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ON THE GLOBAL EXISTENCE THEOREM FOR A FREE BOUNDARY PROBLEM FOR EQUATIONS OF A VISCOUS COMPRESSIBLE HEAT CONDUCTING CAPILLARY FLUID

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Abstract. We consider the motion of a viscous compressible heat conducting fluid in $\mathbb{R}^3$ bounded by a free surface which is under surface tension and constant exterior pressure. Assuming that the initial velocity is sufficiently small, the initial density and the initial temperature are close to constants, the external force, the heat sources and the heat flow vanish, we prove the existence of global–in–time solutions which satisfy, at any moment of time, the properties prescribed at the initial moment.

1. Introduction. In this paper we consider the global motion of a viscous compressible heat conducting fluid in a bounded domain $\Omega_t \subset \mathbb{R}^3$ which depends on time $t \in \mathbb{R}^1_+$. The shape of the free boundary S_t of Ω_t is governed by the surface tension.Then the problem is described by the following system with the boundary and initial conditions (see [4], Chs.2 and 5):

$$\begin{aligned}
&\rho\,[\,v_t + (v\cdot\nabla)v\,] + \nabla\, p - \mu\triangle\, v - \nu\nabla\,\mathrm{div}\, v = \rho f && \text{in } \tilde{\Omega}^T,\\
&\rho_t + \mathrm{div}\,(\rho v) = 0 && \text{in } \tilde{\Omega}^T,\\
&\rho c_v(\theta_t + v\cdot\nabla\theta) + \theta p_\theta\,\mathrm{div}\, v - \kappa\triangle\theta + && (1.1)\\
&-\frac{\mu}{2}\sum_{i,j=1}^{3}(v_{i,x_j} + v_{j,x_i})^2 - (\nu-\mu)(\,\mathrm{div}\, v)^2 = \rho r && \text{in } \tilde{\Omega}^T,\\
&\mathbf{T}\bar{n} - \sigma H\bar{n} = -p_0\,\bar{n} && \text{on } \tilde{S}^T,\\
&v\cdot\bar{n} = -\frac{\phi_t}{|\nabla\phi|} && \text{on } \tilde{S}^T,\\
&\frac{\partial\theta}{\partial n} = \theta_1 && \text{on } \tilde{S}^T,\\
&v|_{t=0} = v_0\,,\ \rho|_{t=0} = \rho_0\,,\ \theta|_{t=0} = \theta_0 && \text{in } \Omega\,,
\end{aligned}$$

where $\phi(x,t) = 0$ describes S_t, $\bar{n}$ is the unit outward vector normal to the boundary $\left(\text{i.e. } \bar{n} = \frac{\nabla\phi}{|\nabla\phi|}\right)$, $\tilde{\Omega}^T = \bigcup_{t\in(0,T)}\Omega_t \times \{t\}$, $\Omega_0 = \Omega$ is an initial domain, $\tilde{S}^T = \bigcup_{t\in(0,T)} S_t \times \{t\}$. Next, $v = v(x,t)$, $\rho = \rho(x,t)$, $\theta = \theta(x,t)$, denote the velocity, the density and the temperature of the fluid, respectively.

Moreover, $f = f(x,t)$ is the external force field per unit mass, $r = r(x,t)$ – the heat sources per unit mass, $\theta_1 = \theta_1(x,t)$ – the heat flux, $p = p(\rho,\theta)$ – the pressure, μ and ν – the viscosity coefficients, κ – the coefficient of the heat conductivity, $c_v = c_v(\rho,\theta)$ – the specific heat at constant valume, p_0 – the external (constant) pressure. We assume that coefficients μ, ν, κ are constans, $\frac{\partial p}{\partial \rho}(\rho,\theta) > 0$, $\frac{\partial p}{\partial \theta}(\rho,\theta) > 0$ and thermodynamic considerations imply that $c_v > 0$, $\kappa > 0$, $\nu \geq \frac{\mu}{3} > 0$.

Next, $\mathbf{T} = \mathbf{T}(v,p)$ denotes the stress tensor of the form

$$\mathbf{T} = \{T_{i,j}\} = \{-p\delta_{ij} + \mu(v_{i,x_j} + v_{j,x_i}) +$$

$$+(\nu-\mu)\delta_{ij}\,\mathrm{div}\,v\} \equiv \{-p\delta_{ij} + D_{ij}(v)\}\,,$$

where $i,j = 1,2,3$, $\mathbf{D}(v) = \{D_{ij}(v)\}$ is the deformation tensor.

By H we denote the double mean curvature of S_t which is negative for convex domains and can be expressed in the form

$$H\bar{n} = \triangle_{S_t}(t)x\,, \quad x = (x_1,x_2,x_3)\,,$$

where $\triangle_{S_t}(t)$ is the Laplace-Beltrami operator on S_t.

Let S_t be determined by

$$x = x(s^1,s^2,t)\,, \quad (s^1,s^2) \in I\!R^2\,. \tag{1.2}$$

Then we have

$$\triangle_{S_t}(t) = g^{-\frac{1}{2}}\frac{\partial}{\partial s^\alpha}g^{-\frac{1}{2}}\hat{g}_{\alpha\beta}\frac{\partial}{\partial s^\beta} = g^{-\frac{1}{2}}\frac{\partial}{\partial s^\alpha}g^{\frac{1}{2}}g^{\alpha\beta}\frac{\partial}{\partial s^\beta}$$

$$(\alpha,\beta = 1,2)\,,$$

where the convention summation over the repeated indices is assumed, $g = \det\{g_{\alpha\beta}\}_{\alpha,\beta=1,2}$, $g_{\alpha\beta} = x_\alpha \cdot x_\beta$ $\left(x_\alpha = \frac{\partial x}{\partial s^\alpha}\right)$, $\{g^{\alpha\beta}\}$ is the inverse matrix to $\{g_{\alpha\beta}\}$ and $\{\hat{g}_{\alpha\beta}\}$ is the matrix of algebraic complements of $\{g_{\alpha\beta}\}$.

Assume that domain Ω is given. Then by $(1.1)_5$, $\Omega_t = \{x \in I\!R^3 : x = x(\xi,t)\,, \xi \in \Omega\}$, where $x = x(\xi,t)$ is the solution of the Cauchy problem

$$\frac{\partial x}{\partial t} = v(x,t)\,, \quad x|_{t=0} = \xi\,, \quad \xi = (\xi_1,\xi_2,\xi_3)\,. \tag{1.3}$$

Hence

$$x = \xi + \int_0^t u(\xi,s)\,ds \equiv X_u(\xi,t)\,, \tag{1.4}$$

where $u(\xi,t) = v(X_u(\xi,t),t)$.

