator of the form

$$P_m(x, \partial_x) = [a(x, \partial_x)]^2$$

which is the simplest double-characteristic case, and would be treated in part II of this paper.

2.3. Characteristic projection

Now we discuss further properties of $k(x, y)$.

It is easy to see that the solutions of (8), (9) enjoy the following homogeneity properties:

$$\begin{aligned} (18) \quad \xi_i(\theta^{1-m}t, \theta\eta, y) &= \theta\xi_i(t, \eta, y), \\ x_i(\theta^{1-m}t, \theta\eta, y) &= x_i(t, \eta, y). \end{aligned}$$

Hence ξ is a characteristic projection in Leray's sense [4] (IV). Set

$$(19) \quad \tau = t^{\frac{1}{m-1}}, \quad \theta = \tau,$$

we have

$$\begin{aligned} \tau\xi_i(t, \eta, y) &= \xi_i(1, \tau\eta, y), \\ x_i(t, \eta, y) &= x_i(1, \tau\eta, y). \end{aligned}$$

Thus, denote $\tau\eta = \zeta$, $\tau\xi = \pi$, we see $x_i(t, \eta, y)$ are actually functions $x_i(1, \zeta, y)$ of (ζ, y) , $\pi_i(t, \eta, y)$ are $\xi_i(1, \zeta, y)$, both are holomorphic in (ζ, y) . Hence we write them hereafter simply as $x(\zeta, y)$, $\pi(\zeta, y)$. (16) and (17) now give

$$(20) \quad \begin{aligned} k(x, y) &= \langle \pi(\zeta, y), x(\zeta, y) \rangle, \\ k_{x_i}(x, y) &= \pi_i(\zeta, y), \end{aligned}$$

and $k(x, y)$ is also holomorphic in (ζ, y) .

The solutions of (8) now give

$$(21) \quad \begin{aligned} x_i &= y_i + P_{m\zeta_i}(x, \zeta) + \dots, \\ \pi_i &= \zeta_i - \tau P_{mx_i}(x, \zeta) + \dots, \end{aligned}$$

“ \dots ” are terms of higher order. Hence, from the implicit function theorem, we have from the second equation of (21),

$$\zeta_i = \pi_i + o(\tau).$$

Substitute it into the first equation of (21) gives

$$x_i = y_i + P_{m\zeta_i}(y, \pi) + o(\tau^m).$$

Differentiate both side with respect to x_j , we have

$$\begin{aligned} \delta_{ij} &= \sum_{l=1}^n P_{mil}(y, \pi) \frac{\partial \pi_l}{\partial x_j} + \dots \\ &= \tau^{m-2} \sum_{l=1}^n P_{mil}(y, \xi) \frac{\partial \pi_l}{\partial x_j} + \dots \\ &= \sum_{l=1}^n P_{mil}(y, k_x) \frac{\partial^2 k}{\partial x_i \partial x_j} + \dots \end{aligned}$$

Thus,

$$(22) \quad \sum_{i,j=1}^n P_{mij}(x, k_x) \frac{\partial^2 k}{\partial x_i \partial x_j} = n + F(\zeta, y).$$

where $F(\zeta, y)$ is holomorphic in (ζ, y) and $F(0, y) = 0$. Henceforth, sub-indices always denote differentiation in fiber variables.

2.4. Construction of Hadamard fundamental solution

Now we will look for a solution to $Pu = 0$ where P is defined in (6) in the form

$$(23) \quad u(x, y) = \sum_{h=0}^{\infty} U_h k^{p+h} / \Gamma(p+h+1),$$

p and U_h are to be decided. Whereas (11) plays the role of eiconal equation, we are to find the transport equations. We proceed first to calculate $P_m(x, \partial_x)U_0 k^p / \Gamma(p+1)$. By generalized Leibniz's formula, we have

$$(24) \quad \begin{aligned} P_m(x, \partial_x)U_0 k^p / \Gamma(p+1) &= U_0 P_m(x, \partial_x)k^p / \Gamma(p+1) \\ &+ \sum_{i=1}^n \frac{\partial U_0}{\partial x_i} P_{mj}(x, \partial_x)k^p / \Gamma(p+1) + \dots, \end{aligned}$$

where the dots stand for terms containing a factor of the form $P_{m\xi}^{(\alpha)}(x, \partial_x)k^p / \Gamma(p+1)$, $|\alpha| \geq 2$. For the first term, a careful calculation gives

$$(25) \quad \begin{aligned} P_m(x, \partial_x)k^p / \Gamma(p+1) &= P_m(x, k_x)k^{p-m} / \Gamma(p-m+1) \\ &+ \frac{1}{2} \sum_{i,j=1}^n P_{mij}(x, k_x) \frac{\partial^2 k}{\partial x_i \partial x_j} \cdot k^{p-m+1} / \Gamma(p-m+2) + \dots \end{aligned}$$

Verification of this result, particularly the coefficient $\frac{1}{2}$ in front of the second term is tedious. But a general procedure would be given in Part II, showing that (25) is correct.

For the second term in (24), noting that

$$P_{mj}(x, \partial_x)k^p / \Gamma(p+1) = P_{mj}(x, k_x)k^{p-m+1} / \Gamma(p-m+2) + \dots ;$$

and $k_x = t^{\frac{1}{m-1}}\xi$, $P_{mj}(x, \xi) = \frac{dx_j}{dt}$, we have

$$\sum_{j=1}^n \frac{\partial U_0}{\partial x_j} P_{mj}(x, \partial_x)k^p / \Gamma(p+1) = t \sum \frac{\partial U_0}{\partial x_j} \frac{dx_j}{dt} k^{p-m+1} / \Gamma(p-m+2) + O(k^{p-m+2}).$$

Thus, using (22) and (11)

$$\begin{aligned} P_m(x, \partial_x)U_0k^p / \Gamma(p+1) &= \left[\frac{p-m+1}{m-1} + \frac{n}{2} \right. \\ &\quad \left. + \frac{1}{2}F(\zeta, y) \right] U_0 k^{p-m+1} / \Gamma(p-m+2) \\ &\quad + t \frac{dU_0}{dt} k^{p-m+1} / \Gamma(p-m+2) + O(k^{p-m+2}) \\ &= \left[t \frac{dU_0}{dt} + \left(\frac{p-m+1}{m-1} + \frac{n}{2} \right. \right. \\ &\quad \left. \left. + \frac{1}{2}F(\zeta, y) \right) U_0 \right] k^{p-m+1} / \Gamma(p-m+2) \\ &\quad + O(k^{p-m+2}). \end{aligned}$$

Since $t \frac{d}{dt} = \frac{1}{m-1}\tau \frac{d}{d\tau}$, $F(\zeta, y) = O(\tau)$, when we are to look for a bounded U_0 , p must be so chosen such that

$$(26) \quad \frac{n}{2} + \frac{p-m+1}{m-1} = 0, \text{ i.e., } p = (1-m)\left(\frac{n}{2} - 1\right).$$

After p is decided, we come next to degree $m-1$ (in ∂_x) part of $P(x, \partial_x)$, i.e. $P_{m-1}(x, \partial_x)$. By similar calculation as above, we have

$$\begin{aligned} P_{m-1}(x, \partial_x)U_0k^p / \Gamma(p+1) &= U_0 P_{m-1}(x, \partial_x)k^p / \Gamma(p+1) \\ &\quad + \sum_{|\alpha|>0} \frac{1}{\alpha!} \frac{\partial^\alpha U_0}{\partial x^\alpha} P_{m-l,\xi}^{(\alpha)}(x, \partial_x)k^p / \Gamma(p+1) \\ &= U_0 P_{m-1}(x, k_x)k^{p-m+1} / \Gamma(p-m+2) \\ &\quad + O(k^{p-m+2}) \\ &= t P_{m-1}(x, \xi)U_0 k^{p-m+1} / \Gamma(p-m+2) \\ &\quad + O(k^{p-m+2}). \end{aligned}$$

Summing up, we have proved

THEOREM 2. *We have the following expansion*

$$\begin{aligned}
 P(x, \partial_x)U_0 k^p / \Gamma(p+1) &= \frac{k^{p-m+1}}{\Gamma(p-m+2)} \left[\frac{1}{m-1} \tau \frac{dU_0}{d\tau} \right. \\
 &\quad \left. + \left(\frac{n}{2} + \frac{p-m+1}{m-1} + O(\tau) \right) U_0 \right] \\
 (27) \quad &\quad + \frac{k^{p-m+2}}{\Gamma(p-m+3)} L_2(U_0) + \dots \\
 &\quad + \frac{k^p}{\Gamma(p+1)} L_m(U_0),
 \end{aligned}$$

where L_2, \dots, L_m are linear holomorphic partial differential operators of order 2, 3, ..., m respectively. Thus for the existence of a bounded U_0 , we must take $p = (1-m)(\frac{n}{2}-1)$ and we have the transport equation for U_0 :

$$(28) \quad \tau \frac{dU_0}{d\tau} + \left[(m-1) \left(\frac{n}{2} + \frac{p-m+1}{m-1} \right) + \tau A(\tau) \right] U_0 = 0.$$

We may take

$$(29) \quad U_0 = \exp \left(\int_0^\tau A(\tau) d\tau \right).$$

The transport equations for U_h are similar to (28) only with $p+h$ to replace h and a linear form of $U_{h-1}, \dots, U_{h-m+1}$ and the derivatives of U_{h-j} up to order $j+1$ as the right hand side:

$$\begin{aligned}
 (30) \quad \tau \frac{dU_h}{d\tau} + \left[p+h-m+1 + \frac{n}{2}(m-1) + \tau A(\tau) \right] U_h \\
 = L(U_{h-1}, \dots, U_{h-m+1}).
 \end{aligned}$$

Setting $p = (1-m)(\frac{n}{2}-1)$ as in (26), we have the transport equation for U_h

$$(31) \quad \tau \frac{dU_h}{d\tau} + [h + \tau A(\tau)] U_h = L(U_{h-1}, \dots, U_{h-m+1}).$$

Its unique bounded solution is

$$(32) \quad U_h = \frac{U_0}{\tau^h} \int_0^\tau \frac{\tau^{h-1}}{U_0} L(U_{h-1}, \dots, U_{h-m+1}) d\tau.$$

The argument above fails when p is a negative integer when we should replace (23) by

$$\begin{aligned}
 (33) \quad u(x, y) &= U_0 k^p + \dots + U_{-p-1} k^{-p-1} + U_{-p} \log k + \dots \\
 &\quad + U_{-p+h} k^{(h)} + \dots + \sum_{h=0}^{\infty} V_h k^h / h!,
 \end{aligned}$$

where $k^{(h)}$ is a h -th primitive of k , i.e.,

$$\begin{aligned}
 k^{(h)} &= \int_0^k \dots \int_0^k \log k dk = \frac{k^h}{h!} \left(\log k - 1 - \dots - \frac{1}{h} \right), \\
 &\quad (h - \text{fold integration}).
 \end{aligned}$$

The transport equations for U_0, \dots, U_{-p-1} are the same as (30), in the deduction of which we should note that

$$kk^{(h)} = (h+1)k^{(h+1)} + k^{h+1}/(h+1)!,$$

while for V_h we have

$$(34) \quad \tau \frac{dV_h}{d\tau} + \left[h + \frac{n}{2}(m-1) + \tau A(\tau) \right] V_h = U_{-p+h} + L(V_{h-1}, \dots, V_{h-m+1}).$$

2.5. Convergence of the formal solution

For the convergence of (23) (that for (33) are quite similar, hence omitted) we use the majorant method as J. Hadamard did in [1].

