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THE REGULARIZED TRACE FOR NUCLEAR PERTURBATIONS OF DISCRETE OPERATORS

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Abstract. The regularized trace of nuclear perturbations for semibounded (positive) discrete operators is found.

1. Introduction

In [**1**] the regularized trace was found for a perturbation of a discrete operator by an operator of finite rank. For this result the behaviour of the function $N(\lambda)$ played an important role, where $N(\lambda)$ denotes the distribution of the eigenvalues for the nonperturbed operator.

In this paper this result is generalized for perturbations of a discrete operator by a nuclear operator which satisfies some additional conditions.

2. Statement of the problem and the main result

Let L_0 be a selfadjoint discrete operator acting on a separable Hilbert space H, with domain $\mathcal{D}(L_0)$. We denote by S a nuclear operator with a Hilbert representation:

$$S = \sum_{k=1}^{\infty} s_k(\,\cdot\,, f_k) g_k, \qquad \Big(\sum_{k=1}^{\infty} s_k < \infty \Big).$$

Let us consider the operator $L = L_0 + SL_0$. In this case μ_ν and λ_ν are eigenvalues of the operators L and L_0 respectively, arranged in the increasing order of their real parts, each of which is repeated according to its multiplicity. The main result is the following theorem.

THEOREM. *Let L_0 be a selfadjoint, discrete, semibounded (from below) operator acting on H. Suppose that the distribution function of its eigenvalues has the form $N(\lambda) = \sum_{\lambda_n \leq \lambda} 1 = c\lambda + O(\lambda^p)$ where c is a constant, $0 < p < 1$, $n = 1, 2, \dots$, $0 \notin \sigma(L_0)$. Suppose that one of the following conditions is satisfied:*

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1° $c = 0$; *for some* $q \in [0,1]$ *we have that* $g_k \in \mathcal{D}(L_0^{1-q})$, $f_k \in \mathcal{D}(L_0^q)$ *for every* k *and in this case* $\sum_{k=1}^{\infty} s_k \|L_0^{1-q} g_k\| \cdot \|L_0^q f_k\| < \infty$;

2° $c \neq 0$; *for some* $q \in [0,2]$ *we have that* $g_k \in \mathcal{D}(L_0^{2-q})$, $f_k \in \mathcal{D}(L_0^q)$ *for every* k *and in this case* $\sum_{k=1}^{\infty} s_k \|L_0^{2-q} g_k\| \cdot \|L_0^q f_k\| < \infty$.

Then the operator L *is discrete and there exists a subsequence of natural numbers* k_n $(k_n \to +\infty)$ *such that*

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} (\mu_k - \lambda_k) = \sum_{k=1}^{\infty} s_k (L_0^{1-q} g_k, L_0^q f_k). \tag{1}$$

Note that formula (1) gives the regularized trace of the operator L because the right-hand side of this equality can be evaluated.

3. Resolvent of the operator L

For the sake of symmetry, we shall introduce new vectors $f_k' = \sqrt{s_k} f_k$, $g_k' = \sqrt{s_k} g_k$. Then

$$S = \sum_{k=1}^{\infty} (\,\cdot\,, f_k') g_k', \qquad (f_k', f_l') = s_k \delta_{kl}, \quad (g_k', g_l') = s_k \delta_{kl}.$$

In order to find the resolvent of the operator L it is necessary to examine the solvability of the equation

$$L_0 u + \sum_{k=1}^{\infty} (L_0 u, f_k') g_k' = \lambda u + f \tag{2}$$

where $u \in \mathcal{D}(L_0)$ and $f \in H$. Suppose $\lambda \notin \sigma(L_0)$. Applying the operator $R_\lambda^0 = (L_0 - \lambda I)^{-1}$ to (2) we have that

$$u + \sum_{j=1}^{\infty} (L_0 u, f_j') R_\lambda^0 g_j' = R_\lambda^0 f.$$

Let us denote $(L_0 u, f_j') = \xi_j$; then we obtain

$$u = -\sum_{j=1}^{\infty} R_\lambda^0 g_j' + R_\lambda^0 f.$$

Substituting the expression for u in the formula for ξ_j we shall get an infinite system of algebraic linear equations:

$$\xi_k + \sum_{j=1}^{\infty} \xi_j (L_0 R_\lambda^0 g_j', f_k') = (L_0 R_\lambda^0 f, f_k'), \quad k = 1,2,3,\dots\ . \tag{3}$$

