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The Abundancy Index of Divisors of Odd Perfect Numbers

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Abstract

If $N = q^k n^2$ is an odd perfect number, where q is the Euler prime, then we show that $\sigma(n) \leq q^k$ is necessary and sufficient for Sorli's conjecture that $k = \nu_q(N) = 1$ to hold. It follows that, if $k = 1$ then the Euler prime q is the largest prime factor of N and that $q > 10^{500}$. We also prove that $q^k < \frac{2}{3}n^2$.

1 Introduction

Perfect numbers are positive integral solutions to the number-theoretic equation $\sigma(N) = 2N$, where σ is the sum-of-divisors function. Euclid derived the general form for the even case; Euler proved that every even perfect number is given in the Euclidean form $N = 2^{p-1}(2^p - 1)$ where p and $2^p - 1$ are prime. On the other hand, it is still an open question to determine existence (or otherwise) for an odd perfect number. Euler proved that every odd perfect number is given in the so-called Eulerian form $N = q^k n^2$ where $q \equiv k \equiv 1 \pmod 4$ and $\gcd(q, n) = 1$. (We call q the Euler prime of the odd perfect number N, and the component q^k will be called the Euler's factor of N.) As of February 2012, only 47 even perfect numbers are known (13 of which were found by the distributed computing project GIMPS [13]), while no single example of an odd perfect number has been found. (Ochem and Rao of CNRS, France are currently orchestrating an effort to push the lower bound for an odd perfect number from the previously known 10^{300} to a significantly improved 10^{1500} (see [7]). Nielsen has obtained the lower bound $\omega(N) \geq 9$, for the number of distinct prime factors of N; and the upper bound $N < 2^{4^{\omega(N)}}$ ([13, 14]).

Let $\sigma(x)$ denote the sum of the divisors of the natural number x. That is, let $\sigma(x) = \sum_{d|x} d$. Let $\omega(x)$ denote the number of distinct prime factors of x. Let $\nu_q(N)$ denote the highest power of q that divides N; that is, if $l = \nu_q(N)$, then $q^l | N$ but $q^{l+1} \nmid N$. Let $I(x) = \sigma(x)/x$ denote the abundancy index of x.

Sorli conjectured in [10] that the exponent $k = \nu_q(N)$ on the Euler prime q for an odd perfect number N given in the Eulerian form $N = q^k n^2$ is one.

Throughout this paper, we will let

$$N = q^k n^2 = \prod_{j=1}^{\omega(N)} {q_j}^{\beta_j}$$

denote the canonical factorization of the odd perfect number N. That is,

$$\min(q_j) = q_1 < q_2 < q_3 < \cdots < q_{\omega(N)} = \max(q_j).$$

Note that q is never the smallest prime divisor of N. This is because q, being congruent to 1 modulo 4, satisfies $(q+1)|\sigma(q^k)|\sigma(N) = 2N$ giving $\frac{q+1}{2}|N$, so N must have a smaller odd prime divisor than q.

2 Odd Perfect Numbers Circa 2008

We begin with the following definition:

Definition 1. An odd perfect number N is said to be given in Eulerian form if $N = q^k n^2$ where $q \equiv k \equiv 1 \pmod 4$ and $\gcd(q, n) = 1$.

The author made the following conjecture [3]:

Conjecture 2. Suppose there is an odd perfect number given in Eulerian form. Then $q^k < n$.

The author formulated Conjecture 2 on the basis of the following result:

Lemma 3. *If an odd perfect number N is given in Eulerian form, then $I(q^k) < \dfrac{5}{4} < \sqrt{\dfrac{8}{5}} < I(n)$.*

Proof. Since q is the Euler prime and

$$I(N) = 2 = I(q^k)I(n^2),$$

we appeal to some quick numerical results. Since

$$I(q^k) < \frac{q}{q-1}$$

and $q \equiv 1 \pmod 4$, we know that $q \geq 5$. Consequently, we have

$$1 < I(q^k) < \frac{5}{4} = 1.25.$$

On the other hand,

$$I(n^2) = \frac{2}{I(q^k)}$$

so that we obtain the bounds

$$1.6 = \frac{8}{5} < I(n^2) < 2.$$

But it is also (fairly) well-known [6, 8, 9] that the abundancy index (as a function) satisfies the inequality

$$I(ab) \leq I(a)I(b)$$

with equality occurring if and only if $\gcd(a, b) = 1$.