Let $\sigma = \frac{\tau}{\alpha} + \eta_1 + \dots + \eta_n$, $0 < \alpha < 1$. Since U_0 is holomorphic in (τ, η) we can find $0 < r < 1$ and constant $A_0 > 0$ such that

$$(35) \quad U_0 \ll A_0 / \left(1 - \frac{\sigma}{r} \right).$$

Assume that for U_1, \dots, U_{h-1} it is already known that

$$U_j \ll M A_j / \left(1 - \frac{\sigma}{r} \right)^{2j+1},$$

then for U_h we should first estimate $L(U_{h-1}, \dots, U_{h-m+1})$. Since L contains derivatives of U_{h-j} up to orders $j+1$, we have

$$\begin{aligned} U_{h-j}^{(j+1)} &\ll M A_{h-j} (2h-2j+1) \dots (2h-2j+j+1) / \left(1 - \frac{\sigma}{r} \right)^{2h-2j+j+2} \\ &\ll M_1 2h (2h+1) (p+h)_{j-1} A_{h-j} / \left(1 - \frac{\sigma}{r} \right)^{2h+2}. \end{aligned}$$

Thus

$$\begin{aligned} L(U_{h-1}, \dots, U_{h-m+1}) &\ll \frac{K 2h (2h+1)}{\left(1 - \frac{\sigma}{r} \right)^{2h+2}} [A_{h-1} + (p+h-1) A_{h-2} + \dots \\ &\quad + (p+h-1)_{(m-1)} A_{h-m+1}]. \end{aligned}$$

Hence

$$\begin{aligned} U_h &\ll \frac{K 2h (2h+1)}{4(h+1)} [A_{h-1} + (p+h-1) A_{h-2} + \dots \\ &\quad + (p+h-1)_{(m-1)} A_{h-m+1}] / \left(1 - \frac{\sigma}{r} \right)^{2h}, \\ (\lambda)_j &= \lambda(\lambda-1) \dots (\lambda-j+1). \end{aligned}$$

Setting

$$\begin{aligned} A_h &= \Lambda K (p+h+1) [A_{h-1} + (p+h-1) A_{h-2} + \dots \\ &\quad + (p+h-1)_{(m-1)} A_{h-m+1}], \\ \Lambda &= \sup_h \frac{2h(2h+1)}{4(h+1)(p+h+1)}. \end{aligned}$$

We have

$$U_h \ll A_h / \left(1 - \frac{\sigma}{r} \right)^{2h}.$$

Since the power series

$$\sum A_h k^{p+h} / \Gamma(p+h+1)$$

converges for $|k|$ small enough, we know that the series (23) converges for $|k|$ small enough. Summing up we have the main result of Part I.

THEOREM 3. *Let*

$$\begin{aligned} P(x, \partial_x) &= \sum_{|\alpha| \leq m} a_\alpha(x) \left(\frac{\partial}{\partial x} \right)^\alpha \\ &= P_m(x, \partial_x) + P_{m-1}(x, \partial_x) + \dots \\ &= \sum_{|\alpha|=m} a_\alpha(x) \left(\frac{\partial}{\partial x} \right)^\alpha + \sum_{|\alpha|=m-1} a_\alpha(x) \left(\frac{\partial}{\partial x} \right)^\alpha + \dots, \quad m \geq 2 \end{aligned}$$

be a linear holomorphic partial differential operator in a domain X of a complex analytic manifold, $y \in X$ a fixed point, and

$$\text{Hess}_\eta P_m(y, \eta) \neq 0 \text{ for } P_m(y, \eta) = 0, \eta \neq 0.$$

Then, there exists an Hadamard fundamental solution with an asymptotic expansion

$$u(x, y) = \sum_{h=0}^{\infty} U_h k^{p+h} / \Gamma(p+h+1),$$

where $p = (1-m)(\frac{n}{2}-1)$. $k(x, y) = 0$ is the equation of characteristic conoid, with k holomorphic for x close enough to y satisfying

$$P_m(x, k_x) = \frac{k}{m-1}.$$

When p is a negative integer, the expansion should be modified as (33).

2.6. Propagation of Singularities

The method above can also be applied to various problems for linear holomorphic partial differential operator, among them, propagation of singularities. J. Leray, Y. Hadamard and C. Wagschal considered in a series of papers [6], [7], [8] the Cauchy problem for

$$(36) \quad P(x, \partial_x)u = v(x),$$

where

$$(37) \quad v(x) = f(x)/[g(x)]^\lambda,$$

$g(x) = 0$ is a holomorphic hypersurface in X which is not characteristic for $P(x, \partial_x)$. Their main results go roughly as follows: the singularities of the solution $u(x)$ propagate along characteristics issuing from the singularities for the right hand side, i.e., points of the hypersurface $g(x) = 0$. But when $g(x) = 0$ is characteristic, the behavior of the solution is quite different. Actually, we have

THEOREM 4. *If the right hand side of (36) is of the form*

$$(38) \quad P(x, \partial_x)u = f(x)/[k(x, y)]^\lambda,$$

where $1 - \lambda \neq 0, -1, -2, \dots$ and $k(x, y) = 0$ is the characteristic conoid with vertex at y . Then there exists a solution of (38) of the form

$$(39) \quad u(x, y) = \sum_{h=0}^{\infty} U_h(x, y)[k(x, y)]^{p+h}/\Gamma(p+h+1).$$

where

$$(40) \quad p = m - \lambda - 1.$$

Proof. Using the procedure as above, we obtain the transport equation for U_0 as

$$\tau \frac{dU_0}{d\tau} + \left[\frac{n}{2}(m-1) - \lambda + \tau A(\tau) \right] U_0 = f(x)\Gamma(1-\lambda).$$

Denote

$$\mu = \frac{n}{2}(m-1) - \lambda,$$

We have

$$\frac{d}{d\tau}(\tau^\mu U_0) + A(\tau)\tau^\mu U_0 = \tau^{\mu-1}f(x)\Gamma(1-\lambda),$$

and

$$(41) \quad U_0 = \tau^{-\mu} \int_0^\tau \tau^{\mu-1} f(x)\Gamma(1-\lambda) e^{-\int_0^\tau A(s) ds} d\tau.$$

When $\mu \leq 0$, the integral should be taken in the distributional sense. U_0 is bounded. For the remaining U_h , $h > 0$, we have similar transport equations,

$$\tau \frac{dU_h}{d\tau} + \left[\frac{n}{2}(m-1) - \lambda - h + \tau A(\tau) \right] U_h = L(U_{h-1}, \dots, U_{h-m+1}),$$

the unique bounded solution of which is

$$(42) \quad U_h = \frac{U_0}{\tau^h} \int_0^\tau \frac{\tau^{h-1}}{U_0} L(U_{h-1}, \dots, U_{h-m+1}) d\tau.$$

After (39) is constructed as a formal solution, convergence can be proved as before. \square

REMARK 1. When $1 - \lambda = 0, -1, -2, \dots$, further modification for (refc1.34) is needed by adding terms containing $\log k$. The details are omitted.

REMARK 2. The implication of this theorem is as follows. If another hyperplane not through y is taken to be the initial hyperplane, its intersection with $k(x, y) = 0$ are new source of singularities. According to results already known on the behavoir of solutions, singularities would propagate along all bicharacteristics issuing from the intersection. But theorem 4 states that they propagate only along $k(x, y) = 0$. This is analogous to the Huygens' principle for the wave equation. According to Huygens' construction, superficially, there would not only be a wave front going forward, but also another wave front going backward. This apparent contradiction was explained by Fresnel in terms of interference [9]. Theorem 4 claims similar results for general partial differential operators.

REMARK 3. Some new problems are also motivated by the procedure above. For instance, if we introduce new variables (z_1, \dots, z_n) with $z_1 = k(x, y)$. Operator $P(x, \partial_x)$ will have the form

$$(43) \quad z_1 \left(\frac{\partial}{\partial z_1} \right)^m + Q(z, \partial_z)$$

where $Q(z, \partial_z)$ contains at most $\left(\frac{\partial}{\partial z_1} \right)^{m-1}$. (43) is of Fuchsian type. In this sense Hadamard's theory is also a preliminary Fuchsian theory.

But the transformation $x \rightarrow z$ is singular at $x = y$. For the simplest case of $m = 2$, take $y = 0$, $k(x, y) = k(x)$ is holomorphic at $x = 0$. Using Morse lemma, which is also valid in complex holomorphic case (for a proof see e.g. [10]), we will arrive at a holomorphic partial differential operator with $\frac{\partial}{\partial z_1}$ appearing in the principal part in the following form:

$$\left(z_1 \frac{\partial}{\partial z_1} + \sum_{j=2}^n z_j \frac{\partial}{\partial z_j} \right) \frac{\partial}{\partial z_1}.$$

There is a critical point in its Hamiltonian field. A preliminary study is given in [11]. For more general case, we must study corresponding uniformization problem.

3. Part II. Multiple-characteristic Problems

3.1. Motivation

In this part, we are to modify the Hadamard's procedure that it would be applicable to multiple-characteristic problems. We are to consider only the simplest case when the principal symbol $P_m(x, \xi)$ is factorizable, more precisely, when

$$P_m(x, \xi) = [a(x, \xi)]^2,$$

and to see where lie the main difficulties. For simplicity, we assume $a(x, \xi)$ to be a homogeneous polynoimal in ξ of order m , such that the original operator $P(x, \partial_x)$ is of order $2m$, thus the principal symbol of $P(x, \partial_x)$ is

$$(44) \quad P_{2m}(x, \xi) = [a(x, \xi)]^2.$$

Here, as in Part I, we also assume

$$(45) \quad \text{Hess}_\eta a(y, \eta) \neq 0, \text{ when } a(y, \eta) = 0, \eta \neq 0.$$

We are to use the Hamiltonian system for $a(x, \xi)$ instead of that for $P_{2m}(x, \xi)$, namely, we are to consider

$$(46) \quad \begin{aligned} \frac{dx}{dt} &= a_\xi(x, \xi), \quad x|_{t=0} = y, \\ \frac{d\xi}{dt} &= -a_x(x, \xi), \quad \xi|_{t=0} = \eta, \\ a(y, \eta) &= 0. \end{aligned}$$

Let $k(x, y) = 0$ be the characteristic conoid with respect to $a(x, \xi)$, hence

$$(47) \quad a(x, k_x) = \frac{1}{m-1} k.$$

Then

$$P_{2m}(x, k_x) = \frac{1}{(m-1)^2} k^2.$$

If we take $z_1 = k(x, y)$ as a new independent variable, the equation $P(x, \partial_x)u = 0$ becomes

$$z_1^2 \frac{\partial^m u}{\partial z_1^m} + A\left(x, \frac{\partial}{\partial x}\right) \frac{\partial^{m-1} u}{\partial z_1^{m-1}} + \dots = 0,$$

or

$$\frac{\partial^m u}{\partial z_1^m} + \frac{1}{z_1^2} A\left(x, \frac{\partial}{\partial x}\right) \frac{\partial^{m-1} u}{\partial z_1^{m-1}} + \dots = 0.$$

This equation looks like an ordinary differential equation with irregular singularity at $z_1 = 0$.

For such equations as

$$\frac{dw}{dt} = t^{-(2+r)} A(t)w, \quad r = 0, 1, 2, \dots,$$

with irregular singularity at $t = 0$, the solution can be of the form

$$w = \exp \left[\sum_{i=1}^{r+1} (a_i t^i) \right] t^p f(t)$$

(see e.g. [12], Chap. 5, Theorem 2.1). Thus we are motivated to find a solution of

$$(48) \quad P(x, \partial_x)u = 0$$

in the following form:

$$(49) \quad u = \exp[A(x, y)/k(x, y)] \left(\sum_{h=0}^{\infty} U_h k^{p+h} \right).$$

where $A(y, y) \neq 0$. In contrast to Part I, now we should need eiconal equations for both A and k . The latter is (47) while that for A is to be sought. Thus, we should have “eiconal system”. Transport equations for U_h are also to be constructed.