Formula (1.4) yields a relation between Eulerian x and Largangian ξ coordinates. Moreover, the kinematic boundary condition $(1.1)_5$ implies that boundary S_t is a material surface. Thus, if $\xi \in S = S_0$ then $X_u(\xi,t) \in S_t$ and $S_t = \{x : x = X_u(\xi,t)\,, \xi \in S\}$.

By the equation of continuity $(1.1)_2$ and $(1.1)_5$ the total mass M of the drop is conserved and the following relation between ρ and Ω_t

$$\int_{\Omega_t} \rho(x,t)\,dx = M$$

holds.

In the paper we prove the existence of global in time solution of problem (1.1) near a constant state. Before introducing the definition of the constant state take $\theta_e = \frac{1}{|\Omega|}\int_\Omega \theta_0\,d\xi$ and consider the equation

$$p\left(\frac{M}{\left(\frac{4}{3}\pi\right)R_e^3}, \theta_e\right) = p_0 + \frac{2\sigma}{R_e}\,. \tag{1.5}$$

We assume that equation (1.5) is solvable with respect to $R_e > 0$.

Definition 1.1. Let $f = r = \theta_1 = 0$. Then by the constant (equilibrium) state we mean a solution $(v, \theta, \rho, \Omega_t)$ of problem (1.1) such that

$$v = 0\,, \quad \theta = \theta_e\,, \quad \rho = \rho_e\,, \quad \Omega_t = \Omega_e \quad \text{for} \quad t \geq 0\,,$$

where $\rho_e = M/(4\pi/3)R_e^3$, Ω_e is a ball of radius R_e which is a solution of the equation (1.5).

The paper consists of five sections. In Section 2 we introduce some notation and auxiliary results used in the paper. In Section 3 we formulate the local existence theorem (see Theorem 3.1) proved in [19]. In Section 4 we present the differential inequality (see Theorem 4.1) derived in [24]. Section 5 is devoted to the global existence theorem (see Theorem 5.5).

The main result of the paper – Theorem 5.5 is proved under the appropriate choice of $\rho_0, v_0, \theta_0, \theta_1, p_0, \sigma, \kappa$ and the form of the internal energy per unit mass $\varepsilon = \varepsilon(\rho, \theta)$ (see assumptions (2.5) and $(5.37)_i - (5.40)$) and under the assumptions that $\phi(0) \leq \alpha_1$ and $\| H(\cdot, 0) + \frac{2}{R_e} \|^2_{2,S^1} \leq \alpha_3$ ($\phi(t)$ is given in (4.5)), where α_1 and α_3 are sufficiently small. In Theorem 5.5 we prove a global solution of (1.1) such that $(v,\, \vartheta_0,\, \vartheta,\, \rho_\sigma,\, \bar\rho_{\Omega_t}) \in \mathcal{M}(t)$ for $t \in \mathbb{R}^1_+$, ($\vartheta_0,\, \vartheta,\, \rho_\sigma,\, \bar\rho_{\Omega_t}$ are defined in (4.1) and $\mathcal{M}(t)$ is defined in the beginning of Section 5) and $S_t \in W_2^{4+\frac{1}{2}}$. The method used to prove Theorem 5.5 is similar to that in paper of W.M. Zajączkowski [29].

The global existence theorem in the case without surface tension, i.e. when $\sigma = 0$ is proved in [25], while conservation laws and a differential inequality for this case are presented in [21] and [23], respectively. Moreover, in [21] we prove that we can choose ρ_0, v_0, θ_0, θ_1, p_0, κ and the form of the internal energy per unit mass $\varepsilon = \varepsilon(\rho, \theta)$, such that $\mathrm{var}_t|\Omega_t| = \sup_t|\Omega_t| - \inf_t|\Omega_t|$ is as small as we need.

The other papers concerning problem (1.1) are [19], [20], [22] and [24].

In [19] the local in time existence and uniqueness of a solution to problem (1.1) in the anisotropic Sobolev–Slobodetskii space is proved.

In [22] conservation laws and global estimates for problem (1.1) are presented, while paper [24] is devoted to a differential inequality to problem (1.1). More precisely, papers [22] and [24] contain Lemmas 2.4, 2.5 and differential inequality (4.6) (see Theorem 4.1), which are fundamental in the proof of Theorem 5.5.

Finally, paper [20] contains the review of all results from [21]–[25] including the main result proved in this paper.

We have to underline that the case $\sigma > 0$ holds for more general data (p_0 can be equal to zero, heat sources and heat flux can be non–vanishing) and the proofs of both the local and the global existence are much more complicated because of appearing the surface tension.

In the case of a compressible barotropic fluid the corresponding drop problem was considered by W.M. Zajączkowski in [26]–[29] and by V.A. Solonnikov and A. Tani in [16]–[17].

Papers of V.A. Solonnikov [13]–[15] are concerned with the motion of a viscous incompressible fluid bounded by a free surface.

The motion of a viscous compressible heat conducting fluid in a fixed domain was examined by A. Matsumura and T. Nishida in [5]–[9] and by A. Valli and W.M. Zajączkowski in [18], while K. Pileckas and W.M. Zajączkowski proved in [11] the existence of stationary motion of a viscous compressible barotropic fluid bounded by a free surface governed by the surface tension.

Finally, papers of J.T. Beale [1]–[2] are devoted to the global existence of solution to free boundary problems, where the free boundary is infinite and the gravitation is taken into account.