This system can be used to determine ξ_j. Put $a_{kj} = -(L_0 R_\lambda^0 g_j', f_k')$ and $c_k = (L_0 R_\lambda^0 f, f_k')$; then the system (3) can be written in the form $\xi_k - \sum_{j=1}^{\infty} a_{kj} \xi_j = c_k$, $k = 1,2,3,\dots$. We can check directly

$$\sum_{k=1}^{\infty} |a_{kk}| < \infty, \qquad \sum_{j,k=1}^{\infty} |a_{kj}|^2 < \infty. \tag{4}$$

Formulas (4) imply that the infinite matrix $I-\mathcal{A}$, where $\mathcal{A}=(a_{kj})_{k,j=1}^{\infty}$ is a Koch matrix. Therefore, by deleting a fixed column and a row from the matrix $I-\mathcal{A}$ we shall get again a Koch matrix.

To the Koch matrix $I-\mathcal{A}$ we can associate its determinant:

$$\det(I-\mathcal{A}) = 1-\sum_{j=1}^{\infty} a_{jj} + \frac{1}{2!}\sum_{j,k=1}^{\infty}\begin{vmatrix} a_{jj} & a_{jk} \\ a_{kj} & a_{kk}\end{vmatrix} - \frac{1}{3!}\sum \quad + \cdots$$

(For more details about the Koch matrix see [**2**] or [**3**])

Since the matrix of the system $\xi_{ik}-\sum_{j=1}^{\infty} a_{kj}\xi_j = c_k$, $k=1,2,3,\ldots$ is a Koch matrix, the solution of the system (3) is given by

$$\xi_k = \sum_{j=1}^{\infty}(-1)^{j+k}\Delta_{jk}c_j/\Delta \tag{5}$$

where Δ_{jk} is the determinant of the matrix obtained from the matrix $I-\mathcal{A}$ by deleting its j-th row and k-th column; $\det(I-\mathcal{A})=\Delta$. Because of $R_\lambda f = u$ ($R_\lambda=(L-\lambda I)^{-1}$) and $u=-\sum_{j=1}^{\infty}\xi_j R_\lambda^0 g_j' + R_\lambda^0 f$, (5) implies that

$$R_\lambda f = R_\lambda^0 f - \sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}(-1)^{j+k}\Delta_{jk}(L_0R_\lambda^0 f, f_j')R_\lambda^0 g_k'\right)\Delta^{-1}. \tag{6}$$

The spectrum of the operator L is discrete because of Theorem 10.1, p. 336, in [**2**]. Suppose $\lambda\notin\sigma(L_0)\cup\sigma(L)$; then

$$L-\lambda I = (I+SL_0R_\lambda^0)(L_0-\lambda I).$$

Since the operator $L-\lambda I$ vanishes only at the point $x=0$ it follows that $-1\notin\sigma(SL_0R_\lambda^0)$. The operator $SL_0R_\lambda^0$ is nuclear, so that $(I+SL_0R_\lambda^0)^{-1}=I+K$, where K is a nuclear operator. But then we have $(L-\lambda I)^{-1}=(L_0-\lambda I)^{-1}(I+K)$ so that $(L-\lambda I)^{-1}$ is a compact operator and L is a discrete operator.

Now, we introduce linear functionals φ_k:

$$\varphi_k(f) = \Delta^{-1}\sum_{j=1}^{\infty}(-1)^{j+k}\Delta_{jk}s_k^{-1/2}(L_0R_\lambda^0 f, f_j'), \qquad f\in H. \tag{7}$$

LEMMA 1. *The sequence of linear functionals* φ_k $(k=1,2,3,\ldots)$ *on the space* H *is uniformly bounded.*

Proof. Since

$$\varphi_k(f) = \Delta^{-1}\sum_{j=1}^{\infty}(-1)^{j+k}\Delta_{jk}s_k^{-1/2}(L_0R_\lambda^0 f, f_j')$$

it follows that

$$\varphi_k(f) = \Delta^{-1}\Delta_{kk}(L_0R_\lambda^0 f, f_k) + \Delta^{-1}\sum_{j\neq k}(-1)^{j+k}\Delta_{jk}s_j^{1/2}s_k^{-1/2}(L_0R_\lambda^0 f, f_j)$$

and

$$|\varphi_k(f)| \le |\Delta|^{-1}|\Delta_{kk}| \cdot \|L_0 R^0_\lambda\| \cdot \|f\| + |\Delta|^{-1}\Bigl(\sum_{j\ne k}|\Delta_{jk}|^2 s_j/s_k \cdot \sum_{j\ne k}|(L_0R^0_\lambda f, f_j)|^2\Bigr)^{1/2}.$$