For brevity in the sequel, we always use sub-indices to denote differentiation with respect to fibre variables such as ξ , η , etc., if applicable.

3.2. Notations

In the following we will use repeatedly Euler's formula for homogeneous functions $P(x, \xi)$ of order l :

$$(50) \quad \begin{aligned} P(x, \xi) &= \frac{1}{l} \sum_{i_1=1}^n \partial_{i_1} P(x, \xi) \xi_{i_1} \\ &= \frac{1}{l(l-1)} \sum_{i_1, i_2=1}^n \partial_{i_1} \partial_{i_2} P(x, \xi) \xi_{i_1} \xi_{i_2} = \dots \\ &= \frac{1}{(l)_r} \sum_{i_1, i_2, \dots, i_r=1}^n \partial_{i_1} \partial_{i_2} \dots \partial_{i_r} P(x, \xi) \xi_{i_1} \xi_{i_2} \dots \xi_{i_r} = \dots \\ &= \frac{1}{l!} \sum_{i_1, \dots, i_l=1}^n \partial_{i_1} \dots \partial_{i_l} P(x, \xi) \xi_{i_1} \dots \xi_{i_l}. \end{aligned}$$

In these formulas, i_1, \dots, i_l run independently from 1 to n . This form of $P(x, \xi)$ is more convenient for our purpose than the usual expression

$$(51) \quad P(x, \xi) = \sum_{|\alpha|=l} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} P(x, \xi) \xi^{\alpha}.$$

Hence, we introduce a new notation

$$(52) \quad D^{\beta} = \partial_{i_1} \dots \partial_{i_l} \sim (\partial_{i_1}, \dots, \partial_{i_l}) \sim (i_1, \dots, i_l),$$

here $\beta = (i_1, i_2, \dots, i_l)$ is an ordered set of l elements from the set $\{1, 2, \dots, n\}$. Elements of β may coincide among themselves. If ∂^{α} can be written as a certain D^{β} , we say D^{β} is a permutation of ∂^{α} , denoted as $\beta \in \sigma(\alpha)$, also $|\beta|$ is defined to be $|\alpha|$, actually $|\beta| = \text{card } \beta$. Corresponding to the addition of multi-index α , such as $\alpha = \alpha' + \alpha''$, hence $\partial^{\alpha} = \partial^{\alpha'} \partial^{\alpha''}$, we have the partition of β into two subsets $\beta = \beta' \cup \beta''$, $\beta' \in \sigma(\alpha')$, $\beta'' \in \sigma(\alpha'')$ and $D^{\beta} = D^{\beta'} D^{\beta''}$. So we also denote $\beta = \beta' + \beta''$. Similarly, we define $\beta' = \beta \setminus \beta''$ to be deleting from β the subset β'' while keep the original order of elements in β' , and also write $\beta' = \beta - \beta''$. Lastly, for multi-index $\alpha = 0$ and $|\alpha| = 0$ means $\alpha = (0, \dots, 0)$, and $\partial^{\alpha} = \partial^0 = \text{id}$. For β , $|\beta| = 0$ means $\beta = \emptyset$ and also $D^{\emptyset} = \text{id}$, hence we also use $\beta = 0$ to denote $\beta = \emptyset$.

DEFINITION 1. When

$$(53) \quad \beta = \beta^1 + \dots + \beta^L = \beta^1 \cup \dots \cup \beta^L,$$

$\beta^i \neq 0$ and elements in β^i keep their original order in β , we say that $(\beta^1, \dots, \beta^L)$ is a partition of β , L the partition number. Denote $\lambda_1 = \text{card} \{ \beta^i, |\beta^i| = l \}$, i.e., the number of l -element-subsets in (53). $[\lambda] = (\lambda_1, \dots, \lambda_{|\beta|})$ is called the partition type of (53).

We have an evident but important

LEMMA 2. For the numbers of subsets of various cardinality, we have

$$(54) \quad 1 \cdot \lambda_1 + 2\lambda_2 + \dots + |\beta| \lambda_{|\beta|} = |\beta|$$

From (54) we see, $\lambda_1 = |\beta|$ is possible ($\lambda_2 = \dots = \lambda_{|\beta|} = 0$), it means β is partitioned into the sum of $|\beta|$ singleton subsets. But $\lambda_1 = |\beta| - 1$ is impossible. In fact, the partition type of β is just a non-negative- integer solution of (54) as a Diophantine equation.

LEMMA 3. The number of partitions (53) (the order of the subsets β^L is irrelevant) with the same partition type $[\lambda]$ is $m! / L! (1!)^{\lambda_1} \dots (m!)^{\lambda_m}$, $|m| = |\beta|$.

Proof. First, construct λ_1 singleton-subsets. There are

$$\binom{m}{1} \binom{m-1}{1} \dots \binom{m-\lambda_1+1}{1} = \frac{(m)_{\lambda_1}}{1!^{\lambda_1}}$$

different ways. Next for the first doubleton-subsets, we take 2 arbitrary elements from the remaining $m - \lambda_1$ elements in $(m - \lambda_1)_2 / 2!$ ways. (The chosen elements should keep their original order in β , thus there shouldn't be $(m - \lambda_1)_2$ ways, which is the permutation number and hence contains twice the same set with elements in opposite orders). The second doubleton-subsets can be chosen in $(m - \lambda_1 - 2)_2 / 2!$ ways and so on. Summing up, the doubleton-subsets can be

chosen in $(m - \lambda_1)_{2\lambda_2}/(2!)^{\lambda_2}$ ways. After the $(m - 1)$ -element subsets are chosen, there remain $m - [1 \cdot \lambda_1 + 2 \cdot \lambda_2 + \dots + (m - 1) \cdot \lambda_{m-1}] = m\lambda_m$ elements for m -element subsets, the number of which is λ_m . So they can be chosen in $(m\lambda_m)_{\lambda_m}/(m!)^{\lambda_m}$ ways. Thus we have

$$\frac{(m)_{\lambda_1}(m - \lambda_1)_{2\lambda_2} \cdots (m - [1 \cdot \lambda_1 + 2 \cdot \lambda_2 + \dots + (m - 1) \cdot \lambda_{m-1}])_{m\lambda_m}}{(1!)^{\lambda_1}(2!)^{\lambda_2} \cdots (m!)^{\lambda_m}}$$

ways to generate a partition of type $[\lambda]$. But the permutation of L subsets in the partition is irrelevant, hence the number of partitions of β with the same partition type $[\lambda]$ is

$$\frac{m!}{L!(1!)^{\lambda_1} \cdots (m!)^{\lambda_m}}.$$

□

3.3. Lemmas on differentiation of composite functions

The computation below relies on differentiation of composite functions. We are to use a modification of the Faa-de Bruno formula [16] (p. 78, Exer. 16). To facilitate the computation, we put the techniques needed as three lemmas.

LEMMA 4. *Let $P(x, \xi)$ be a homogeneous polynomial in ξ of degree l , then*

$$\begin{aligned} (55) \quad P(x, \xi) &= \sum_{|\alpha|=l} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} P(x, \xi) \partial_x^{\alpha} \\ &= \frac{1}{l!} \sum_{1 \leq i \leq n} D_{\xi}^{\beta} P(x, \xi) D_x^{\beta} \end{aligned}$$

Particularly, for $P(x, \xi) = \partial_x^{\alpha}$, we have

$$(56) \quad \partial_x^{\alpha} = \frac{1}{l!} \sum_{\beta \in \sigma(\alpha)} D_x^{\beta}.$$

For D_x^{α} , $\beta \in \sigma(\alpha)$, we have

$$(57) \quad D_x^{\beta} = \partial_x^{\alpha}$$

This lemma is quite trivial. (57) shows that D_x^{β} and ∂_x^{α} are the same when acting on a function; (56) shows that ∂_x^{α} is the average of D_x^{β} 's with $\beta \in \sigma(\alpha)$; while (55) makes the Euler's formula for homogeneous function more symmetric.

Assume $\Phi(k)$ to be a smooth function of $k(x)$. By chain-rule, $\partial_x^{\alpha} \Phi(k) = \{ \text{linear combination of } \Phi^{\gamma}(k) \partial_{i_1} k \cdots \partial_{i_L}^2 k \cdots \}$.

But to write down explicitly the coefficients is much easier for D_x^{β} than for ∂_x^{α} . In fact we have

LEMMA 5. *Let $\Pi(\beta)$ be the set of partitions of β , then*

$$(58) \quad D_x^{\beta} \Phi(k) = \sum_{\Pi(\beta)} \Phi^{(L)}(k) D^{\beta^1} k \cdots D^{\beta^L} k,$$

where $\beta = \beta^1 + \cdots + \beta^L$.

Proof. We proceed by induction. For $|\beta| = 1$, e.g., $D_x^\beta = D_x$, (58) becomes

$$D_{x_1} \Phi(k) = \Phi^{(l)}(k) D_{x_1} k$$

which holds evidently. Suppose that the lemma holds for $|\beta| = m - 1$. For $|\beta| = m$, we may assume that $D_x^\beta = D_1 D_x^{\beta'}$, $|\beta'| = m - 1$. Then the partitions of β may be one of two kinds. The first is $\beta = \beta^1 \cup (\beta^2 \cup \dots \cup \beta^L)$, $\beta^1 = \{D_1\}$ and $\beta^2 \cup \dots \cup \beta^L = \beta'$ is a partition of β' with partition number $L - 1$. The second kind are those not containing the singleton-subset $\{D_1\}$, hence D_1 is contained in another subset, say $D^{\beta^1} : D_x^{\beta^1} = D_1 \cdot D_x^{\beta''}$, $|\beta''| = |\beta'| - 1$. Thus $D_x^\beta = D_1 (D_x^{\beta''} D_x^{\beta^2} \dots D_x^{\beta^L})$ and the factor $D_x^{\beta''} D_x^{\beta^2} \dots D_x^{\beta^L}$ constitutes a partition of $D_x^{\beta'}$ of partition number also L . $\Pi_1(\beta') = \{\beta^2 \cup \dots \cup \beta^L\}$ and $\Pi_2(\beta') = \{\beta'' \cup \beta^2 \cup \dots \cup \beta^L\}$ exhaust all possible partitions of β' . Hence, by induction hypothesis,

$$\begin{aligned} D_x^\beta \Phi(k) &= D_1 \cdot [D_x^{\beta'} \Phi(k)] \\ &= D_1 \sum_{\Pi(\beta')} \Phi^{(L')} D^{\beta'} k \dots D^{\beta^L} k \\ &= \sum_{\Pi_1(\beta')} \Phi^{(L'+1)}(k) D_1 k D^{\beta^2} k \dots D^{\beta^L} k \\ &\quad + \sum_{\Pi_2(\beta')} \Phi^{(L')}(k) (D_1 \cdot D^{\beta'}) k \dots D^{\beta^L} k \\ &= \sum_{\Pi_1(\beta')} \Phi^{(L)}(k) D_1 k D^{\beta^2} k \dots D^{\beta^L} k \\ &\quad + \sum_{\Pi_2(\beta')} \Phi^{(L)}(k) (D_1 \cdot D^{\beta'} k) \dots D^{\beta^L} k \\ &= \sum_{\Pi(\beta)} \Phi^{(L)}(k) D^{\beta^1} k D^{\beta^2} k \dots D^{\beta^L} k. \end{aligned}$$

□

The generalized Leibniz's formula for D also takes a simpler form, namely, we have

LEMMA 6 (LEIBNIZ'S FORMULA).

$$(59) \quad D^\beta u v = \sum_{\beta=\beta^1+\beta^2} D^{\beta^1} u \cdot D^{\beta^2} v.$$

The binomial coefficients disappear hence make the calculation easier. The proof is omitted for its simplicity.