2. Notation and auxiliary results. By $W_2^{l,\frac{l}{2}}(Q_T)$ (where $l \in \mathbb{R}^1_+$) we denote the anisotropic Sobolev–Slobodetskii spaces of functions (see [3]) defined in Q_T, where $Q_T = \Omega^T \equiv \Omega \times (0,T)$ ($\Omega \subset \mathbb{R}^3$ is a domain, $T < \infty$ or $T = \infty$) or $Q_T = S^T \equiv S \times (0,T)$, $S = \partial\Omega$. We define $W_2^{l,\frac{l}{2}}(\Omega^T)$ as the space of functions u such that

$$\| u \|_{W_2^{l,\frac{l}{2}}(\Omega^T)} = \Bigg[\sum_{|\alpha|+2i\le[l]} \| D_\xi^\alpha \partial_t^i u \|^2_{L_2(\Omega^T)} +$$

$$+ \sum_{|\alpha|+2i=[l]} \bigg(\int_0^T \int_\Omega \int_\Omega \frac{|D_\xi^\alpha \partial_t^i u(\xi,t) - D_{\xi'}^\alpha \partial_t^i u(\xi',t)|^2}{|\xi-\xi'|^{3+2(l-[l])}} d\xi d\xi' dt +$$

$$+ \int_\Omega \int_0^T \int_0^T \frac{|D_\xi^\alpha \partial_t^i u(\xi,t) - D_\xi^\alpha \partial_{t'}^i u(\xi,t')|^2}{|t-t'|^{1+2(\frac{l}{2}-[\frac{l}{2}])}} dt dt' d\xi \bigg) \Bigg]^{\frac{1}{2}} < \infty ,$$

where we use generalized (Sobolev) derivatives, $D_\xi^\alpha = \partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}$, $\partial_{\xi_j}^{\alpha_j} = \frac{\partial^{\alpha j}}{\partial \xi_j^{\alpha j}}$ $(j = 1,2,3)$, $\alpha = (\alpha_1,\alpha_2,\alpha_3)$ is a multiindex, $|\alpha| = \alpha_1+\alpha_2+\alpha_3$, $\partial_t^i = \frac{\partial^i}{\partial t^i}$ and $[l]$ is the integer part of l. In the case where l is integer the second terms in the above formula must be omitted, while in the case $\frac{l}{2}$ integer the last terms in the above formula must be omitted, as well.

Similarly as $W_2^{l,\frac{l}{2}}(\Omega^T)$ using local mappings and a partition of unity we introduce the normed space $W_2^{l,\frac{l}{2}}(S^T)$ of functions defined on $S^T = S \times (0,T)$, where $S = \partial\Omega$.

By $W_2^l(Q)$, where $l \in \mathbb{R}_+^1$, $Q = \Omega$ ($\Omega \in \mathbb{R}^3$ is a bounded domain) or $Q = S$, we denote usual Sobolev spaces.

To simplify notation we write

$$\| u \|_{l,Q} = \| u \|_{W_2^{l,\frac{l}{2}}(Q)} \quad \text{if} \quad Q = \Omega^T \quad \text{or} \quad Q = S^T ,$$

$$\| u \|_{l,Q} = \| u \|_{W_2^l(Q)} \quad \text{if} \quad Q = \Omega \quad \text{or} \quad Q = S .$$

Moreover, $\| u \|_{L_p(Q)} = |u|_{p,Q}$, $1 \le p \le \infty$.

To prove the global existence of solutions of problem (1.1) we need the spaces $\Gamma_k^l(\Omega)$ and $\Gamma_k^{l,\frac{l}{2}}(\Omega)$ of functions u defined on $\Omega \times (0,T)$ ($T < \infty$ or $T = \infty$) such that

$$|u|_{l,k,\Omega} \equiv \| u \|_{\Gamma_k^l(\Omega)} = \sum_{i\le l-k} \| \partial_t^i u \|_{l-i,\Omega} < \infty$$

and

$$\left| u \right|_{l,k,\Omega} \equiv \| u \|_{\Gamma_k^{l,\frac{l}{2}}(\Omega)} = \sum_{2i\le l-k} \| \partial_t^i u \|_{l-2i,\Omega} < \infty ,$$

where $l \in \mathbb{R}_+^1$, $k \ge 0$.

Next, define the space $L_p(0,T;\Gamma_0^{l,\frac{l}{2}}(\Omega))$ with the norm $\| u \|_{L_p(0,T;\Gamma_0^{l,\frac{l}{2}}(\Omega))} \equiv$ $|u|_{l,0,p,\Omega^T}$ (where $1 \le p \le \infty$).

Moreover, let $C^{2,1}(Q)\,(C_B^{2,1}(Q))\,(Q \subset \mathbb{R}^3 \times [0,+\infty))$ denotes the space of functions such that $D_x^\alpha \partial_t^i u \in C^0(Q)\,(D_x^\alpha \partial_t^i u \in C_B^0(Q))$ for $|\alpha| + 2i \le 2$ ($C_B^0(Q)$ is the space of continuous bounded functions on Q).

Finally, the following seminorm is used

$$|u|_{\kappa,Q^T} = \left(\int_0^T \frac{|u|_{2,Q}^2}{t^{2\kappa}} dt\right)^{\frac{1}{2}}, \quad \text{where} \quad Q = \partial\Omega .$$

Let X be whichever of the function spaces mentioned above. We say that a vector-valued function $u = (u_1, u_2, \dots u_\nu)$ belongs to X if $u_i \in X$ for any $1 \le i \le \nu$.

In this paper we shall use the following lemmas.

Lemma 2.1. *The following imbedding holds*

$$W_r^l(\Omega) \subset L_p^\alpha(\Omega) \ \ (\Omega \subset \mathbb{R}^3,\ \Omega \textit{ has the cone property}),$$

where $|\alpha| + \frac{3}{r} - \frac{3}{p} \le l$, $l \in \mathbf{Z}$, $1 < r \le p < \infty$; $L_p^\alpha(\Omega)$ *is the space of functions* u *such that* $|D_x^\alpha u|_{p,\Omega} < \infty$; $W_r^l(\Omega)$ *is the Sobolev space.*

Moreover, the following interpolation inequalities are true:

$$|D_x^\alpha u|_{p,\Omega} \le c\varepsilon^{1-\kappa} |D_x^l u|_{r,\Omega} + c\varepsilon^{-\kappa} |u|_{r,\Omega}, \tag{2.1}$$

where $\kappa = \frac{|\alpha|}{l} + \frac{3}{lr} - \frac{3}{lp} < 1$, $1 \le r \le p \le \infty$, ε *is a parameter,* $c > 0$ *is a constant independent of* u *and* ε;

$$|D_x^\alpha u|_{q,S} \le c\varepsilon^{1-\kappa} |D_x^l u|_{r,\Omega} + c\varepsilon^{-\kappa} |u|_{r,\Omega}, \tag{2.2}$$

where $\kappa = \frac{|\alpha|}{l} + \frac{3}{lr} - \frac{2}{lq} < 1$, $1 \le r \le q \le \infty$, ε *is a parameter,* $c > 0$ *is a constant independent of* u *and* ε.

The above lemma follows from Theorem 10.2 of [3]. The next lemma is proved in [12].