This formulas imply, by using the Bessel inequality, that

$$|\varphi_k(f)| \le \|f\|\biggl(|\Delta_{kk}| \cdot |\Delta|^{-1} \cdot \|L_0 R^0_\lambda\| + \|L_0 R^0_\lambda\| \cdot |\Delta|^{-1} \cdot \Bigl(\sum_{j\ne k}|\Delta_{jk}|^2 s_j/s_k\Bigr)^{1/2}\biggr).$$

We show that $\sup_{j,k}|\Delta_{jk}|/s_k < \infty$ and $\sup_k|\Delta_{kk}| < \infty$. By the Hadamard inequality

$$|\Delta_{jk}|^2 \le \prod_{p\ne j,\, p\ne k}\Bigl(\sum_{q\ne k}|\delta_{pq} - a_{pq}|^2\Bigr)\Bigl(\sum_{q\ne k}|\delta_{kq} - a_{kq}|^2\Bigr)$$

and by using that $a_{kj} = -(s_k s_j)^{1/2}(L_0R^0_\lambda g_j, f_k)$ we get

$$\sum_{q\ne k}|\delta_{kq} - a_{kq}|^2 = \sum_{q\ne k}|a_{kq}|^2 \le s_k\|L_0R^0_\lambda\|^2|S|_1$$

($|S|_1$ is the nuclear norm of operator S). The last formula implies that

$$|\Delta_{jk}|^2/s_k \le \|L_0R^0_\lambda\|^2 \cdot |S|_1 \cdot \prod_{p\ne j,\, p\ne k}\Bigl(\sum_{q\ne k}|\delta_{pq} - a_{pq}|^2\Bigr).$$

Because of

$$\prod_{p\ne j,\, p\ne k}\Bigl(\sum_{q\ne k}|\delta_{pq} - a_{pq}|^2\Bigr) \le \exp\Bigl(2\sum_{p=1}^{\infty}|a_{pp}| + \sum_{p,q=1}^{\infty}|a_{pq}|^2\Bigr) = M < \infty$$

we have $|\Delta_{jk}|^2/s_k \le M|S|_1 \cdot \|L_0R^0_\lambda\|^2$.

We can show in a similar way that $|\Delta_{kk}| \le \sqrt{M}$. This completes the proof of Lemma 1.

Using (7), the equality (6) can be written in the form

$$R_\lambda - R^0_\lambda = -\sum_{k=1}^{\infty} s_k\varphi_k(\,\cdot\,)R^0_\lambda g_k.$$

Since Lemma 1 implies $\sum_{k=1}^{\infty} s_k\|\varphi_k\|\,\|R^0_\lambda g_k\| < \infty$, the operator

$$\sum_{k=1}^{\infty} s_k\varphi_k(\,\cdot\,)R^0_\lambda g_k$$

is nuclear. So we have

$$\mathrm{Sp}(R_\lambda - R^0_\lambda) = -\sum_{k=1}^{\infty} s_k\varphi_k(R^0_\lambda g_k)$$

and

$$\mathrm{Sp}(R_\lambda - R^0_\lambda) = -\Delta^{-1} \sum_{k,j=1}^{\infty} (-1)^{j+k} \Delta_{jk} (L_0 (R^0_\lambda)^2 g'_k, f'_j). \tag{8}$$

Using that $a_{jk} = -(L_0 R^0_\lambda g'_k, f'_j)$, we can get $a'_{jk}(\lambda) = -(L_0 (R^0_\lambda)^2 g'_k, f'_j)$ and (8) can be written as

$$\mathrm{Sp}(R_\lambda - R^0_\lambda) = \Delta^{-1} \sum_{k,j=1}^{\infty} (-1)^{j+k} \Delta_{jk} a'_{jk}(\lambda). \tag{9}$$

For the Koch matrix we have

$$\sum_{j,k=1}^{\infty} (-1)^{j+k} \Delta_{jk} a'_{jk}(\lambda) = \Delta'(\lambda),$$

so (9) implies

$$\mathrm{Sp}(R_\lambda - R^0_\lambda) = \Delta'(\lambda)/\Delta(\lambda). \tag{10}$$