3.4. Computation of $P_{2m}(x, \xi) E_p$ (General plan)

In order to compute $P(x, \partial_x)u$ where u is expressed as (49), we proceed term by term and start from $P(x, \partial_x)E_p U_0$, where $E_p = \exp[A/k] \cdot k^p$. Since $P = P_{2m} + P_{2m-1} + \dots + P_0$ where $P_{2m-j}(x, \xi)$ is a homogeneous polynomial in ξ of degree $2m - j$, we first compute

$P_{2m}(x, \partial_x)E_p U_0$ which by Leibniz's formula can be written as

$$(60) \quad P_{2m}(x, \partial_x)E_p U_0 = U_0 P_{2m}(x, \partial_x)E_p + \sum_{i=1}^n \partial_{\xi_i} P_{2m}(x, \partial_x)E_p \cdot \frac{\partial U_0}{\partial x_i} + \dots$$

Thus our plan is, first, compute $P_{2m}(x, \partial_x)E_p$, then lower order terms $P_{2m-j}(x, \partial_x)E_p$ and further $\sum_{i=1}^n \partial_{\xi_i} P_{2m}(x, \partial_x)E_p \cdot \frac{\partial U_0}{\partial x_i}$ etc. Every term should be expressed as a series in powers of k . It is to be verified, that the lowest order terms are of the order k^{2-4m} and k^{3-4m} for determining $A(x, y)$, p and U_0 . We also denote $P_{2m}(x, \xi)$ by $g(x, \xi)$ and note that the results are applicable also to $P_{2m-j}(x, \xi)$ when replacing g by a lower degree polynomial.

Denote $E_p = \Phi(k_p)$, where $\Phi(\cdot) = \exp(\cdot)$, $k_p = Ak^{-1} + p \log k$. From Lemma 4,

$$g(x, \partial_x)E_p = \frac{1}{(2m)!} \sum_{1 \leq i \leq n} D_{\xi}^{\beta} P(x, \xi) \cdot D_x^{\beta} E_p, \quad |\beta| = 2m,$$

$\beta = (i_1, i_2, \dots, i_{2m})$ and the i 's take arbitrary value from $(1, 2, \dots, n)$. Using Lemma 5, we have

$$D_x^{\beta} E_p = \sum_{\Pi(\beta)} \Phi^{(L)}(k_p) D^{\beta^1} k_p \cdots D^{\beta^L} k_p,$$

and $\Phi^{(L)}(k_p) = \Phi(k_p)$ since $\Phi(\cdot) = \exp(\cdot)$, hence

$$(61) \quad D_x^{\beta} E_p = E_p \sum_{\Pi(\beta)} D^{\beta^1} k_p \cdots D^{\beta^L} k_p.$$

For $D^{\beta^i} k_p$, for instance, for $D^{\beta^1} k_p$

$$D^{\beta^1} k_p = D^{\beta^1}(Ak^{-1}) + p D^{\beta^1} \log k.$$

Apply Leibniz's formula for the first term, we have

$$D^{\beta^1}(Ak^{-1}) = \sum_{\beta^1 = \eta^1 + \gamma^1} D^{\eta^1} A D^{\gamma^1} k^{-1}.$$

Apply Lemma 5 to $D^{\gamma^1} k^{-1}$. γ^1 is partitioned as $\gamma^1 = \delta^{11} + \dots + \delta^{1l_1}$, then

$$\begin{aligned} D^{\gamma^1} k^{-1} &= \sum_{\Pi(\gamma')} \Phi^{(l_1)}(k) D^{\delta^{11}} k \cdots D^{\delta^{1l_1}} k, \quad \Phi(\cdot) = (\cdot)^{-1} \\ &= \sum_{\Pi(\gamma')} (-1)^{l_1} l_1! k^{-(l_1+1)} D^{\delta^{11}} k \cdots D^{\delta^{1l_1}} k. \end{aligned}$$

Hence

$$(62) \quad D^{\beta^1}(Ak^{-1}) = \sum_{\beta^1 = \eta^1 + \gamma^1} \sum_{\Pi(\gamma^1)} D^{\eta^1} A (-1)^{l_1} (l_1!) k^{-(l_1+1)} D^{\delta^{11}} k \cdots D^{\delta^{1l_1}} k.$$

Similarly, apply lemma 5 to compute $D^{\beta^1}(\log k)$, we have

$$(63) \quad D^{\beta^1}(\log k) = \sum_{\Pi(\beta^1)} (-1)^{l_1-1} (l_1-1)! k^{-l_1} D^{\delta^{11}} k \cdots D^{\delta^{1l_1}} k.$$

In order to combine (62) and (63), consider the part of (62) where $\eta^1 = 0$, i. e., $\beta^1 = \gamma^1$, and they can be combined into

$$A \sum_{\Pi(\beta^1)} (-1)^{l_1} l_1! k^{-(l_1+1)} D^{\delta^{11}} k \cdots D^{\delta^{1l_1}} k,$$

and the sum of (62) and (63) gives

$$(64) \quad D^{\beta^1}(k_p) = \sum_{\beta^1 = \eta^1 + \gamma^1} \sum_{\Pi(\gamma^1)} (D^{\eta^1} A)^* (-1)^{l_1} (l_1!) k^{-(l_1+1)} D^{\delta^{11}} k \cdots D^{\delta^{1l_1}} k,$$

$(D^{\eta^1} A)^*$ denotes $D^{\eta^1} A$ when $\eta^1 \neq 0$ and $(D^0 A)^* = (A - pk/l_1)$.

Substuting these results into (61), and denote $l = (l_1, \dots, l_L)$, $|l| = l_1 + \cdots + l_L$, $l! = l_1! \cdots l_L!$, l_i is the partition number of $\eta^i = (\delta^{i1}, \dots, \delta^{il_i})$. Re-numbering $D^{\delta^{11}} k, \dots, D^{\delta^{il_i}} k$, \dots as $D^{\delta^1} k, \dots, D^{\delta^N} k$, N is the total partition number. Then by lemma 4,

$$(65) \quad g(x, \partial_x) E_p = \frac{E_p}{(2m)!} \sum_{\beta} D_{\xi}^{\beta} g(x, \xi) \cdot \sum_{\Pi(\beta)} (D^{\eta^1} A)^* \cdots (D^{\eta^L} A)^* \cdot (-1)^{|l|} l! k^{-(N+L)} \cdot D^{\delta^1} k \cdots D^{\delta^N} k, \quad |\beta| = 2m.$$

$\Pi(\beta)$ under the second \sum denotes β twice partitioned $\beta = \beta^1 + \cdots + \beta^L = \sum_{i=1}^L (\eta^i + \sum_{j=1}^{l_i} \delta^{ij})$. If we write $D^{\beta} = D_{i_1} D_{i_2} \cdots D_{i_{2m}}$, (65) can be further simplified:

$$(66) \quad g(x, \partial_x) E_p = \frac{E_p}{(2m)!} \sum_{1 \leq i \leq n} D_{\xi}^{\beta} g(x, \xi) (-1)^{|l|} l! k^{-(N+L)} \cdot (D^{\eta^1} A)^* \cdots (D^{\eta^L} A)^* \cdot D^{\delta^1} k \cdots D^{\delta^N} k.$$

To organize further computation, we proceed in the descending order of the 2 partition numbers L and N .

In the first partition $\beta = \beta^1 + \cdots + \beta^L$, all $|\beta^i| > 0$ hence

$$(67) \quad L \leq |\beta^1| + \cdots + |\beta^L| = |\beta| = 2m.$$

In the second partition $\beta^i = \eta^i + \gamma^i$, either $\gamma^i = 0$ and $\beta^i = \eta^i$, either $\gamma^i \neq 0$, $\gamma^i = \delta^{i1} + \cdots + \delta^{il_i}$, all $\delta^{ij} \neq 0$, thus $|\delta^i| > 0$ and

$$(68) \quad N \leq |\delta^1| + \cdots + |\delta^N| + |\eta^1| + \cdots + |\eta^L| = \sum_{i=1}^L |\beta^i| = 2m.$$

We should also take into consideration that

$$g(x, \xi) = P_{2m}(x, \xi) = [a(x, \xi)]^2,$$

hence

$$(69) \quad \begin{aligned} D_i g(x, \xi) &= 2a(x, \xi) a_i(x, \xi), \\ D_i D_j g(x, \xi) &= 2a(x, \xi) a_{ij}(x, \xi) + 2a_i(x, \xi) a_j(x, \xi), \\ D_i D_j D_l g(x, \xi) &= 2a(x, \xi) a_{ijl}(x, \xi) \\ &\quad + 2[a_i(x, \xi) a_{jl}(x, \xi) + a_j(x, \xi) a_{il}(x, \xi) + a_l(x, \xi) a_{ij}(x, \xi)]. \end{aligned}$$

Since $a(x, k_x) = \frac{k}{m-1}$, we have

$$\begin{aligned}
 g(x, k_x) &= \frac{k^2}{(m-1)^2}, \\
 D_i g(x, k_x) &= \frac{2k}{m-1} a_i(x, k_i), \\
 D_i D_j g(x, k_x) &= \frac{2k}{m-1} a_{ij}(x, k_i) + 2a_i(x, k_x) a_j(x, k_x), \\
 D_i D_j D_l g(x, k_x) &= \frac{2k}{m-1} a_{ijl}(x, k_i) + 2 \sum a_i(x, k_x) a_{jl}(x, k_x).
 \end{aligned} \tag{70}$$

Thus in the *coefficients* of (66), there may appear powers of k up to k^2 .

3.5. Computation of $P_{2m}(x, \partial_x)E_p$ (cont')

I. First consider those terms in (66) where $L = 2m$. By (67) all $|\beta^i| = 1$, and since $\beta^i = \eta^i + \gamma^i$, then either $|\gamma^i| = 1$ or $|\gamma^i| = 0$. Set $\{1, 2, \dots, 2m\} = I \cup J$, $I \cap J = \emptyset$, such that for $i \in I$, $|\gamma^i| = 1$, $|\eta^i| = 0$ hence $|I| = N$; for $i \in J$ $|\gamma^i| = 0$, $|\eta^i| = 1$. In short, terms in (66) corresponding to $L = 2m$ are

$$\begin{aligned}
 \frac{E_p}{(2m)!} \sum_{|\beta|=2m} D_\xi^\beta g(x, \xi) \sum_{I, J} (-1)^{|I|} l! k^{-(N+2m)} D^{\delta^1} k \dots \\
 \cdot D^{\delta^N} k (D^{\eta^1} A)^* \dots (D^{\eta^L} A)^*.
 \end{aligned}$$

(I-A). $N = 2m$. From (68), this is the extreme case when all $\eta^i = 0$, $|\gamma^i| = 1$, thus $\gamma^i = \delta^i$, and the partition number $l_i = 1$, $|l| = 2m$, $l! = 1$, $D^{\delta^1} k \dots D^{\delta^N} k = k_{i_1} \dots k_{i_{2m}}$. By Euler's formula, these terms can be summed up to give

$$\begin{aligned}
 E_p g(x, k_x) (A - pk)^{2m} k^{-4m} &= \frac{E_p}{(m-1)^2} (A - pk)^{2m} k^{2-4m} \\
 (71) \quad &= E_p \left[\frac{A^{2m}}{(m-1)^2} k^{2-4m} - \frac{2mpA^{2m-1}}{(m-1)^2} k^{3-4m} \right. \\
 &\quad \left. + O(k^{4-4m}) \right].
 \end{aligned}$$