Lemma 2.2. *(see* [12]*) For a sufficiently regular* u *we have*

$$\| \partial_t^i u(t) \|_{2l-1-2i,\Omega} \le c\left(\| u \|_{2l,\Omega^T} + \| \partial_t^i u(0) \|_{2l-1-2i,\Omega}\right),$$

where $0 \le 2i \le 2l - 1$, $l \in \mathbf{N}$, $c > 0$ *is a constant independent of* T.

Now, consider problem (1.1). We assume the following condition:

(A) Ω_t is diffeomorphic to a ball, so S_t can be described by

$$|x| \equiv r = R(\omega, t), \quad \omega \in S^1, \tag{2.3}$$

where S^1 is the unit sphere.

The following lemma is true.

Lemma 2.3. *(see [29], Lemma 2.4) Let S_t be determinated by (2.3) and assume that the origin of coordinates coincides with the barycentre of Ω_t. Let $\rho(x,t)$ be a density defined for $x \in \bar{\Omega}_t$ and let $t \in [T_1, T_2]$. Assume that there exists a maximum and a minimum of the density for $t \in [T_1, T_2]$ denoted by*

$$\rho_* = \min_{t\in[T_1,T_2]} \min_{\bar{\Omega}_t} \rho\,(x,t)\,, \quad \rho^* = \max_{t\in[T_1,T_2]} \max_{\bar{\Omega}_t} \rho\,(x,t)\,.$$

Set $|\Omega^*| = M/\rho^*$, $|\Omega_*| = M/\rho_*$, $\bar{\rho}_t = M/|\Omega_t|$. Then there exists a constant $\hat{\delta} \in (0, \frac{1}{2})$ such that if

$$\sup_{S^1} |R(\omega,t) - R_t| + \sup_{S^1} |\nabla R| \le \hat{\delta} R_t\,, \quad t \in [T_1, T_2]\,, \tag{2.4}$$

where $|\nabla R|^2 = R_\theta^2 + (\sin\theta)^{-2} R_\phi^2$ in spherical coordinates, $R_t = ((3/4\pi)\ |\Omega_t|)^{\frac{1}{3}}$, then

$$\int_{S^1} \Big(|R(\omega,t) - R_t|^2 + |\nabla R(\omega,t)|^2\Big)\, d\omega \le$$

$$\le c_1 \Big(|S_t| - 4\pi R_t^2\Big) + c_2 R_t^2 |\Omega_*|^{-2} \left(|\Omega_t| - |\Omega_*|\right)^2\,,$$

where $c_1, c_2 > 0$ are constants which do not depend on $\hat{\delta}$ and R_t.

Let the internal energy per unit mass $\varepsilon = \varepsilon(\rho, \theta)$ has the form

$$\varepsilon(\rho,\theta) = a_0 \rho^\alpha + h(\rho,\theta)\,, \tag{2.5}$$

where $a_0 > 0$, $\alpha > 0$, $h(\rho,\theta) \ge h_* \ge 0$, a_0, α, h_* are constants, $h(\rho,\theta)$ is a sufficiently regular function of its arguments. Moreover, we assume that $h(\rho,\theta)$ has at (ρ_e, θ_e) (ρ_e and θ_e are introduced in Definition 1.1) the only minimum point equal to h_*, i.e. $\min_{\rho,\theta} h(\rho,\theta) = h(\rho_e,\theta_e) = h_*$.

In [22] it is shown that assumption (2.5) and the thermodynamical relation

$$d\varepsilon = \theta\, ds + \frac{p}{\rho^2}\, d\rho$$

(where s in this formula is the density of entropy per unit mass) imply the following relations between h, p and c_v

$$\alpha a_0 \rho^{\alpha+1} + \rho^2 h_\rho = p - \theta p_\theta \tag{2.6}$$

and

$$c_v = \frac{\partial \varepsilon}{\partial \theta} = h_\theta\,. \tag{2.7}$$

In order to formulate the next lemma we need some notation. Introduce

$$D = \nu_0(\nu_0 - 2\mu_0^3)\,,$$

where

$$\mu_0 = \frac{\tilde{c}\sigma(\beta - \frac{1}{3})}{3p_0\beta}\,, \quad \nu_0 = \frac{d_*(\beta - 1)}{2p_0\beta}\,, \quad \tilde{c} = (36\pi)^{\frac{1}{3}}\,,$$

$$\beta = \alpha + 1\,, \quad d_* = d - h_* M\,,$$

$$d = \int_\Omega \rho_0 \left(\frac{v_0^2}{2} + \varepsilon(\rho_0, \theta_0) \right) d\xi + p_0|\Omega| + \sigma|S| \; +$$
$$+ \; \kappa \sup_t \int_0^t dt' \int_{S_{t'}} \theta_1(s, t') ds\,.$$

Notice that $d_* > 0$ because

$$\int_\Omega \rho_0 h(\rho_0, \theta_0) - h_* M = \int_\Omega \rho_0 (h(\rho_0, \theta_0) - h_*)\, d\xi \geq 0$$

and the sum of the other terms in d_* is positive.
We have the following possibilities :

$$\nu_0 \in (2\mu_0^3, \infty) \equiv I_1\,, \quad \text{then} \quad D > 0\,; \tag{2.8}$$

$$\nu_0 \in (\mu_0^3, 2\mu_0^3] \equiv I_2\,, \quad \text{then} \quad D \leq 0\,; \tag{2.9}$$

$$\nu_0 \in (0, \mu_0^3] \equiv I_3\,, \quad \text{then} \quad D < 0\,. \tag{2.10}$$

For $\nu_0 \in I_i$ $(i = 1, 2, 3)$ we define ϕ_i $(i = 1, 2, 3)$ by

$$\cos h\phi_1 \equiv \frac{\nu_0}{\mu_0^3} - 1\,, \quad \text{where} \quad \nu_0 \in I_1\,;$$

$$\cos \phi_2 \equiv \frac{\nu_0}{\mu_0^3} - 1\,, \quad \text{where} \quad \nu_0 \in I_2\,;$$

$$\cos \phi_3 \equiv 1 - \frac{\nu_0}{\mu_0^3}\,, \quad \text{where} \quad \nu_0 \in I_3\,.$$