4. The proof of the theorem in the case when $c = 0$ in the formula for $N(\lambda)$

Let $\{\lambda_k\}$ be the eigenvalues of the operator L_0 and $\{\psi_k\}$ an orthonormal base in the space H, where ψ_k are eigenvectors (corresponding to λ_k) of the operator L_0. We denote $c^j_n = (g_j, \psi_n)$, $b^j_n = (f_j, \psi_n)$; then $g_k = \sum_{n=1}^{\infty} c^k_n \psi_n$, $f_j = \sum_{n=1}^{\infty} b^j_n \psi_n$. The spectral theorem implies

$$(L_0 R^0_\lambda g_k, f_j) = \sum_{n=1}^{\infty} \frac{\lambda_n c^k_n \overline{b^j_n}}{\lambda - \lambda_n}. \tag{11}$$

Since the assumptions are $g_k \in \mathcal{D}(L_0^{1-q})$, $f_j \in \mathcal{D}(L_0^q)$, the Bessel inequality implies

$$\sum_{n=1}^{\infty} \lambda_n |c^k_n b^j_n| \le \|L_0^{1-q} g_k\| \, \|L_0^q f_j\|. \tag{12}$$

For the case $c = 0$ we have $N(\lambda) = O(\lambda^p)$ $(0 < p < 1)$ and then there exists a sequence of real numbers r_n $(r_n \to \infty)$ such that $d_n \to \infty$, where $d_n = d(\Gamma_n, \sigma(L_0))$ is the distance between the circle $\Gamma_n = \{\lambda : |\lambda| = r_n\}$ and the spectrum of the operator L [**1**].

LEMMA 2. *The operators L and L_0 have the same number of eigenvalues inside the circle Γ_n, for n large enough.*

Proof. We multiply equality (10) by $1/2\pi i$ and integrate it over Γ_n. Note that the well-known Riesz theorem implies

$$\frac{1}{2\pi i} \int_{\Gamma_n} \mathrm{Sp}(R_\lambda - R^0_\lambda)\, d\lambda = N_1 - N_2,$$

where N_1 and N_2 denote the numbers of eigenvalues inside the contour Γ_n (counting their multiplicity) of the operators L and L_0, respectively. So, this implies that

$$N_1 - N_2 = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{\Delta'(\lambda)}{\Delta(\lambda)}\, d\lambda = \frac{1}{2\pi i} \int_{\Gamma_n} \frac{S'(\lambda)}{1 + S(\lambda)}\, d\lambda \tag{13}$$

where $S(\lambda) = \Delta(\lambda) - 1$. Let us estimate $S(\lambda)$. Since $a_{jk} = -(s_j s_k)^{1/2}(L_0 R_\lambda^0 g_k, f_j)$ by using (11) and (12) we have

$$|a_{jk}| \leq (s_j s_k)^{1/2} \|L_0^{1-q} g_k\| \cdot \|L_0^q f_j\| / d(\lambda) \tag{14}$$

where $d(\lambda)$ denotes the distance between a point λ and $\sigma(L_0)$. Note that for $\lambda \in \Gamma_n$ we have $d(\lambda) \geq d_n$. The definition of the function $S(\lambda)$ and (14) imply the following estimate

$$\left.\begin{aligned} |S(\lambda)| &\leq \sum_{k=1}^{\infty} M(\lambda)^k \\ \text{where} \quad M(\lambda) &= \sum_{j=1}^{\infty} s_j \|L_0^q f_j\| \cdot \|L_0^{1-q} g_j\| / d(\lambda) \end{aligned}\right\} \tag{15}$$

From $\sum_{j=1}^{\infty} s_j \|L_0^q f_j\| \, \|L_0^{1-q} g_j\| < \infty$, and (15) it follows that $S(\lambda) \to 0$ when $n \to \infty$ and $\lambda \in \Gamma_n$; the convergence is uniform with respect to λ. Then (13) implies

$$N_1 - N_2 = \sum_{\nu=0}^{\infty} (-1)^\nu \frac{1}{2\pi i} \int_{\Gamma_n} S'(\lambda) S^\nu(\lambda) \, d\lambda. \tag{16}$$

From (15) we get

$$|S^\nu(\lambda)| \leq K_1^\nu / d(\lambda)^\nu \tag{17}$$

for $\lambda \in \Gamma_n$ and n is large enough (K_1 does not depend on λ and ν).