(I-B). $N = 2m - 1$. Delete any element from $\{i_1, i_2, \dots, i_{2m}\}$ and let the remaining correspond to $\delta^1, \dots, \delta^N$. There are $2m$ such chooses and the following is one typical, thus we should multiply the results by $2m$. Let $\{\delta^1, \dots, \delta^N\} = \{i_2, \dots, i_{2m}\}$, then $l_2 = \dots = l_{2m} = 1$, $l! = 1$, $|l| = 2m - 1$; also $(D^{\eta^1} A)^* = \partial_{i_1} A$, $(D^{\eta^2} A)^* = \dots = (D^{\eta^L} A)^* = A - pk$. By Euler's formula,

$$\begin{aligned}
 \frac{1}{(2m)!} \sum_{|\beta|=2m} D_\xi^\beta g(x, \xi) D^{\delta^1} k \dots D^{\delta^N} k (D^{\eta^1} A)^* (-1)^{2m-1} k^{-(4m-1)} \\
 \cdot (A - pk)^{2m-1} = \frac{1}{2m} \sum_{i=1}^n \partial_i g(x, k_x) \partial_i A (-1)^{2m-1} k^{-(4m-1)} (A - pk)^{2m-1} \\
 = -\frac{2k}{2m(m-1)} \sum_{i=1}^n a_i(x, k_x) \partial_i A k^{-(4m-1)} (A - pk)^{2m-1}.
 \end{aligned}$$

Since $k_x = \tau \xi$ and from (46) $a_i(x, \xi) = \frac{dx_i}{dt}$, the final results for terms corresponding to $L = 2m, N = 2m - 1$ is

$$(72) \quad E_p \left[-\frac{2A^{2m-1}}{m-1} t \frac{dA}{dt} k^{2-4m} + \frac{2A^{2m-2}}{m-1} t \frac{dA}{dt} (2m-1) p k^{3-4m} + O(k^{4-4m}) \right].$$

(I-C). $N = 2m - 2$. Delete for instance $\{i_1, i_2\}$ from $\{i_1, i_2, \dots, i_{2m}\}$ and let the remaining be $\delta^1, \dots, \delta^N$ ($\frac{1}{2}(2m)_2$ such chooses), then $i_3 = \dots = i_{2m} = 1$, $|l| = 2m - 2$, $l! = 1$, $(D^{\eta_1} A)^* = \partial_{i_1} A$, $(D^{\eta_2} A)^* = \partial_{i_2} A$, all other $(D^\eta A)^* = A - pk$. Use also Euler's formula and multiply the results by $\frac{1}{2}(2m)_2$, the final result is

$$(73) \quad \begin{aligned} & \frac{1}{2} E_p \sum_{i_1, i_2=1}^n g_{i_1 i_2}(x, k_x) \partial_{i_1} A \partial_{i_2} A k^{2-4m} (A - pk)^{2m-2} \\ &= E_p \left[\sum_{i_1, i_2=1}^n \frac{1}{m-1} a_{i_1, i_2}(x, k_x) \partial_{i_1} A \partial_{i_2} A \cdot A^{2m-2} k^{3-4m} \right. \\ & \quad + \left(t \frac{dA}{dt} \right)^2 A^{2m-2} k^{2-4m} - p \left(t \frac{dA}{dt} \right)^2 (2m-2) A^{2m-3} k^{3-4m} \\ & \quad \left. + O(k^{4-4m}) \right]. \end{aligned}$$

(I-D). $N = 2m - 3$. A typical example is deleting $\{i_1, i_2, i_3\}$ from $\{i_1, i_2, \dots, i_{2m}\}$ and let $\{\delta^1, \dots, \delta^N\} = \{i_4, i_5, \dots, i_{2m}\}$. A factor $\frac{(2m)_3}{3!}$ should be added to the result and the final result obtained would be

$$(74) \quad \begin{aligned} & \frac{(2m)_3}{3!} E_p \left\{ \frac{1}{(2m)_3} 2 \sum_{i_1, i_2, i_3} [a_{i_1}(x, k_x) a_{i_2 i_3}(x, k_x) + a_{i_2}(x, k_x) a_{i_1 i_3}(x, k_x) \right. \\ & \quad + a_{i_3}(x, k_x) a_{i_1 i_2}(x, k_x)] \partial_{i_1} A \partial_{i_2} A \partial_{i_3} A (-1)^{2m-3} A^{2m-3} k^{3-4m} \\ & \quad \left. + O(k^{4-4m}) \right\} \\ &= -E_p \left[\sum_{i_1, i_2} a_{i_1, i_2}(x, k_x) \partial_{i_1} A \partial_{i_2} A \cdot t \frac{dA}{dt} A^{2m-3} k^{3-4m} \right. \\ & \quad \left. + O(k^{4-4m}) \right], \end{aligned}$$

Our computation would stop here. Since for $N \leq 2m - 4$, $k^{-(N+L)} = O(k^{4-4m})$ and hence omitted.

Summing up, when $L = 2m$, we need to consider the cases $N = 2m, 2m - 1, 2m - 2$ and $2m - 3$ to obtain (71), (72), (73) and (74), giving an asymptotic expansion of $P_{2m}(x, \partial_x) E_p$ up

to $O(k^{4-4m})$ as

$$\begin{aligned}
 (75) \quad & E_p k^{2-4m} A^{2m-2} \left(t \frac{dA}{dt} - \frac{A}{m-1} \right)^2 \\
 & - E_p k^{3-4m} A^{2m-3} \left[(2m-2)p \left(t \frac{dA}{dt} - \frac{A}{m-1} \right)^2 \right. \\
 & - \frac{2p}{m-1} \left(t \frac{dA}{dt} - \frac{A}{m-1} \right) A \\
 & + \left(\sum_{i_1, i_2} a_{i_1, i_2}(x, k_x) \partial_{i_1} A \partial_{i_2} A \right) \left(t \frac{dA}{dt} - \frac{A}{m-1} \right) \left. \right] \\
 & + E_p O(k^{4-4m}).
 \end{aligned}$$

II. $L = 2m - 1$. From (67),

$$|\beta^1| + \dots + |\beta^{2m-1}| = 2m,$$

The only solution is: one of β^i , say $|\beta^{2m-1}| = 2$ and other $|\beta^i| = 1$. A typical choice for the β^i 's is $\beta^j = i_j$, $j = 1, \dots, 2m-2$, $\beta^{2m-1} = (i_{2m-1}, i_{2m})$. There are $\frac{1}{2}(2m)_2$ similar choices, so we must add a factor $\frac{1}{2}(2m)_2$ to the results of all particular case.

(II-A). $N = 2m$. This case can be attained only when for $\beta^i = \eta^i + \gamma^i$, set $\eta^i = 0$, and $\gamma^i = \delta^i, \dots, \gamma^{2m-2} = \delta^{2m-2}$. Hence $l = (l_1, \dots, l_L) = (1, \dots, 1, 2)$, $|l| = 2m$, $l! = 2$. The terms in (66) arising from this pair of particular (L, N) are

$$\begin{aligned}
 (76) \quad & \frac{(2m)_2}{2} E_p \frac{1}{(2m)!} \sum_{1 \leq i \leq n} D_\xi^\beta g(x, \xi) \partial_{i_1} k \dots \partial_{i_{2m}} k \cdot (-1)^{2m} \cdot 2 \cdot k^{-(4m-1)} \cdot \\
 & \cdot (A - pk)^{2m} = (2m)_2 E_p g(x, k_x) k^{-(4m-1)} (A - pk)^{2m} \\
 & = E_p \left[\frac{(2m)_2}{(m-1)^2} k^{3-4m} A^{2m} + O(k^{4-4m}) \right].
 \end{aligned}$$

(II-B). $N = 2m - 1$. Now, we have 2 possibilities for the first partition, one is the same as above, $l = (1, \dots, 1, 2)$, $|l| = 2m - 1$, $l! = 2$. In this case, roughly speaking, one of D^δ (say ∂_j) should be acted on A , giving $\partial_j A$. Since there are $2m$ choices for D^δ , we obtain corresponding terms in (66) as

$$\begin{aligned}
 (77) \quad & -\frac{(2m)_2}{2} \cdot 2m \cdot E_p \frac{1}{(2m)!} \sum_{1 \leq i \leq n} D_\xi^\beta g(x, \xi) \partial_{i_1} k \dots \widehat{\partial_j k} \dots \\
 & \cdot \partial_{i_{2m}} k \cdot \partial_j A \cdot 2 \cdot k^{2-4m} (A - pk)^{2m-1} \\
 & = -(2m)_2 E_p \sum_{j=1}^n \partial_{\xi_j} g(x, k_x) \frac{\partial A}{\partial x_j} k^{2-4m} (A - pk)^{2m-1} \\
 & = -\frac{2(2m)_2}{m-1} E_p \left[t \frac{dA}{dt} k^{3-4m} A^{2m-1} + O(k^{4-4m}) \right]
 \end{aligned}$$

(“ \wedge ” means wanting).

The second possibility is that $l = (1, 1, \dots, 1)$ ($2m - 1$ entries), $|l| = 2m - 1$, $l! = 1$. The first $2m - 2$ D^{β^i} (in $D^{\beta^i} k$) all act on k^{-1} and the last D^{β^L} in $D^{\beta^L} k$ also acts on k^{-1} giving a

second order derivature of k . Thus we have

$$\begin{aligned}
& \frac{(2m)_2}{2} E_p \frac{1}{(2m)!} \sum_{1 \leq i \leq n} D_\xi^\beta g(x, \xi) \partial_{i_1} \cdots \partial_{i_{2m-2}} k \partial_{i_{2m-1} i_{2m}}^2 k \cdot (-1)^{2m-1} \cdot 1 \cdot \\
& \quad \cdot (A - pk)^{2m-1} k^{2m-4} \\
& = -\frac{E_p}{2} \sum_{i,j} g_{ij}(x, k_x) \partial_{ij}^2 k \cdot A^{2m-1} k^{2-4m} \\
& \quad + \frac{E_p}{2} \cdot p \sum_{i,j} g_{ij}(x, k_x) \partial_{ij}^2 k \cdot A^{2m-2} k^{3-4m} + E_p O(k^{4-4m}).
\end{aligned}$$

Since it holds $g_{ij}(x, k_x) = 2a(x, k_x)a_{ij}(x, k_x) + 2a_i(x, k_x)a_j(x, k_x) = \frac{2k}{m-1}a_{ij}(x, k_x) + 2a_i(x, k_x)a_j(x, k_x)$, we have

$$\begin{aligned}
& -E_p \sum_{i,j} a_i(x, k_x) a_j(x, k_x) \partial_{ij}^2 k A^{2m-1} k^{2-4m} \\
& - E_p \cdot \frac{1}{m-1} \sum_{i,j} a_{ij}(x, k_x) \partial_{ij}^2 k A^{2m-1} k^{3-4m} \\
& + p E_p \sum_{i,j} a_i(x, k_x) a_j(x, k_x) \partial_{ij}^2 k A^{2m-2} k^{3-4m} + E_p O(k^{4-4m}).
\end{aligned}$$

But (22) in part I shows that $\sum_{i,j} a_{ij}(x, k_x) \partial_{ij}^2 k = \delta_{il}(1 + O(t))$, and Euler's formula gives $a_j(x, k_x) = \frac{1}{m-1} \sum_{l=1}^n a_{jl}(x, k_x) \partial_{jl} k$, hence the result above can be simplified further to give

$$\begin{aligned}
& -\frac{E_p}{m-1} \sum_{i,j,l} a_i(x, k_x) a_{jl}(x, k_x) \partial_{ij}^2 k \partial_{jl} k A^{2m-1} k^{2-4m} \\
& - \frac{E_p}{m-1} [n + O(t)] A^{2m-1} k^{3-4m} \\
& + \frac{p E_p}{m-1} \cdot \sum_{i,j,l} a_i(x, k_x) a_{jl}(x, k_x) \partial_{ij}^2 k \partial_{jl} k \cdot A^{2m-2} k^{3-4m} \\
& + E_p O(k^{4-4m}) \\
(78) \quad & = -\frac{E_p}{m-1} \sum_{l} a_l(x, k_x) \partial_l k (1 + O(t)) A^{2m-1} k^{2-2m} \\
& - \frac{E_p}{m-1} [n + O(t)] A^{2m-1} k^{3-4m} \\
& + \frac{p E_p}{m-1} \sum_{l} a_l(x, k_x) \partial_l k (1 + O(t)) A^{2m-2} k^{3-4m} + O(k^{4-4m}) \\
& = -\frac{E_p}{m-1} \left(\frac{m}{m-1} + n \right) (1 + O(t)) A^{2m-1} k^{3-4m} + E_p O(k^{4-4m}).
\end{aligned}$$