Next, denote

$$\Phi_1(\mu_0,\phi_1,p_0,\beta,a_0,M)\equiv\frac{p_0\mu_0^{3\beta}}{\beta-1}\left(2\cos h\frac{\phi_1}{3}-1\right)^{3(\beta-1)}\cdot$$
$$\cdot\left[2(\cos h\phi_1+1)-\frac{\beta-1}{\beta-\frac{1}{3}}\left(2\cos h\frac{\phi_1}{3}-1\right)^2\right]-a_0M^\beta\,,$$
$$\Phi_2(\mu_0,\phi_2,p_0,\beta,a_0,M)\equiv\frac{p_0\mu_0^{3\beta}}{\beta-1}\left(2\cos\frac{\phi_2}{3}-1\right)^{3(\beta-1)}\cdot \tag{2.11}$$
$$\cdot\left[2(\cos\phi_2+1)-\frac{\beta-1}{\beta-\frac{1}{3}}\left(2\cos\frac{\phi_2}{3}-1\right)^2\right]-a_0M^\beta\,,$$
$$\Phi_3(\mu_0,\phi_3,p_0,\beta,a_0,M)\equiv\frac{p_0\mu_0^{3\beta}}{\beta-1}\left[2\cos\left(\frac{\pi}{3}-\frac{\phi_3}{3}\right)-1\right]^{3(\beta-1)}\cdot$$
$$\cdot\left\{2(1-\cos\phi_3)-\frac{\beta-1}{\beta-\frac{1}{3}}\left[2\cos\left(\frac{\pi}{3}-\frac{\phi_3}{3}\right)-1\right]^2\right\}-a_0M^\beta\,.$$

In [22] the following result is proved.

Lemma 2.4. *(see* [22]*, Theorem 1) Let conditions (2.5)–(2.7) be satisfied. Let*

$$f=0\,,\quad \theta_1\ge 0\,. \tag{2.12}$$

Assume that parameters μ_0, ν_0, p_0, β, a_0, M satisfy one of the relations

$$\nu_0\in I_i\,,\quad 0<\Phi_i(\mu_0,\phi_i,p_0,\beta,a_0,M)\le\delta_0\,, \tag*{(2.13)$_i$}$$

where i=1,2,3, I_i are defined in (2.8)-(2.10), Φ_i are given by (2.11) and $\delta_0>0$ is a sufficiently small constant. Then there exists a constant c_1 independent of δ_0 (it can depend on the parameters) such that

$$var_t|\Omega_t|\le c_1\delta\,,$$

where $c_1>0$ is a constant, $\delta^2=c\delta_0$, $c>0$ is a constant, $var_t|\Omega_t|=\sup_t|\Omega_t|-\inf_t|\Omega_t|$, $t\in[0,T]$, T is the time of the existence of solution (v,ρ,θ) of (1.1). Moreover, in the case $(2.13)_i$ we have

$$||\Omega_t|-Q_i|\le c_2\delta\quad\forall t\in[0,T]\,, \tag*{(2.14)$_i$}$$

where i =1,2,3, $Q_1=\mu_0^3\left(2\cos h\frac{\phi_1}{3}-1\right)^3$, $Q_2=\mu_0^3\left(2\cos\frac{\phi_2}{3}-1\right)^3$, $Q_3=\mu_0^3\left[2\cos\left(\frac{\pi}{3}-\frac{\phi_3}{3}\right)-1\right]^3$, $c_2>0$ is a constant.

Next, we have

Lemma 2.5. *Let assumptions of Lemmas 2.3 and 2.4 be satisfied. Then*

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_t}\rho v^2\,dx + a_0\left(\int_{\Omega_t}\rho^\beta\,dx - \inf_t\int_{\Omega_t}\rho^\beta\,dx\right)+\\
&+\int_{\Omega_t}\rho(h(\rho,\theta)-h_*)\,dx + p_0(|\Omega_t|-|\Omega_*|)+\sigma(|S_t|-|S_*|)\le\\
&\le\frac{1}{2}\int_\Omega\rho_0 v_0^2\,d\xi + a_0\left(\int_\Omega\rho_0^\beta\,d\xi - \frac{M}{(c_2\delta+Q_i)^{\beta-1}}\right)+\\
&+\int_\Omega\rho_0(h(\rho_0,\theta_0)-h_*)\,d\xi + p_0[|\Omega|-(Q_i-c_2\delta)]+\\
&+\sigma[|S|-\tilde c(Q_i-c_2\delta)^{\frac{2}{3}}]+\kappa\sup_t\int_0^t dt'\int_{S_{t'}}\theta_1(s,t')\,ds,
\end{aligned}$$

where $|\Omega_*| = \inf_t|\Omega_t|$, $|S_*| = 4\pi R_*^2$ *(*R_* *is determined by* $\frac{4}{3}\pi R_*^3 = |\Omega_*|$*),* $\tilde c = (36\pi)^{\frac{1}{3}}$, c_2 *and* δ *are constants from (2.14)*$_i$*;* $t\in[0,T]$, T *is the time of the existence of solution* (v,ρ,θ) *of (1.1). Moreover, if*

$$\begin{aligned}
&\frac{1}{2}\int_\Omega\rho_0 v_0^2\,d\xi + a_0\left(\int_\Omega\rho_0^\beta\,d\xi - \frac{M}{(c_2\delta+Q_i)^{\beta-1}}\right)+\\
&+\int_\Omega\rho_0\left(h(\rho_0,\theta_0)-h_*\right)\,d\xi + \rho_0\left[|\Omega|-(Q_i-c_2\delta)\right]+ \qquad (2.15)\\
&+\sigma\left[|S|-\tilde c(Q_i-c_2\delta)^{\frac{2}{3}}\right]+\kappa\sup_t\int_0^t dt'\int_{S_{t'}}\theta_1(s,t')\,ds\le\tilde\delta,
\end{aligned}$$

then

$$\| R(\omega,t)-R_t\|_{1,S^1}\le\varepsilon_0\,, \qquad (2.16)$$

where $R_t = \left(\frac{3}{4\pi}|\Omega_t|\right)^{\frac{1}{3}}$, $\varepsilon_0=\varepsilon_0(\tilde\delta)$, $\varepsilon_0\to 0$ *as* $\tilde\delta\to 0$.

The proof of the above lemma is presented in [22] (see Corollary 1).

Remark 2.6. In [22] it is proved that there exist μ_0, ν_0, p_0, β, a_0, M such that assumption *(2.13)*$_i$ is satisfied. For example if $\nu_0 = 2\mu_0^3$ then $0<\Phi_2\le\delta_0$ if β is sufficiently close to 1. Similarly, if $\nu_0=\mu_0^3$ then $0<\Phi_3\le\delta_0$ for β sufficiently close to 1.