In particular, by setting $a = b = n$, we get

$$\frac{2}{I(q^k)} = I(n^2) < (I(n))^2$$

whereupon we get the lower bound

$$\sqrt{\frac{8}{5}} < \sqrt{\frac{2}{I(q^k)}} = \sqrt{I(n^2)} < I(n).$$

We get the rational approximation $\sqrt{8/5} \approx 1.264911$. $\square$

Remark 4. When Conjecture 2 was formulated in 2008, the author was under the naive impression that the divisibility constraint $\gcd(q, n) = 1$ induced an "ordering" property for the Euler prime-power q^k and the component $n = \sqrt{N/q^k}$, in the sense that the related inequality $q^k < n^2$ followed from the result $I(q^k) < I(n^2)$. (Indeed, the author was able to derive the (slightly) stronger result $q^k < \sigma(q^k) \leq (2/3)n^2$ [3]).

We reproduce the proof for a generalization of the author's result mentioned in Remark 4 in the following theorem.

Theorem 5. *Suppose there is an odd perfect number with canonical factorization*

$$N = \prod_{i=1}^{\omega(N)} {q_i}^{\alpha_i}$$

where the q_i's are primes and $q_1 < q_2 < \cdots < q_{\omega(N)}$. Then, for all i with $1 \leq i \leq \omega(N)$, the numbers $\rho_i = \sigma(N/{q_i}^{\alpha_i})/{q_i}^{\alpha_i}$ are positive integers and satisfy $\rho_i \geq 3$.

Proof. Since

$$N = \prod_{i=1}^{\omega(N)} {q_i}^{\alpha_i}$$

is an odd perfect number and ${q_i}^{\alpha_i} || N \ \forall i$, then the quantity $\rho_i = \sigma(N/{q_i}^{\alpha_i})/{q_i}^{\alpha_i}$ is an integer (because $\gcd({q_i}^{\alpha_i}, \sigma({q_i}^{\alpha_i})) = 1$).

Suppose $\rho_i = 1$. Then $\sigma(N/{q_i}^{\alpha_i}) = {q_i}^{\alpha_i}$ and $\sigma({q_i}^{\alpha_i}) = 2N/{q_i}^{\alpha_i}$. Since N is an odd perfect number, q_i is odd, whereupon we have an odd α_i by considering parity conditions from the last equation. But this means that q_i is the Euler prime q, and we rewrite the equations using ${q_i}^{\alpha_i} = q^k$ and $N/{q_i}^{\alpha_i} = N/q^k = n^2$, giving $\sigma(q^k) = 2n^2$ and $\sigma(n^2) = q^k$. This contradicts Dandapat, et. al. [1] who showed in 1975 that no odd perfect number satisfies these constraints. This implies that $\rho_i \neq 1$.

Suppose $\rho_i = 2$. Then $\sigma(N/{q_i}^{\alpha_i}) = 2{q_i}^{\alpha_i}$ and $\sigma({q_i}^{\alpha_i}) = N/{q_i}^{\alpha_i}$. Since $N/{q_i}^{\alpha_i}$ is odd, then the last equation gives α_i is even. Applying the σ function to both sides of the last equation, we get $\sigma(\sigma({q_i}^{\alpha_i})) = \sigma(N/{q_i}^{\alpha_i}) = 2{q_i}^{\alpha_i}$. This last equation implies that ${q_i}^{\alpha_i}$ is superperfect. This contradicts Suryanarayana [11] who showed in 1973 that "There is no odd superperfect number of the form $p^{2\alpha}$" (where p is prime). This implies that $\rho_i \neq 2$. Since $\rho_i \in \mathbb{N}$, $\rho_i \geq 3$ and we are done. □

Corollary 6. *If an odd perfect number N is given in Eulerian form, then $q^k < (2/3)n^2$.*

Next, we define the functions $L(q)$ and $U(q)$.

Definition 7. If q is the Euler prime of an odd perfect number N given in Eulerian form, then

$$L(q) = (3q^2 - 4q + 2)/(q(q-1))$$

and

$$U(q) = (3q^2 + 2q + 1)/(q(q+1)).$$

The author obtained the following results in the same year (2008).