By differentiating (11) with respect to λ and using (12) we have

$$|a'_{jk}(\lambda)| \leq (s_j s_k)^{1/2} \|L_0^{1-q} g_k\| \cdot \|L_0^q f_j\| / d^2(\lambda). \tag{18}$$

From (14) and (18) and the definition of the function $S(\lambda)$ we can get the following estimate

$$|S'(\lambda)| \leq \frac{1}{d(\lambda)} \cdot \sum_{n=1}^{\infty} n M^n(\lambda) \leq \frac{K_2}{d^2(\lambda)} \tag{19}$$

where the constant K_2 does not depend on λ, $\lambda \in \Gamma_n$ and n is large enough. Then (17) and (19) imply

$$\left| \int_{\Gamma_n} S'(\lambda) S^\nu(\lambda) \, d\lambda \right| \leq K_2 K_1^\nu \int_{\Gamma_n} \frac{|d\lambda|}{d^{2+\nu}(\lambda)}.$$

Next we can easily check that

$$\int_{\Gamma_n} \frac{|d\lambda|}{d^2(\lambda)} \leq \frac{C_0}{d_n}$$

(the constant C_0 does not depend on n). Because of $d(\lambda) \geq d_n$ on Γ_n the inequality

$$\int_{\Gamma_n} \frac{|d\lambda|}{d^{2+\nu}(\lambda)} \leq \frac{1}{d_n^\nu} \int_{\Gamma_n} \frac{|d\lambda|}{d^2(\lambda)}$$

holds and consequently

$$\left| \int_{\Gamma_n} S'(\lambda) S^\nu(\lambda) \, d\lambda \right| \leq C_0 K_2 K_1^\nu / d_n^{\nu+1} \tag{20}$$

for n large enough. Formula (16) implies $|N_1 - N_2| \le \sum_{\nu=0}^{\infty} C_0 K_2 K_1^{\nu}/d_n^{\nu+1}$ (n large enough) and $|N_1 - N_2| = O(1/d_n)$. Since $|N_1 - N_2|$ is a natural number or 0, this is possible if and only if $N_1 = N_2$. So the lemma is proved.

Let us now prove the Theorem in the case $c = 0$. If we multiply (10) by $\lambda/2\pi i$ and integrate it over Γ_n we have

$$\frac{1}{2\pi i}\int_{\Gamma_n} \lambda\,\mathrm{Sp}(R_\lambda - R_\lambda^0)\,d\lambda = \frac{1}{2\pi i}\int_{\Gamma_n} \lambda\frac{\Delta'(\lambda)}{\Delta(\lambda)}\,d\lambda. \tag{21}$$

The properties of Riesz projectors imply that

$$\frac{1}{2\pi i}\int_{\Gamma_n} \lambda\,\mathrm{Sp}(R_\lambda - R_\lambda^0)\,d\lambda = \sum_{\nu=1}^{k_n}(\mu_\nu - \lambda_\nu) \tag{22}$$

where k_n denotes the number of eigenvalues for operators L and L_0 inside the contour Γ_n (this number is the same for both operators by Lemma 2). From (21) and (22) it follows that

$$\sum_{\nu=1}^{k_n}(\mu_\nu - \lambda_\nu) = \frac{1}{2\pi i}\int_{\Gamma_n} \lambda\frac{\Delta'(\lambda)}{\Delta(\lambda)}\,d\lambda. \tag{23}$$

Since

$$\frac{1}{2\pi i}\int_{\Gamma_n} \lambda\frac{\Delta'(\lambda)}{\Delta(\lambda)}\,d\lambda = \sum_{\nu=0}^{\infty}(-1)^{\nu}\frac{1}{2\pi i}\int_{\Gamma_n} \lambda S'(\lambda)S^{\nu}(\lambda)\,d\lambda \tag{24}$$

for n large enough (because of $S(\lambda) \rightrightarrows 0$, as $n \to \infty$, $\lambda \in \Gamma_n$), it is enough to estimate each of the integrals $\int_{\Gamma_n} \lambda S'(\lambda)S^{\nu}(\lambda)\,d\lambda$.

By the definition of the function S (because of the structure of a_{jk}) we know that S is a holomorphic function on $\mathbf{C}$ except at the points $\lambda_1, \lambda_2, \ldots$. Inside the contour Γ_n, the function S has singularities at the points $\lambda_1, \lambda_2, \ldots, \lambda_{k_n}$. We can easily check that

$$\int_{\Gamma_n} \lambda S'(\lambda)S^{\nu}(\lambda)\,d\lambda = -\frac{1}{\nu+1}\int_{\Gamma_n} S^{\nu+1}(\lambda)\,d\lambda. \tag{25}$$

Formulas (24) and (25) imply

$$\frac{1}{2\pi i}\int_{\Gamma_n} \lambda\frac{\Delta'(\lambda)}{\Delta(\lambda)}\,d\lambda = -\frac{1}{2\pi i}\int_{\Gamma_n} S(\lambda)\,d\lambda + \sum_{\nu=1}^{\infty}\frac{(-1)^{\nu+1}}{\nu+1}\frac{1}{2\pi i}\int_{\Gamma_n} S^{\nu+1}(\lambda)\,d\lambda. \tag{26}$$