Remark 2.7. Since $a_0\left(\int_\Omega\rho_0^\beta\,d\xi - \frac{M^\beta}{(c_2\delta+Q_i)^{\beta-1}}\right)\to 0$ as $\beta\to 1$ we see that for β sufficiently close to 1 $a_0\left(\int_\Omega\rho_0^\beta\,d\xi - \frac{M^\beta}{(c_2\delta+Q_i)^{\beta-1}}\right)$ is as small as needed.

In the case $p_0 = 0$ instead of Lemmas 2.4 and 2.5 we obtain the following lemmas.

Lemma 2.8. *(see* [22], *Theorem 2) Let $p_0 = 0$ and let conditions (2.5)–(2.7) and (2.12) be satisfied. Moreover, assume that the following relations hold:*

$$\int_\Omega \rho_0 \frac{v_0^2}{2}\, d\xi + \int_\Omega \rho_0 \left(h(\rho_0,\theta_0) - h_*\right)\, d\xi + \tag{2.17}$$

$$+\kappa \sup_t \int_0^t dt' \int_{S_{t'}} \theta_1(s,t')\, ds \le \delta_0\,,$$

$$\int_\Omega |\rho_e - \rho_0|\, d\xi \le \delta_0\,, \tag{2.18}$$

$$||S| - |S_e|| \le \delta_0\,, \tag{2.19}$$

$$0 < \left[\frac{2}{3}(\beta-1)^{\frac{3(\beta-1)}{2}} \left(\beta - \frac{1}{3}\right)^{\frac{-(3\beta-1)}{2}} (\tilde{c}\sigma)^{\frac{-3(\beta-1)}{2}} \cdot \right. \tag{2.20}$$

$$\left. \cdot \left(a_0\rho_e^\beta + \tilde{c}\sigma|\Omega_e|^{-\frac{1}{3}}\right)^{\frac{3\beta-1}{2}} |\Omega_e|^{\frac{\beta-1}{2}} - a_0\rho_e^\beta\right] |\Omega_e|^\beta \le \delta_0\,,$$

where $\delta_0 > 0$ is a sufficiently small constant, $|S_e| = 4\pi R_e^2$, ρ_e, R_e, Ω_e are introduced in Definition 1.1. Then

$$var_t|\Omega_t| \le c_3\delta\,,$$

where $t \in [0,T]$ (T is the time of the existence of solution (v,ρ,θ) of (1.1)), $c_3 > 0$ is a constant independent of δ_0, $\delta^2 = c\delta_0$, $c > 0$ is a constant. where $c_3 > 0$ is a constant independent of δ_0, $\delta^2 = c\delta_0$, $c > 0$ is a constant.

Lemma 2.9. *(see* [22], *Corollary 2) Let assumptions of Lemmas 2.8 and 2.3 be satisfied. Then*

$$\frac{1}{2}\int_{\Omega_t} \rho v^2\, dx + a_0\left(\int_{\Omega_t} \rho^\beta\, dx - \inf_t \int_{\Omega_t} \rho^\beta\, dx\right) +$$

$$+\int_{\Omega_t} \rho\left(h(\rho,\theta) - h_*\right)\, dx + \sigma\left(|S_t| - |S_*|\right) \le$$

$$\le \frac{1}{2}\int_\Omega \rho_0 v_0^2\, d\xi + a_0\left(\int_\Omega \rho_0^\beta\, d\xi - \frac{M^\beta}{(c_2\delta + Q_i)^{\beta-1}}\right) +$$

$$+\int_\Omega \rho_0\left(h(\rho_0,\theta_0)-h_*\right)d\xi+\sigma\left[|S|-\tilde{c}(Q_i-c_2\delta)^{\frac{2}{3}}\right]+$$

$$+\kappa\sup_t\int_0^t dt'\int_{S_{t'}}\theta_1(s,t')\,ds\,,$$

where $|S_*|$ is defined in Lemma 2.5, $\tilde{c}=(36\pi)^{\frac{1}{3}}$, c_2 and δ are constants from (2.14)$_i$. Moreover, if

$$\frac{1}{2}\int_\Omega \rho_0 v_0^2\,d\xi+a_0\left(\int_\Omega \rho_0^\beta\,d\xi-\frac{M^\beta}{(c_2\delta+Q_i)^{\beta-1}}\right)+$$

$$+\int_\Omega \rho_0\left(h(\rho_0,\theta_0)-h_*\right)d\xi+\sigma\left[|S|-\tilde{c}(Q_i-c_2\delta)^{\frac{2}{3}}\right]+$$

$$+\kappa\sup_t\int_0^t dt'\int_{S_{t'}}\theta_1(s,t')\,ds\ \le\tilde{\delta},$$

then estimate (2.16) holds.

Remark 2.10. It is shown in [22] that there exist β, σ, a_0, ρ_e, $|\Omega_e|$ such that assumption (2.20) is satisfied with δ_0 as small as we wish.

The double curvature of S_t in spherical coordinates has the form

$$\mathcal{H}=\frac{1}{R\sin\theta}\left(\frac{\partial}{\partial\phi}\frac{R_\phi}{\sin\theta\sqrt{R^2+|\nabla R|^2}}+\frac{\partial}{\partial\theta}\frac{R_\phi\sin\theta}{\sqrt{R^2+|\nabla R|^2}}\right)+$$

$$-\frac{2}{\sqrt{R^2|\nabla R|^2}}\,.$$

Consider the equation

$$\mathcal{H}(R)+\frac{2}{R_t}=h(\omega)\,, \tag{2.21}$$

where $h(\omega)=\frac{1}{\sigma}\bar{n}\mathbf{T}(v,p_\sigma)\bar{n}|_{S_t}$, $p_\sigma=p-p_0-\frac{2\sigma}{R_t}$. From Theorem 2.4 in [13] we have