Lemma 8. *Let N be an odd perfect number given in Eulerian form. Then we have the bounds $L(q) < I(q^k) + I(n^2) \leq U(q)$.*

Proof. Starting from the (trivial) inequalities

$$\frac{q+1}{q} \leq I(q^k) < \frac{q}{q-1}$$

we get

$$\frac{2(q-1)}{q} < I(n^2) = \frac{2}{I(q^k)} \leq \frac{2q}{q+1}.$$

Notice that

$$\frac{q}{q-1} < \frac{2(q-1)}{q}$$

for q an Euler prime. Consequently

$$I(q^k) < I(n^2)$$

a result which was mentioned earlier in Remark 4.

Consider the product $\left(I(q^k) - \frac{q+1}{q}\right)\left(I(n^2) - \frac{q+1}{q}\right)$. This product is nonnegative since $\frac{q+1}{q} \leq I(q^k) < I(n^2)$. Expanding the product and simplifying using the equation $I(q^k)I(n^2) = 2$, we get the upper bound $U(q) = \frac{3q^2+2q+1}{q(q+1)}$ for the sum $I(q^k) + I(n^2)$.

Next, consider the product $\left(I(q^k) - \frac{q}{q-1}\right)\left(I(n^2) - \frac{q}{q-1}\right)$. This product is negative since $I(q^k) < \frac{q}{q-1} < I(n^2)$. Again, expanding the product and simplifying using the equation $I(q^k)I(n^2) = 2$, we get the lower bound $L(q) = \frac{3q^2-4q+2}{q(q-1)}$ for the same sum $I(q^k) + I(n^2)$.

A quick double-check gives that, indeed, the lower bound $L(q)$ is less than the upper bound $U(q)$, if q is an Euler prime. □

Remark 9. Notice that, from the proof of Lemma 8, we have

$$\frac{q}{q-1} < \frac{2(q-1)}{q}$$

which implies that $\left(\frac{q}{q-1}\right)^2 < 2$. Thus

$$1 < I(q^k) < \frac{q}{q-1} < \sqrt{2} = \frac{2}{\sqrt{2}} < \frac{2(q-1)}{q} < I(n^2) < 2.$$

Also, observe from Lemma 3 that

$$I(q^k) < \frac{5}{4} < \sqrt{\frac{8}{5}} < \sqrt{\frac{2}{I(q^k)}}$$

which implies that $I(q^k)\sqrt{I(q^k)} < \sqrt{2}$. It follows that

$$I(q^k) < \sqrt[3]{2}.$$

We get the rational approximation $\sqrt[3]{2} \approx 1.259921$.

We give explicit bounds for the sum $I(q^k) + I(n^2)$ in the following corollary.

Corollary 10. *Let N be an odd perfect number given in Eulerian form. Then we have the following (explicit) numerical bounds.*

$$2.85 = \frac{57}{20} < I(q^k) + I(n^2) < 3$$

with the further result that they are best-possible.

Proof. This corollary can be proved using Lemma 8 and basic differential calculus, and is left as an exercise to the interested reader. □

Remark 11. As remarked by Joshua Zelinsky a few years back, "Any improvement on the upper bound of 3 would have (similar) implications for all arbitrarily large primes and thus would be a very major result." (e.g., $U(q) < 2.99$ implies $q \leq 97$.) In this sense, the inequality

$$2.85 = \frac{57}{20} < I(q^k) + I(n^2) < 3$$

is best-possible.

Remark 12. Note that, from Lemma 8,

$$L(q) = \frac{3q^2 - 4q + 2}{q(q-1)} = 3 - \frac{q-2}{q(q-1)}$$

and

$$U(q) = \frac{3q^2 + 2q + 1}{q(q+1)} = 3 - \frac{q-1}{q(q+1)}.$$

Observe that, when $L(x)$ and $U(x)$ are viewed as functions on the domain $D = \mathbb{R} \setminus \{-1, 0, 1\}$, then

$$L(x+1) = U(x)$$

and

$$U(2) = U(3) = L(3) = \frac{17}{6} < 2.84.$$

3 Sorli's Conjecture

We now state Sorli's conjecture on odd perfect numbers.