From (17), for n large enough, it follows that

$$\left|\frac{1}{2\pi i}\int_{\Gamma_n} S^{\nu+1}(\lambda)\,d\lambda\right| \le \frac{C_0 K_1}{2\pi}\left(\frac{K_1}{d_n}\right)^{\nu},$$

therefore

$$\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu+1}}{\nu+1}\frac{1}{2\pi i}\int_{\Gamma_n} S^{\nu+1}(\lambda)\,d\lambda = O(d_n^{-1}). \tag{27}$$

Next, (26) and (27) imply

$$\frac{1}{2\pi i}\int_{\Gamma_n}\lambda\frac{\Delta'(\lambda)}{\Delta(\lambda)}\,d\lambda=-\frac{1}{2\pi i}\int_{\Gamma_n}S(\lambda)\,d\lambda+O(d_n^{-1}).\tag{28}$$

Let us consider the function R given by

$$R(\lambda)=S(\lambda)+\sum_{j=1}^{\infty}a_{jj}(\lambda).\tag{29}$$

In a way similar to the proof of inequality (15), we can prove that $|R(\lambda)|\le P\cdot(d(\lambda))^{-2}$ for $\lambda\in\Gamma_n$, for n large enough and a constant P which does not depend on λ. This inequality implies

$$-\frac{1}{2\pi i}\int_{\Gamma_n}R(\lambda)\,d\lambda=O(d_n^{-1}).\tag{30}$$

Then, from (28), (29) and (30) it follows that

$$\frac{1}{2\pi i}\int_{\Gamma_n}\lambda\frac{\Delta'(\lambda)}{\Delta(\lambda)}\,d\lambda=\frac{1}{2\pi i}\int_{\Gamma_n}\left(\sum_{j=1}^{\infty}a_{jj}(\lambda)\right)d\lambda+O(d_n^{-1}).\tag{31}$$

Since the series $\sum_{j=1}^{\infty}a_{jj}(\lambda)$ converges uniformly on Γ_n, we have

$$\frac{1}{2\pi i}\int_{\Gamma_n}\lambda\frac{\Delta'(\lambda)}{\Delta(\lambda)}\,d\lambda=\sum_{j=1}^{\infty}\frac{1}{2\pi i}\int_{\Gamma_n}a_{jj}(\lambda)\,d\lambda+O(d_n^{-1}).\tag{32}$$

Because of $a_{jj}=-s_j\sum_{\nu=1}^{\infty}\lambda_\nu a_\nu^j\overline{b_\nu^j}/(\lambda_\nu-\lambda)$, (32) gives

$$\frac{1}{2\pi i}\int_{\Gamma_n}\lambda\frac{\Delta'(\lambda)}{\Delta(\lambda)}\,d\lambda=\sum_{j=1}^{\infty}\left(s_j\sum_{\nu=1}^{k_n}\lambda_\nu a_\nu^j\overline{b_\nu^j}\right)+O(d_n^{-1}).\tag{33}$$

From (23) and (24) we can get

$$\sum_{\nu=1}^{k_n}(\mu_\nu-\lambda_\nu)=\sum_{j=1}^{\infty}\left(s_j\sum_{\nu=1}^{k_n}\lambda_\nu a_\nu^j\overline{b_\nu^j}\right)+O(d_n^{-1}).\tag{34}$$

Since $\sum_{\nu=1}^{\infty}\lambda_\nu|a_\nu^jb_\nu^j|\le\|L_0^{1-q}g_j\|\cdot\|L_0^qf_j\|$ the series $\sum_{j=1}^{\infty}s_j\|L_0^{1-q}g_j\|\cdot\|L_0^qf_j\|$ is convergent and $\lim_{n\to\infty}\sum_{\nu=1}^{k_n}\lambda_\nu a_\nu^j\overline{b_\nu^j}=(L_0^{1-q}g_j,L_0^qf_j)$, then we have from (34), when $n\to\infty$,

$$\lim_{n\to\infty}\sum_{\nu=1}^{k_n}(\mu_\nu-\lambda_\nu)=\sum_{j=1}^{\infty}s_j(L_0^{1-q}g_j,L_0^qf_j).$$

So, the theorem is proved in the case $c=0$.