(II-C). $N = 2m - 2$. There are also 2 possiblities for the partition $\beta = \beta^1 + \cdots + \beta^L$. The first is the same as above, i.e., $\frac{(2m)_2}{2}$ choices of the type $l = (1, 1, \dots, 2)$ ($2m - 1$ entries), $|l| = 2m$, $l! = 2$. In order that $N = 2m - 2$, among the differentiations D^β (total order $2m$), 2 must be

chosen acting on A giving terms containing $\partial_{i_1} A \partial_{i_2} A$ (again $\frac{(2m)_2}{2}$ choices for i and j among $\{i_1, i_2, \dots, i_{2m}\}$). They give

$$\begin{aligned}
 & \frac{(2m)_2}{2} \cdot \frac{(2m)_2}{2} E_p \frac{1}{(2m)!} \sum_{1 \leq i \leq n} D_\xi^\beta g(x, \xi) \partial_{i_1} k \cdots \widehat{\partial}_{j_1} k \cdots \widehat{\partial}_{j_2} k \cdots \\
 & \quad \partial_{i_{2m}} k \partial_{j_1} A \partial_{j_2} A \cdot (-1)^{2m} \cdot 2 \cdot k^{3-4m} (A - pk)^{2m-2} \\
 (79) \quad & = \frac{(2m)_2}{2} E_p \sum g_{ij}(x, k_x) \partial_i A \partial_j A \cdot A^{2m-2} k^{3-4m} + E_p O(k^{4-4m}) \\
 & = (2m)_2 E_p \left(t \frac{dA}{dt} \right)^2 A^{2m-2} k^{3-4m} + E_p O(k^{4-4m}).
 \end{aligned}$$

The other possibility is again $l = (1, 1, \dots, 1)$ ($2m - 1$ entries), $|l| = 2m - 1$, $l! = 1$. Thus one and only one of $|\beta^i|$ should be 2. There are $\frac{(2m)_2}{2}$ choices for this subset, $\beta^L = (i_{2m-1}, i_{2m})$ is a typical one. Since all entries in l are 1, there would be no further partition for D^{β^L} , either it acts just on A , giving terms containing $\partial^2 A$, all other D^β act on k^{-1} giving $N = 2m - 2$ ∂k 's; either D^{β^L} acts on k^{-1} giving $\partial^2 k$ and among the remaining D^{β^i} , choose one and only one acting on A ($2m - 2$ such choices) and the remaining $2m - 3$ act on k^{-1} giving $2m - 3$ ∂k 's. Thus, there are totally $1 + (2m - 3) = 2m - 2$ derivatives on k agreeing with $N = 2m - 2$. Hence, these terms are

$$\begin{aligned}
 & \frac{(2m)_2}{2} E_p \frac{1}{(2m)!} \sum_{1 \leq i \leq n} D_\xi^\beta g(x, \xi) \partial_{i_1} k \cdots \partial_{i_{2m-2}} k \partial_{2m-1, 2m}^2 A \cdot \\
 & \quad (-1)^{2m-1} \cdot 1 \cdot (A - pk)^{2m-2} k^{3-4m} \\
 & \quad + \frac{(2m)_2}{2} (2m - 2) E_p \frac{1}{(2m)!} \sum_{1 \leq i \leq n} D_\xi^\beta g(x, \xi) \partial_{i_1} A \partial_{i_2} k \cdots \partial_{i_{2m-2}} k \cdot \\
 & \quad \cdot \partial_{2m-1, 2m}^2 k \cdot (-1)^{2m-1} \cdot 1 \cdot (A - pk)^{2m-2} k^{3-4m} \\
 & = -E_p \sum_{i, j} a_i(x, k_x) a_j(x, k_x) \partial_{ij}^2 A \cdot A^{2m-2} k^{3-4m} \\
 & \quad - \frac{1}{2} E_p \sum_{i, j, l} g_{i, j, l}(x, k_x) \partial_i A \partial_{jl}^2 k A^{2m-2} k^{3-4m} + E_p O(k^{4-4m}).
 \end{aligned}$$

But

$$\begin{aligned}
g_{i,j,l}(x, k_x) &= 2a(x, k_x)a_{jl}(x, k_x) + 2[a_i(x, k_x)a_{jl}(x, k_x) \\
&\quad + a_j(x, k_x)a_{il}(x, k_x) + a_l(x, k_x)a_{ij}(x, k_x)] \\
&= \frac{2k}{m-1}a_{ijk}(x, k_x) + 2[a_i(x, k_x)a_{jl}(x, k_x) \\
&\quad + a_j(x, k_x)a_{il}(x, k_x) + a_l(x, k_x)a_{ij}(x, k_x)], \\
\sum_{j,l} a_{jl}(x, k_x) \partial_{jl}^2 k &= n(1 + O(t)), \\
\sum_j a_j(x, k_x) \partial_{jl}^2 k &= \frac{1}{m-1} \sum_{i,l} a_{ij}(x, k_x) \partial_i k \partial_{jl}^2 k \\
&= \frac{1}{m-1} \sum_i \delta_{il} \partial_i k (1 + O(t)) \\
&= \frac{k_l}{m-1} (1 + O(t)), \\
\sum_{j,l} a_j(x, k_x) a_{il}(x, k_x) \partial_{jl}^2 k &= \frac{1}{m-1} \sum_l a_{il}(x, k_x) k_l (1 + O(t)) \\
&= a_i(x, k_x) (1 + O(t)).
\end{aligned}$$

Hence,

$$\begin{aligned}
&-E_p \sum_{i,j} a_i(x, k_x) a_j(x, k_x) \partial_{ij}^2 A \cdot A^{2m-2} k^{3-4m} \\
&- E_p \left[\sum_{i,j,l} a_i(x, k_x) a_{jl}(x, k_x) \partial_i A \partial_{jl}^2 k \right. \\
&\quad \left. + \sum_{i,j,l} a_j(x, k_x) a_{il}(x, k_x) \partial_i A \partial_{jl}^2 k \right. \\
&\quad \left. + \sum_{i,j,l} a_l(x, k_x) a_{ij}(x, k_x) \partial_i A \partial_{jl}^2 k \right] A^{2m-2} k^{3-4m} + E_p O(k^{4-4m}) \\
(80) \quad &= -E_p \sum_{i,j} a_i(x, k_x) a_j(x, k_x) \partial_{ij}^2 A \cdot A^{2m-2} k^{3-4m} \\
&- E_p \left[n \left(t \frac{dA}{dt} \right) (1 + O(t)) \right. \\
&\quad \left. + \frac{2}{m-1} \sum_{i,l} a_{il}(x, k_x) \partial_l k \partial_i A \right] A^{2m-2} k^{3-4m} + E_p O(k^{4-4m}) \\
&= -E_p \sum_{i,j} a_i(x, k_x) a_j(x, k_x) \partial_{ij}^2 A \cdot A^{2m-2} k^{3-4m} \\
&- E_p (n+2) t \frac{dA}{dt} (1 + O(t)) A^{2m-2} k^{3-4m} + E_p O(k^{4-4m}).
\end{aligned}$$

For $L = 2n - 1$, $N \leq 2m - 3$, $k^{-(L+N)} = O(k^{4-4m})$, hence omitted.

III. $L = 2m - 2$, and $L < 2m - 2$. (A) gives now

$$|\beta^1| + \cdots + |\beta^{2m-2}| = 2m.$$

For $N = 2m$, all $D^{\beta^i} k$ must be partitioned into ∂ acting on k^{-1} , hence there must appear $\partial_{i_1} k \cdots \partial_{i_{2m}} k$, joining them to $D_\xi^\beta g(x, \xi)$ gives $g(x, k_x) = O(k^2)$, hence all these terms are $O(k^{4-4m})$ and omitted. For $N = 2m - 1$, there would appear $2m - 1$ ∂k 's, and Euler's formula associates them to $D_\xi^\beta g(x, \xi)$ to give $\partial_i g(x, k_x) = 2a(x, k_x) a_i(x, k_x) = O(k)$, hence the total order of such terms are again $O(k^{4-4m})$. $N \leq 2m - 2$ can also be neglected.

$L \leq 2m - 3$ gives only terms $O(k^{4-4m})$ hence neglected.

Our computation of $g(x, \partial_x) E_p$ terminates here. We tabulate our results as follows:

I. $L = 2m$

(I-A) $L = 2m, N = 2m$, (71).

(I-B) $L = 2m, N = 2m - 1$, (72).

(I-C) $L = 2m, N = 2m - 2$, (73).

(I-D) $L = 2m, N = 2m - 3$, (74).

$L = 2m, N \leq 2m - 4$ give terms of order $O(k^{4-4m})$, hence negligible.

Summing up, we have (75) for the case $L = 2m$.

II. $L = 2m - 1$

(II-A) $L = 2m - 1, N = 2m$, (76).

(II-B) $L = 2m - 1, N = 2m - 1$, (77).

After simplifying terms containing the factors $\partial_{ij}^2 k$, we have (78).

(II-C) $L = 2m - 1, N = 2m - 2$, (79).

Simplifying terms containing $\partial_{ij}^2 k$, but reserving $\partial_{ij}^2 A$, we have (80).

$L = 2m - 1, N \leq 2m - 3$, negligible, since they are terms of the order $O(k^{4-4m})$.

III. $L \leq 2m - 2$ gives terms of the order $O(k^{4-4m})$, hence negligible.

These results are summed as the

PROPOSITION 1. *The terms of order k^{2-4m} and k^{3-4m} in the asymptotic expansion of*

$P_{2m}(x, \partial_x)E_p$ are the following:

$$\begin{aligned}
 (81) \quad & k^{2-4m} : E_p A^{2m-2} \left(t \frac{dA}{dt} - \frac{A}{m-1} \right)^2. \\
 & k^{3-4m} : -E_p p \left[2(m-1)A^{2m-3} \left(t \frac{dA}{dt} - \frac{A}{m-1} \right)^2 \right. \\
 & \quad - \frac{2}{m-1} A^{2m-2} \left(t \frac{dA}{dt} - \frac{A}{m-1} \right) \\
 & \quad \left. + E_p \left[(2m)_2 A^{2m-2} \left(t \frac{dA}{dt} - \frac{A}{m-1} \right)^2 \right. \right. \\
 & \quad - 2A^{2m-3} \sum_{i,j} a_{ij}(x, k_x) \partial_i A \partial_j A \left(t \frac{dA}{dt} - \frac{A}{m-1} \right) \\
 & \quad - \frac{n(m-1)+m}{(m-1)^2} (1 + O(t)) A^{2m-1} \\
 & \quad - (n+2)t \frac{dA}{dt} \cdot A^{2m-2} (1 + O(t)) \\
 & \quad \left. \left. - \sum_{i,j} a_i(x, k_x) a_j(x, k_x) \partial_{ij}^2 A \cdot A^{2m-2} \right] \right].
 \end{aligned}$$