Conjecture 13. If N is an odd perfect number with Euler prime q then $q || N$.

Remark 14. In other words, if the odd perfect number N is given in the Eulerian form $N = q^k n^2$, then Sorli's conjecture predicts that $k = \nu_q(N) = 1$. Note that, in general by Remark 4 we have

$$q^k < \sqrt{N} = q^{k/2} n$$

which gives $q^{k/2} < n$. Furthermore, the inequality $\sigma(q^k) \leq (2/3)n^2$ from Theorem 5 gives us $(\sqrt{6}/2)q^{k/2} < n$. Together, these two inequalities imply $(\sqrt{6}/2)q^k < n^2$. Note that for further reference, we get the rational approximation $\frac{\sqrt{6}}{2} \approx 1.22474487$.

We give a set of conditions equivalent to Sorli's conjecture. (In that direction, recall that the components q^k and n^2 of the odd perfect number $N = q^k n^2$ are related via the inequality $q^k < n^2$, as mentioned in Remark 4.)

We begin with the following lemma, which gives a sufficient condition for Sorli's conjecture to hold.

Lemma 15. *Let N be an odd perfect number given in Eulerian form. If $n < q$, then $k = 1$.*

Proof. If $n < q$, then by Corollary 6, $q \leq q^k < n^2 < q^2$ so $k = 1$. □

Remark 16. Via a similar argument, we get that $n < q^2$ also implies $k = 1$.

It turns out that $n < q$ is also a necessary condition for Sorli's conjecture to hold.

Lemma 17. *Let N be an odd perfect number given in Eulerian form. If Conjecture 2 is true, then Sorli's conjecture is false.*

Proof. If Conjecture 2 is true, then $q^k < n$. Suppose Sorli's conjecture is also true. Then $k = 1$. It follows that $q < n$. Thus, $k = 1$ implies that $q < n$. The contrapositive of the last statement is $n < q$ implies that $k > 1$, which contradicts Lemma 15. □

The last lemma in this section gives another condition equivalent to Sorli's conjecture.

Lemma 18. *Let N be an odd perfect number given in Eulerian form. Then $n < \sigma(q)$ if and only if $n < q$.*

Proof. Assume that $n < \sigma(q)$. If $q < n$, then $q < n < \sigma(q) = q + 1$, contradicting the fact that $n \in \mathbb{N}$. Now, assume that $n < q$. Since

$$q < q + 1 = \sigma(q),$$

it follows that $n < \sigma(q)$. □

Theorem 19. *Let N be an odd perfect number given in Eulerian form. Then Sorli's conjecture is true if and only if $\sigma(n) \leq q^k$.*

Hence if Sorli's conjecture is true, q is the largest prime which divides N.

Using a lower bound for the largest prime factor of an odd perfect number obtained by Goto and Ohno [5], we have the following theorem.

Theorem 20. *Let N be an odd perfect number given in Eulerian form. If Sorli's conjecture is true, then $q > 10^8$.*

Proof. Suppose that Sorli's conjecture is true. Then $k = 1$, and by Theorem 19, $n < q$. This means that the Euler prime q is the largest prime factor of the odd perfect number $N = q^k n^2$. By [5], we get $q > 10^8$. □

We can improve on Theorem 20 by using a recent lower bound of 10^{1500} for the magnitude of an odd perfect number obtained by Ochem and Rao.

Theorem 21. *Let N be an odd perfect number given in Eulerian form. If Sorli's conjecture is true, then $q > 10^{500}$.*

Proof. Suppose that Sorli's conjecture is true. Then $k = 1$, and by Theorem 19, $n < q$. Consequently, $n^2 < q^2$, which gives $\bar{N} = qn^2 < q^3$, where $\bar{N}$ is the same odd perfect number as $N = q^k n^2$ when $k = 1$. Since $\bar{N} > 10^{1500}$, we get $q > 10^{500}$. □

4 Conclusion

Sorli's conjecture, if proved, will enable easier computations with odd perfect numbers because then the abundancy index $I(q^k)$ for the Euler prime q collapses to $I(q) = (q+1)/q$. In addition, the Euler prime becomes the largest prime factor.

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