5. The proof of the theorem in the case when $c \neq 0$ in formula for $N(\lambda)$

If $g_k \in \mathcal{D}(L_0^{2-q})$ the spectral theorem implies $g_k \in \mathcal{D}(L_0^{1-q})$ and

$$\|L_0^{1-q} g_k\| \leq \lambda_1^{-1} \|L_0^{2-q} g_k\|. \tag{35}$$

Because of the assumption $\sum_{k=1}^{\infty} s_k \|L_0^{2-q} g_k\| \cdot \|L_0^q f_k\| < \infty$ (which holds in the case $c \neq 0$) (35) implies

$$\sum_{k=1}^{\infty} s_k \|L_0^{1-q} g_k\| \cdot \|L_0^q f_k\| < \infty.$$

If $g_k \in \mathcal{D}(L_0^{2-q})$, $f_j \in \mathcal{D}(L_0^q)$ using the Bessel inequality, we have

$$\sum_{n=1}^{\infty} \lambda_n^2 |c_n^k \overline{b_n^j}| \leq \|L_0^{2-q} g_k\| \cdot \|L_0^q f_j\|. \tag{36}$$

In the case $c \neq 0$, the formula for $N(\lambda)$ can be written in the form

$$\lambda_n = n(k + o(1)) \tag{37}$$

where $k = c^{-1}$.

The following lemma was proved in [**1**].

Lemma 3. *Suppose that $\alpha > 0$, N_α is so large that $N_\alpha > 8/\alpha$ and for $o(1)$ in (37) the following holds: $|o(1)| \leq \alpha k / 8(1+\alpha)$, for every $n > N_\alpha$. Then between λ_n and $\lambda_{n'}$, where $n > N_\alpha$ and $n' = [n(1+\alpha)] > N_\alpha$, there are eigenvalues λ_{n_ν} and $\lambda_{n_\nu+1}$ of the operator L_0 such that*

$$\lambda_{n_\nu+1} - \lambda_{n_\nu} \geq k/2. \tag{38}$$

Let us define $\Gamma_{n_\nu} = \{\lambda : |\lambda| = r_{n_\nu}\}$, where $r_{n_\nu} = (\lambda_{n_\nu} + \lambda_{n_\nu+1})/2$. Since

$$(L_0 R_\lambda^0 g_k, f_j) = \sum_{n=1}^{\infty} \frac{\lambda_n c_n^k \overline{b_n^j}}{\lambda_n - \lambda},$$

we have

$$(L_0 R_\lambda^0 g_k, f_j) + \frac{1}{\lambda} (L_0^{1-q} g_k, L_0^q f_j) = \sum_{n=1}^{\infty} \frac{\lambda_n^2 c_n^k \overline{b_n^j}}{\lambda(\lambda_n - \lambda)}. \tag{39}$$

From (35), (36), and (39) it follows that

$$|(L_0 R_\lambda^0 g_k, f_j)| \leq \frac{\|L_0^{2-q} g_k\| \cdot \|L_0^q f_j\|}{|\lambda| d(\lambda)} + \frac{1}{\lambda_1} \frac{\|L_0^{2-q} g_k\| \cdot \|L_0^q f_j\|}{|\lambda|}$$

and then, $d(\lambda) \geq k/4$ on Γ_{n_ν}, implies

$$|a_{jk}(\lambda)| \leq K (s_j s_k)^{1/2} \cdot \|L_0^{2-q} g_k\| \cdot \|L_0^q f_j\| \cdot |\lambda|^{-1} \tag{40}$$

on Γ_{n_ν} $(K = 4 \cdot k^{-1} + \lambda_1^{-1})$. Using (40), we can get an estimate for the function

$$|S(\lambda)| \leq \sum_{n=1}^{\infty} M_1^n(\lambda) \tag{41}$$

where $M_1(\lambda) = |\lambda|^{-1} K \sum_{j=1}^{\infty} s_j \|L_0^{2-q} g_j\| \cdot \|L_0^q f_j\|$.

Clearly, for $\lambda \in \Gamma_{n_\nu}$ and ν large enough, we have $|S(\lambda)| \leq Q/|\lambda|$ where the constant Q does not depend on λ $(\lambda \in \Gamma_{n_\nu})$. This implies that $S(\lambda) \to 0$, for $\lambda \in \Gamma_{n_\nu}$, $\nu \to \infty$, therefore the function $\Delta(\lambda)$ does not have zeros in region $\{\lambda : |\lambda| > r_{n_\nu}\}$ for ν large enough.

In a similar way, as it was done in Lemma 2, we can show that the operators L and L_0 have the same number of eigenvalues inside the contour Γ_{n_ν}, for ν large enough. This number is equal to n_ν.