3.6. Computation of $P_{2m-j}(x, \partial_x)E_p$

The symbol $P(x, \xi)$ can be written as the sum of homogeneous polynomials in ξ of degree $2m-j$:

$$(82) \quad P(x, \xi) = P_{2m}(x, \xi) + P_{2m-1}(x, \xi) + \dots.$$

$P_{2m}(x, \partial_x)E_p$ has been discussed in detail, now we proceed to compute

$$\begin{aligned}
 (83) \quad & P_{2m-j}(x, \partial_x)E_p U_0 = P_{2m-j}(x, \partial_x)E_p \cdot U_0 \\
 & + \sum_{i=1}^n \frac{\partial}{\partial \xi_i} P_{2m-j}(x, \partial_x)E_p \cdot \frac{\partial U_0}{\partial x_i} + \dots, \quad (j \geq 1).
 \end{aligned}$$

First consider $P_{2m-j}(x, \partial_x)E_p$, it is almost the same as above, only replacing $g(x, \xi)$ by $P_{2m-j}(x, \xi)$, thus for (66) we have now

$$\begin{aligned}
 (84) \quad & P_{2m-j}(x, \partial_x)E_p = \frac{E_p}{(2m-j)!} \sum_{1 \leq i \leq n} D_{\xi}^{\beta} P_{2m-j}(x, \xi) (-1)^{|l|} l! k^{-(N+L)} \cdot \\
 & (D^{\eta^1} A)^* \dots (D^{\eta^L} A)^* D^{\delta^1} k \dots D^{\delta^N} k,
 \end{aligned}$$

for (67) and (68), we have now

$$(85) \quad L \leq |\beta^1| + \dots + |\beta^L| = 2m-j, \quad (j \geq 1).$$

$$(86) \quad N \leq |\delta^1| + \dots + |\delta^N| + |\beta^1 - \gamma^1| + \dots + |\beta^L - \gamma^L| = 2m-j, \quad (j \geq 1).$$

Thus, the lowest term in (84) is

$$k^{2j-4m}. \quad (j \geq 1)$$

Similarly, $\partial_{\xi_i} P_{2m-j}(x, \xi)$ is a polynomial in ξ of degree $2m-j-1$, hence

$$(87) \quad \frac{\partial}{\partial \xi_i} P_{2m-j}(x, \partial_x) E_p = O(k^{2+2j-4m}), \quad (j \geq 1).$$

Since we need only terms of order k^{2-4m} and k^{3-4m} , so from the contributions of lower order operators $P_{2m-j}(x, \partial_x) E_p U_0$, we need to consider only $P_{2m-1}(x, \partial) E_p \cdot U_0$.

From (84), we have

$$(88) \quad \begin{aligned} P_{2m-1}(x, \partial_x) E_p &= \frac{E_p}{(2m-1)!} \sum_{1 \leq i \leq n} D_{\xi}^{\beta} P_{2m-1}(x, \xi) (-1)^{|l|} l! k^{-(N+L)} \cdot \\ &\quad \cdot (D^{\eta^1} A)^* \cdots (D^{\eta^L} A)^* D^{\delta^1} k \cdots D^{\delta^N} k. \end{aligned}$$

Consider the case $L = 2m-1$. For $N = 2m-1$, we must have $l = (1, 1, \dots, 1)$ ($L = 2m-1$ entries), $|l| = 2m-1$, $l! = 1$, $\delta^1 = \cdots = \delta^N = 1$, corresponding terms in (88) are

$$(89) \quad \begin{aligned} E_p \cdot P_{2m-1}(x, k_x) (-1)^{2m-1} (A - pk)^{2m-1} k^{2-4m} \\ = E_p [-P_{2m-1}(x, k_x) A^{2m-1} k^{2-4m} \\ + (2m-1)p P_{2m-1}(x, k_x) A^{2m-2} k^{3-4m} + O(k^{4-4m})]. \end{aligned}$$

Next, let $N = 2m-2$. A typical case is to choose $(\delta^1, \dots, \delta^{2m-2}) = (i_2, \dots, i_{2m-1})$, thus $(D^{\eta^1} A)^* = \partial_{i_1} A$, $(D^{\eta^2} A)^* = \cdots = (D^{\eta^L} A)^* = A - pk$. There are $2m-1$ such choices, so we should multiply the results obtained by $2m-1$ and obtain terms in (88) corresponding to $L = 2m-1$, $N = 2m-2$. They are

$$(90) \quad \begin{aligned} E_p \sum_{i=1}^n \partial_i P_{2m-1}(x, k_x) \partial_i A k^{3-4m} (A - pk)^{2m-2} \\ = E_p \sum_{i=1}^n \partial_i P_{2m-1}(x, k_x) \partial_i A \cdot A^{m-2} k^{3-4m} + E_p O(k^{4-4m}). \end{aligned}$$

When $N \leq 2m-3$, $k^{-(N+L)} = O(k^{4-4m})$ and is negligible.

Consider the case $L = 2m-2$. From (67), we have

$$2m-2 \leq |\beta^1| + \cdots + |\beta^{2m-2}| = 2m-1.$$

There is one and only one $|\beta^i| = 2$, the others: $|\beta^i| = 1$. We may choose $\beta^{2m-2} = (i_{2m-2}, i_{2m-1})$, $\beta^1 = i_1, \dots, \beta^{2m-3} = i_{2m-3}$. Since there are $\frac{1}{2}(2m-1)_2$ ways of choosing $|\beta^i| = 2$, the results should be multiplied by $\frac{1}{2}(2m-1)_2$.

We need only consider the case $N = 2m-1$. This would require that both differentiations in $D^{\beta^{2m-2}} k$ acting on k and giving $\partial_{i_{2m-2}} k \partial_{i_{2m-1}} k$. Combining the consideration of L and N we obtain

$$(91) \quad \begin{aligned} \frac{1}{2}(2m-1)_2 E_p \frac{1}{(2m-1)!} \sum_{1 \leq i \leq n} D_{\xi}^{\beta} P_{2m-1}(x, \xi) \partial_{i_1} k \cdots \partial_{i_{2m-1}} k \cdot \\ \cdot k^{3-4m} (-1)^{2m-1} (A - pk)^{2m-1} \\ = -\frac{(2m-1)_2}{2} E_p P_{2m-1}(x, k_x) A^{2m-1} k^{3-4m} + E_p O(k^{4-4m}). \end{aligned}$$

All other terms are neglected. The final result is

$$\begin{aligned}
 P_{2m-1}(x, \partial_x)E_p &= E_p \left[-P_{2m-1}(x, k_x)A^{2m-1}k^{2-4m} \right. \\
 &\quad + (2m-1)pP_{2m-1}(x, k_x)A^{2m-2}k^{3-4m} \\
 (92) \quad &\quad + \sum_{i=1}^n \partial_i P_{2m-1}(x, k_x) \frac{\partial A}{\partial x_i} \cdot A^{2m-2}k^{3-4m} \\
 &\quad \left. + O(k^{4-4m}) \right].
 \end{aligned}$$

3.7. Computation of $\sum_{i=1}^n \partial_i P_{2m}(x, \partial_x)E^p \cdot \frac{\partial U_0}{\partial x_i}$

Here we should find the first term in the asymptotic expansion of

$$\partial_i P_{2m}(x, \partial_x)E^p = Q_i(x, \partial_x)E^p,$$

$Q_i(x, \xi)$ is a homogeneous polynomial of ξ of order $2m-1$. Hence (92) is applicable, with $Q_i(x, \xi)$ replacing $P_{2m-1}(x, \xi)$. But

$$Q_i(x, \xi) = \partial_i P_{2m}(x, \xi) = \partial_i a^2(x, \xi) = 2a(x, \xi)a_i(x, \xi),$$

hence

$$Q_i(x, k_x) = \frac{2k}{m-1}a_i(x, k_x).$$

Substitute this into (92) with Q_i replacing P_{2m-1} , we have

$$\begin{aligned}
 \sum_{i=1}^n \partial_i P_{2m}(x, \xi)E^p \cdot \frac{\partial U_0}{\partial x_i} &= E^p \left[-\frac{2}{m-1}A^{2m-1}k^{3-4m} \sum_i a_i(x, k_x) \frac{\partial U_0}{\partial x_i} \right. \\
 &\quad + 2A^{2m-2}k^{3-4m}t \frac{dA}{dt} \sum_i A_i(x, k_x) \frac{\partial U_0}{\partial x_i} \\
 (93) \quad &\quad \left. + O(k^{4-4m}) \right] \\
 &= E^p \left[2 \left(t \frac{dA}{dt} - \frac{A}{m-1} \right) A^{2m-2}k^{3-4m}t \frac{dU_0}{dt} \right. \\
 &\quad \left. + O(k^{4-4m}) \right].
 \end{aligned}$$

After this long computation, combining (81), (92) and (93), we have

PROPOSITION 2. *The first terms in the asymptotic expansion of $P(x, \partial_x)E_p U_0$ is*

$$\begin{aligned}
 (94) \quad P(x, \partial_x)E_p U_0 &= E^p [A^{2m-2}\mathcal{E}(A)U_0 k^{2-4m} + A^{2m-3}\mathcal{J}(A, U_0)k^{3-4m} \\
 &\quad + \mathcal{R}(A, U_0)].
 \end{aligned}$$

where

$$\begin{aligned}
 (95) \quad \mathcal{E}(A) &= \left(t \frac{dA}{dt} - \frac{A}{m-1} \right)^2 - A P_{2m-1}(x, k_x), \\
 \mathcal{J}(A, U_0) &= A^{2m-3} [p\mathcal{J}_1(A, U_0) + \mathcal{J}_2(A, U_0)],
 \end{aligned}$$

$$(96) \quad \mathcal{J}_1(A, U_0) = - \left[2(m-1) \left(t \frac{dA}{dt} - \frac{A}{m-1} \right)^2 - \frac{2}{m-1} A \left(t \frac{dA}{dt} - \frac{A}{m-1} \right) \right. \\ \left. - (2m-1) A P_{2m-1}(x, k_x) \right] U_0 ,$$

$$(97) \quad \mathcal{J}_2(A, U_0) = 2A \left(t \frac{dA}{dt} - \frac{A}{m-1} \right) t \frac{dU_0}{dt} + \left[(2m)_2 A \left(t \frac{dA}{dt} - \frac{A}{m-1} \right)^2 \right. \\ \left. - \frac{n(m-1)+m}{(m-1)^2} (1+O(t)) A^2 - (n+2) t \frac{dA}{dt} \cdot A (1+O(t)) \right. \\ \left. - \sum_{i,j} a_{ij}(x, k_x) \partial_i A \partial_j A \left(t \frac{dA}{dt} - \frac{A}{m-1} \right) \right. \\ \left. - A \sum_{i,j} a_{ij}(x, k_x) a_j(x, k_x) \partial_{ij}^2 A \right. \\ \left. + A \sum_i \partial_i P_{2m-1}(x, k_x) \frac{\partial A}{\partial x_i} \right] U_0 ,$$

$\mathcal{R}(A, U_0)$ is the remainder which is of order $O(k^{4-4m})$.

3.8. Eiconal system for A and k

As we mentioned earlier, we should have two equations to determinate k and A . The first is Leray's result

$$a(x, k_x) = \frac{k}{m-1} .$$

$\mathcal{E}(A) = 0$ gives the eiconal equation for A :

$$(98) \quad \left(t \frac{dA}{dt} - \frac{A}{m-1} \right)^2 = P_{2m-1}(x, k_x) A .$$

These two equations constitute the eiconal system.