Theorem 2.11. *Let $R\in W_2^{l+\frac{3}{2}}(S^1)$ $(l\in[k+\frac{1}{2},k+1]\,,\ k\in\mathbf{Z}_+\cup\{0\})$ be a solution to (2.21) which satisfies (2.4) with sufficiently small $\hat{\delta}$. If $h\in W_2^\mu(S^1)\,,\ \mu\in[k,k+1]$, then*

$$\| R-R_t\|_{2+\mu,S^1}\le c_1\| h\|_{\mu,S^1}+c_2\| R-R_t\|_{0,S^1}\ .$$

3. Local existence. In order to prove the local existence of solutions to problem (1.1) we rewrite it in the Lagrangian coordinates introduced by (1.3) and (1.4) :

$$
\begin{aligned}
&\eta u_t - \mu\nabla_u^2 u - \nu\nabla_u\nabla_u\cdot u + \nabla_u p(\eta,\Gamma) = \eta g \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \Omega^T \equiv \Omega\times(0,T)\,,\\
&\eta_t + \eta\nabla_u\cdot u = 0 \qquad\qquad \text{in } \Omega^T\,,\\
&\eta c_v(\eta,\Gamma)\Gamma_t - \kappa\nabla_u^2\Gamma = -\Gamma p_\Gamma(\eta,\Gamma)\nabla_u\cdot u + \\
&+\frac{\mu}{2}\sum_{i,j=1}^3\Big(\xi_{x_i}\cdot\nabla_\xi u_j + \xi_{x_j}\cdot\nabla_\xi u_i\Big)^2 + \\
&+(\nu-\mu)(\nabla_u\cdot u)^2 = \eta k \qquad\qquad \text{in } \Omega^T\,,\\
&\mathbf{T}_u(u,p)\bar n - \sigma\triangle_{S_t}(t)X_u(\xi,t) = -p_0\bar n \qquad \text{on } S^T\,,\\
&\bar n\cdot\nabla_u\Gamma = \Gamma_1 \qquad\qquad \text{on } S^T\,,\\
&u|_{t=0} = v_0\,,\ \eta|_{t=0} = \rho_0\,,\ \Gamma|_{t=0} = \theta_0 \qquad \text{in } \Omega\,,
\end{aligned}
\tag{3.1}
$$

where $u(\xi,t) = v(X_u(\xi,t),t)$, $\Gamma(\xi,t) = \theta(X_u(\xi,t),t)$, $\eta(\xi,t) = \rho(X_u(\xi,t),t)$, $g(\xi,t) = f(X_u(\xi,t),t)$, $k(\xi,t) = r(X_u(\xi,t),t)$, $\nabla_u = \xi_x\nabla_\xi \equiv \xi_{ix}\partial_{\xi_i}$, $\mathbf{T}_u(u,p) = -pI + \mathbf{D}_u(u)$, $\mathbf{D}_u(u) = \{\,\mu\,(\,\xi_{kx_i}\partial_{\xi_k}u_j + \xi_{kx_j}\partial_{\xi_k}u_i\,) + (\nu-\mu)\delta_{ij}\nabla_u\cdot u\}$ (here the summation convention over the repeated indices is assumed), $\Gamma_1(\xi,t) = \theta_1(X_u(\xi,t),t)$.

Let $A = \{a_{ij}\}$ be the Jacobi matrix of transformation $x = X_u(\xi,t)$, where $a_{ij} = \delta_{ij} + \int_0^t \partial_{\xi_j}u_i(\xi,t')\,dt'$. Assuming that $|\nabla_\xi u|_{\infty,\Omega^T} \le M$ we obtain

$$0 < c_1(1-Mt)^3 \le \det\{x_\xi\} \le c_2(1+Mt)^3\,, \quad t \le T\,,$$

where $c_1, c_2 > 0$ are constants and $T > 0$ is sufficiently small. Moreover, $\det A = \exp\left(\int_0^t \nabla_u\cdot u\,dt'\right) = \rho_0/\eta$.

Let S_t be determinated (at least locally) by the equation $\phi(x,t) = 0$. Then S is described by $\phi(x(\xi,t),t)|_{t=0} = \tilde\phi(\xi) = 0$. Thus, we have

$$\bar n(x(\xi,t),t) = -\frac{\nabla_x\phi(x,t)}{|\nabla_x\phi(x,t)|}\bigg|_{x=x(\xi,t)}$$

and

$$\bar{n}_0(\xi) = -\frac{\nabla_\xi \tilde{\phi}(\xi,t)}{|\nabla_\xi \tilde{\phi}(\xi)|}\,.$$

Now, we formulate the local existence theorem.

Theorem 3.1. *(see* [19]*, Theorem 4.2) Let* $S \in W_2^{4+\frac{1}{2}}$, $f \in C^{2,1}(\mathbb{R}^3 \times [0,T])$, $r \in C^{2,1}(\mathbb{R}^3 \times [0,T])$, $\theta_1 \in C^{2,1}(\mathbb{R}^3 \times [0,T])$, $v_0 \in W_2^3(\Omega)$, $\theta_0 \in W_2^3(\Omega)$, $\frac{1}{\theta_0} \in L_\infty(\Omega)$, $\theta_0 > 0$, $\rho_0 \in W_2^3(\Omega)$, $\frac{1}{\rho_0} \in L_\infty(\Omega)$, $\rho_0 > 0$, $c_v \in C^2(\mathbb{R}_+^2)$, $c_v > 0$, $p \in C^3(\mathbb{R}_+^2)$. *Moreover, assume that the following compatibility conditions are satisfied:*

$$D_\xi^\alpha\left[\mathbf{D}_\xi(v_0)\bar{n}_0 - p(\rho_0,\theta_0)\bar{n}_0 - \sigma\Delta_{S_t}(\theta)\xi\right] = -D^\alpha(p_0\bar{n}_0)\,, \quad |\alpha| \le 1\,, \quad \textit{on } S$$

and

$$D_\xi^\alpha(\bar{n}_0\cdot\nabla_\xi\theta_0) = D_\xi^\alpha(\theta_1(\xi,0))\,, \quad |\alpha|\le 1\,, \quad \textit{on } S.$$