As in the previous case, we have

$$\sum_{k=1}^{n_\nu} (\mu_k - \lambda_k) = -\frac{1}{2\pi i} \int_{\Gamma_{n_\nu}} S(\lambda)\, d\lambda + \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r+1} \frac{1}{2\pi i} \int_{\Gamma_{n_\nu}} S(\lambda)^{r+1}\, d\lambda. \tag{42}$$

Using the estimate $|S(\lambda)| \leq Q/|\lambda|$, which holds for $\lambda \in \Gamma_{n_\nu}$ and ν large enough, we can get easily

$$\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r+1} \frac{1}{2\pi i} \int_{\Gamma_{n_\nu}} S(\lambda)^{r+1}\, d\lambda = O(r_{n_\nu}^{-1}), \qquad \nu \to \infty$$

and consequently (42) implies

$$\sum_{k=1}^{n_\nu} (\mu_k - \lambda_k) = -\frac{1}{2\pi i} \int_{\Gamma_{n_\nu}} S(\lambda)\, d\lambda + O(r_{n_\lambda}^{-1})\,. \tag{43}$$

Since $\left| S(\lambda) + \sum_{j=1}^{\infty} a_{jj}(\lambda) \right| \leq \mathrm{const}/|\lambda|^2$ on Γ_{n_ν} (ν large enough) we have

$$\frac{1}{2\pi i} \int_{\Gamma_{n_\nu}} \left(S(\lambda) + \sum_{j=1}^{\infty} a_{jj}(\lambda) \right) d\lambda = O(r_{n_\nu}^{-1}). \tag{44}$$

From (43) and (44) it follows that

$$\sum_{k=1}^{n_\nu} (\mu_k - \lambda_k) = \frac{1}{2\pi i} \int_{\Gamma_{n_\nu}} \sum_{j=1}^{\infty} a_{jj}(\lambda)\, d\lambda + O(r_{n_\nu}^{-1}). \tag{45}$$

Because of

$$\frac{1}{2\pi i} \int_{\Gamma_{n_\nu}} a_{jj}(\lambda)\, d\lambda = s_j \sum_{j=1}^{n_\nu} \lambda_s a_s^j \overline{b_s^j}$$

and the uniform convergence of the series $\sum a_{jj}(\lambda)$ on Γ_{n_ν}, (45) becomes

$$\sum_{k=1}^{n_\nu} (\mu_k - \lambda_k) = \sum_{j=1}^{\infty} \left(s_j \sum_{s=1}^{n_\nu} \lambda_s a_s^j \overline{b_s^j} \right) + O(r_{n_\nu}^{-1}). \tag{46}$$

Hence

$$\sum_{j=1}^{\infty} s_j \sum_{s=1}^{\infty} \lambda_s |a_s^j b_s^j| \leq \sum_{j=1}^{\infty} s_j \|L_0^{1-q} g_j\| \cdot \|L_0^q f_j\| < \infty$$

(46) and the fact that $\lim_{\nu\to\infty} \sum_{s=1}^{n_\nu} (\mu_s - \lambda_s) = (L_0^{1-q} g_j, L_0^q f_j)$ imply

$$\lim_{\nu\to\infty} \sum_{s=1}^{n_\nu} (\mu_s - \lambda_s) = \sum_{j=1}^{\infty} s_j (L_0^{1-q} g_j, L_0^q f_j).$$

So, the proof of the theorem in case $c \neq 0$ is completed.

Remark. Under the condition stated in the theorem can get directly that

$$\sum_{j=1}^{\infty} s_j (L_0^{1-q} g_j, L_0^q f_j) = \sum_{j=1}^{\infty} \lambda_j (S\varphi_j, \varphi_j) \tag{47}$$

so the statement of the theorem can be formulated in terms of the operator S by formula (47).

Example. Let L_0 be a semibounded ($L_0 > 0$), discrete, selfadjoint operator which acts on H. Suppose that $S > 0$ is a nuclear operator which has properties $R(S) \subset \mathcal{D}(L_0)$ and $\sum \lambda_n (S\varphi_n, \varphi_n) < \infty$ (λ_n denotes the eigenvalues and φ_n the corresponding eigenvectors of the operator L_0). Then

$$\lim_{n\to\infty} \sum_{\nu=1}^{k_n} (\mu_\nu - \lambda_\nu) = \sum_{\nu=1}^{\infty} \lambda_\nu (S\varphi_\nu, \varphi_\nu).$$

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