Let $t = \tau^{m-1}$ and $A = \tau B$, we have

$$\tau^3 \left(\frac{dB}{d\tau} \right)^2 = (m-1)^2 P_{2m-1}(x, k_x) B .$$

Since $k_x = \tau \xi$, using Taylor's formula, we have

$$\left(\frac{dB}{d\tau} \right)^2 = (m-1)^2 \tau^{2(m-2)} [P_{2m-1}(y, \eta) + O(\tau)] B .$$

Integrating this ODE along the Hamiltonian orbit for $a(x, \xi)$, we have

$$2\sqrt{B} = (m-1) \int_0^\tau \tau^{m-2} \sqrt{P_{2m-1}(y, \eta) + O(\tau)} d\tau + C .$$

Let $C = 0$ and assume that

$$(99) \quad P_{2m-1}(y, \eta) \neq 0 ,$$

We have

$$(100) \quad \begin{aligned} B &= \frac{1}{4} \tau^{2m-2} P_{2m-1}(y, \eta) (1+O(\tau)) , \\ A &= \frac{1}{4} \tau^{2m-1} P_{2m-1}(y, \eta) (1+O(\tau)) . \end{aligned}$$

We are to obtain transport equation for U_0 from $\mathcal{J}(A, U_0) = 0$. In order to simplify it, substitute (100) into it and we have

$$(101) \quad \begin{aligned} \mathcal{J}_1(A, U_0) &= -\left[2(m-1)P_{2m-1}(x, k_x)A - \frac{2}{m-1}P_{2m-1}^{1/2}(x, k_x)A^{3/2}\right. \\ &\quad \left.- (2m-1)P_{2m-1}(x, k_x)A\right]U_0 \\ &= \frac{m}{4(m-1)}\tau^{4m-2}P_{2m-1}^2(y, \eta)(1 + O(t))U_0, \end{aligned}$$

$$(102) \quad \begin{aligned} \mathcal{J}_2(A, U_0) &= \frac{1}{4(m-1)}\tau^{4m-2}P_{2m-1}^2(y, \eta)(1 + O(t))\tau \frac{dU_0}{d\tau} \\ &\quad + \frac{(2m)_2}{16}[\tau^{2m-1}P_{2m-1}(y, \tau)]^3(1 + O(\tau))U_0 \\ &\quad - \frac{1}{16(m-1)^2}[2m^2n - 2mn + 4m^2 - 5m + 2] \cdot \\ &\quad \cdot [\tau^{2m-1}P_{2m-1}(y, \eta)]^2(1 + O(\tau))U_0 \\ &\quad - \frac{1}{4}\tau^{2m-1}P_{2m-1}(y, \eta)(1 + O(\tau))\left[2 \sum_{i,j} a_{ij}(x, k_x)\partial_i A \partial_j A\right. \\ &\quad \left.+ \sum_{i,j} a_i(x, k_x)a_j(x, k_x)\partial_{ij}^2 A - \sum_i \partial_i P_{2m-1}(x, k_x)\frac{\partial A}{\partial x_i}\right]U_0. \end{aligned}$$

3.9. Transport equation for U_0

$\mathcal{J}(A, U_0) = 0$ is the transport equation for U_0 which is its bounded solution. For this purpose, (102) should be further simplified.

Note that

$$\frac{\partial A}{\partial \eta_i} = \sum_{j=1}^n \frac{\partial A}{\partial x_j} \frac{\partial x_j}{\partial \eta_i}$$

and the Hamiltonian system for x gives

$$x_j = y_j + a_j(y, \eta)t + O(t),$$

hence

$$\begin{aligned} \frac{\partial x_j}{\partial \eta_i} &= a_{ij}(y, \eta)t + O(t), \\ \frac{D(x)}{D(\eta)} &= t \text{Hess}_\eta a(y, \eta)(1 + O(t)). \end{aligned}$$

Denote $[\text{Hess}_\eta a(y, \eta)]^{-1} = (b_{ij}(y, \eta))$. We have

$$\frac{\partial A}{\partial x_j} = t^{-1} \sum_i b_{ij}(y, \eta)(1 + O(t)) \frac{\partial A}{\partial \eta_j}.$$

But (100) gives A as a function of η , we have

$$(103) \quad \frac{\partial A}{\partial x_j} = \frac{1}{4}t^{-1} \sum_i b_{ij}(y, \eta)(1 + O(t))P_{2m-1,j}(y, \eta) = O(\tau^m),$$

thus,

$$(104) \quad \sum_{i,j} a_{ij}(x, k_x) \partial_i A \partial_j A = O(\tau^{m-2+2m}) = \tau^{2m-1} O(\tau^{m-1}).$$

Next, denote $D = t \frac{d}{dt} = \sum_i a_i(x, k_x) \frac{\partial}{\partial x_i}$, we have

$$\begin{aligned} \sum_{i,j} a_i(x, k_x) a_j(x, k_x) \partial_{ij} A &= \sum_i a_i(x, k_x) D(\partial_i A) \\ &= \sum_i D(a_i(x, k_x) \partial_i A) - \sum_i \partial_i A \cdot Da_i \\ &= D^2 A - \sum_i \partial_i A Da_i \\ &= D\left(\frac{A}{m-1} + [P_{2m-1}(x, k_x) A]^{1/2}\right) - \sum_i \partial_i A Da_i \\ &= \frac{1}{m-1} DA + \frac{PDA + ADP}{2\sqrt{PA}} - \sum_i \partial_i A Da_i, \\ &\quad (\text{here } P = P_{2m-1}(x, k_x)). \end{aligned}$$

Since

$$DA = t \frac{dA}{dt} = \frac{A}{m-1} + \sqrt{PA},$$

then, the expression above can be simplified further:

$$\begin{aligned} &= \frac{\tau^{2m-1}}{4(m-1)^2} P_{2m-1}(y, \eta)(1 + O(\tau)) + \frac{3\tau^{2m-1}}{4(m-1)} P_{2m-1}(y, \eta)(1 + O(\tau)) \\ &\quad + \frac{\tau^{2m-1}}{2} P_{2m-1}(y, \eta)(1 + O(\tau)) + \frac{1}{4} D P_{2m-1}(x, k_x) - \sum_i \partial_i A Da_i \\ &= \frac{(2m-1)^2}{4(m-1)^2} \tau^{2m-1} P_{2m-1}(y, \eta) - \sum_i \partial_i A Da_i. \end{aligned}$$

Now

$$\begin{aligned}
\sum_i \partial_i A D a_i &= \sum_{i,j} \frac{\partial A}{\partial x_i} a_j(x, k_x) \frac{\partial}{\partial x_j} a_i(x, k_x) \\
&= \sum_{i,j} \frac{\partial A}{\partial x_i} a_j(x, k_x) \left(a_{ix_j}(x, k_x) + \sum_l a_{il}(x, k_x) \frac{\partial^2 k}{\partial x_j \partial x_l} \right) \\
&= \sum_{i,j} \frac{\partial A}{\partial x_i} a_j(x, k_x) (a_{ix_j}(x, k_x) + \delta_{ij}(1 + O(\tau))) \\
&= \sum_{i,j} \frac{\partial A}{\partial x_i} a_j(x, k_x) a_{ix_j}(x, k_x) + \sum_i a_i(x, k_x) \frac{\partial A}{\partial x_i} (1 + O(\tau)) \\
&= \sum_{i,j} \frac{\partial A}{\partial x_i} a_j(x, k_x) a_{ix_j}(x, k_x) + t \frac{dA}{dt} (1 + O(\tau)) \\
&= \tau^{2m-1} O(\tau) + \frac{A}{m-1} + \sqrt{P_{2m-1}(x, k_x) A} (1 + O(\tau)) \\
&= \tau^{2m-1} O(\tau) + \left(\frac{1}{4(m-1)} + \frac{1}{2} \right) \tau^{2m-1} P_{2m-1}(y, \eta) (1 + O(\tau)) \\
&= \frac{2m-1}{4(m-1)} \tau^{2m-1} P_{2m-1}(y, \eta) (1 + O(\tau)).
\end{aligned} \tag{105}$$

For the last term in (102), we have

$$\sum_i \partial_i P_{2m-1}(x, k_x) \frac{\partial A}{\partial x_i} = O(\tau^{3m-2}) = \tau^{2m-1} O(\tau^{m-1}) \tag{106}$$

Substituting (104), (105), (106) into (102), we have

$$\begin{aligned}
\mathcal{J}_2(A, U_0) &= \frac{1}{4(m-1)} [\tau^{2m-1} P_{2m-1}(y, \eta)]^2 (1 + O(\tau)) \\
&\quad \left[\tau \frac{dU_0}{d\tau} - \left(\frac{M}{4(m-1)} + \tau f(\tau, \eta, y) \right) U_0 \right],
\end{aligned} \tag{107}$$

here $M = 2m(m-1)(n+1)$, $f(\tau, \eta, y)$ is holomorphic in (τ, η) near $\tau = 0$.

Combining (101) for $\mathcal{J}_1(A, U_0)$ and (107) for $\mathcal{J}_2(A, U_0)$, we obtain finally the transport equation for U_0 :

$$\tau \frac{dU_0}{d\tau} + \left[mp - \frac{M}{4(m-1)} - \tau f(\tau, \eta, y) \right] U_0 = 0. \tag{108}$$

For U_0 to be bounded near τ , we must choose

$$p = \frac{M}{4m(m-1)} = \frac{n+1}{2}, \tag{109}$$

and

$$U_0 = \exp \left(\int_0^\tau f(s, \eta, y) ds \right). \tag{110}$$

3.10. Main result

We have been seeking a solution for the equation $P(x, \partial_x)u = 0$ in the form of (49):

$$u(x, y) = \exp[A(x, y)/k(x, y)] \sum_{h=0}^{\infty} U_h h^{p+h},$$

what we have attained up to present is only the solution of eiconal system for $k(x, y)$ ((47)) and $A(x, y)$ ((98)) and also the transport equation for U_0 ((108)). Hence, p is defined by (109), U_0 by (110) and $A(x, y)$ by (100).

For U_h , $h \geq 1$, we would obtain similar equation as (108) only with $p + h$ replacing p and the right hand side would be an expression linear in U_{h-1}, \dots, U_{h-2m} and their derivatives, i.e.

$$(111) \quad \tau \frac{dU_h}{d\tau} + [mh - \tau f(\tau, \eta, y)]U_h = L(U_{h-1}, \dots, U_{h-2m+1}).$$

The only bounded solution of it is

$$(112) \quad U_h = \frac{U_0}{\tau^{mh}} \int_0^{\tau} \frac{s^{mh-1}}{U_0} L(U_{h-1}, \dots, U_{h-2m+1}) ds.$$

Thus we are brought back to the case of simple-characteristic problems as discussed in Part I and we come to our main result:

THEOREM 5. *Let $P(x, \partial_x)$ be a holomorphic linear partial differential operator of order $2m$ ($m > 1$) in a domain X of an analytic complex manifold. Let the principal symbol of $P(x, \partial_x)$ be of the form*

$$P_{2m}(x, \xi) = [a(x, \xi)]^2,$$

such that for $y \in X$,

$$\text{Hess}_{\eta}[a(y, \eta)] \neq 0 \text{ for } a(y, \eta) = 0, \eta \neq 0.$$

Further, we also assume that

$$P_{2m-1}(y, \eta) \neq 0 \text{ and } m \geq 2.$$

Then there exist solutions of $P(x, \partial_x)u = 0$ with the asymptotic expansion

$$u(x, y) = \exp[A(x, y)/k(x, y)] \sum_{h=0}^{\infty} U_h h^{p+h}.$$

Here $k(x, y) = 0$ is an equation of the characteristic conoid, which satisfies

$$a(x, k_x) = \frac{k}{m-1}.$$

$A(x, y)$ is given by (98):

$$A(x, y) = \frac{1}{4} \tau^{2m-1} P_{2m-1}(y, \eta) (1 + O(\tau)),$$

U_0 and U_h , $h = 1, 2, \dots$, are determined by transport equation (108) and (??), p is determined by (109). Thus U_0 and U_h are given by (110) and (112). This asymptotic expansion converges for $|k(x, y)|$ small enough.

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