Let $T^* > 0$ *be so small that* $0 < c_1(1-CK_0T^*)^3 \le \det\{x_\xi\} \le c_2(1+CK_0T^*)^3$ *(where* $x(\xi,t) = \xi + \int_0^t u_0(\xi,t')\,dt'$ *for* $t \le T^*$, u_0 *is given by (3.74) from* [19], $K_0 \le c\,(\| \rho_0 \|_{3,\Omega} + |\rho_0|_{\infty,\Omega} + |\frac{1}{\rho_0}|_{\infty,\Omega} + \| v_0 \|_{3,\Omega} + \| \theta_0 \|_{3,\Omega} + \| u_t(0) \|_{1,\Omega} + \| \Gamma_t(0) \|_{1,\Omega})$, $c > 0$ *is a constant,* $C = C(K_0)$ *is a nondecreasing continuous function of* K_0 *given by (3.94) from* [19]*). Then there exists* T^{**}, $0 < T^{**} \le T^*$ *such that for* $T \le T^{**}$ *there exists a unique solution* $(u,\Gamma,\eta) \in W_2^{4,2}(\Omega^T) \times W_2^{4,2}(\Omega^T) \times C^0(0,T;\Gamma_0^{3,\frac{3}{2}}(\Omega))$ *of problem (3.1). Moreover,*

$$\eta_t \in C^0(0,T;W_2^2(\Omega)) \cap L_2(0,T;W_2^3(\Omega))\,,$$

$$\eta_{tt} \in L_2(0,T;W_2^1(\Omega))$$

and

$$\| u \|_{4,\Omega^T} + \| \Gamma \|_{4,\Omega^T} \le CK_0\,,$$

$$\sup_t \| \eta \|_{3,\Omega} + \sup_t \| \eta_t \|_{2,\Omega} + \| \eta_t \|_{L_2(0,T;W_2^3(\Omega))} +$$

$$+ \| \eta_{tt} \|_{L_2(0,T;W_2^1(\Omega))} \le \Phi_1(T, T^a K_0) \| \rho_0 \|_{3,\Omega}\,,$$

$$\left|\frac{1}{\eta}\right|_{\infty,\Omega^T} + |\eta|_{\infty,\Omega^T} \le \Phi_2\left(T^{\frac{1}{2}}K_0\right)\left|\frac{1}{\rho_0}\right|_{\infty,\Omega} +$$

$$+ \Phi_3\left(T^{\frac{1}{2}}K_0\right)|\rho_0|_{\infty,\Omega}\,,$$

where Φ_1 *and* Φ_2 *are increasing continuous functions,* $a > 0$.

In order to consider the global existence we need

Remark 3.2. Assume that $g = 0$. Denoting $p_\sigma = p - p_0 - q_0$, $\gamma_0 = \Gamma - \theta_e$, $\eta_\sigma = \eta - \rho_e$ (where q_0, θ_e and ρ_e are introduced in Definition 1.1). Then problem (1.1) can be written in the form

$$\begin{aligned}
&\eta u_t - \mu\nabla_u^2 u - \nu\nabla_u\nabla_u\cdot u + \nabla_u p_\sigma = 0\,,\\
&\eta_{\sigma t} + \eta\nabla_u\cdot u = 0\,,\\
&\eta c_v(\eta,\Gamma)\gamma_{0t} - \kappa\nabla_u^2\gamma_0 + \Gamma p_\Gamma(\eta,\Gamma)\nabla_u\cdot u\ +\\
&+\frac{\mu}{2}\sum_{i,j=1}^{3}\left(\xi_{x_i}\cdot\nabla_\xi u_j + \xi_{x_j}\cdot\nabla_\xi u_i\right)^2 +\\
&+(\nu-\mu)(\nabla_u\cdot u)^2 = \eta k\,,\\
&\Pi_0\Pi\mathbf{D}_u(u)\bar n = 0\,,\\
&\bar n_0\mathbf{D}_u(u)\bar n - \sigma\triangle(t)\int_0^t u(t')\,dt' = \bar n_0\cdot\bar n p_\sigma\ +\\
&+\sigma\bar n_0\left(\triangle(t)-\triangle(0)\right)\xi + \sigma\left(H(\xi,0)+\frac{2}{R_e}\right),\\
&\bar n\cdot\nabla_u\gamma_0 = \Gamma_1\,,\\
&u|_{t=0} = v_0\,,\ \eta_\sigma|_{t=0} = \rho_{\sigma0}\,,\ \gamma_0|_{t=0} = \vartheta_{00}\,,
\end{aligned}\tag{3.2}$$

where Π_0 and Π are the projections defined by $\Pi_0 g = g - (g\cdot\bar n_0)\bar n_0$ and $\Pi g = g - (g\cdot\bar n)\bar n$, $\rho_{\sigma0} = \rho_0 - \rho_e$, $\vartheta_{00} = \theta_0 - \theta_e$ and R_e is defined in Definition 1.1.

Let the assumptions of Theorem 3.1 be satisfied and let (u,Γ,η) be the corresponding local solution of problem (3.1). Then by Theorems 3.6, 4.1 and Lemma 3.3 from [19] for solution (u,γ_0,η_σ) of (3.2) such that

$$T^a\left(\|\, u\,\|_{4,\Omega^T} + \|\, v_0\,\|_{3,\Omega} + \|\,\vartheta_{00}\,\|_{3,\Omega} + \|\,\rho_{\sigma0}\,\|_{3,\Omega}\right)\phi_1(T,K_0)\le\delta$$

(where $a > 0$ is a constant, ϕ_1 is an increasing continuous function of its arguments, $\delta > 0$ is sufficiently small) the following estimate holds

$$\| u \|_{4,\Omega^T} + \| \eta_\sigma \|_{3,\Omega^T} + |\eta_\sigma|_{3,0,\infty,\Omega^T} + \| \gamma_0 \|_{4,\Omega^T} \leq$$

$$\leq \phi_2(T, K_0) \Big(\| v_0 \|_{3,\Omega} + \| \rho_{\sigma 0} \|_{3,\Omega} + \| \vartheta_{00} \|_{3,\Omega} + \| k \|_{2,\Omega^T} +$$

$$+ \| \Gamma_1 \|_{3-\frac{1}{2},S^T} + |D^2_{\xi,t}\Gamma_1|_{\frac{1}{4},S^T} + \| k(0) \|_{1,\Omega} +$$

$$+ \| H(\xi,0) + \frac{2}{R_e} \|_{2+\frac{1}{2},S^1} \Big) ,$$

where $\phi_2(T, K_0)$ is an increasing continuous function of its arguments.

4. Differential inequality. In order to prove the global existence of solution to problem (1.1) we need the differential inequality obtained in [24] (see Theorem 3.13). Assume that the existence of a sufficiently smooth local solution of problem (1.1) has been proved and consider the motion near the constant state (see Definition 1.1) $v_e = 0$, $p_e = p_0 + \frac{2\sigma}{R_e}$, $\theta_e = \frac{1}{|\Omega|}\int_\Omega \theta_0 \, d\xi$, $\rho_e = \frac{M}{(4\pi/3)R_e^3}$, where R_e is a solution of the equation

$$p\left(\frac{M}{(4\pi/3)R_e^3}, \theta_e\right) = p_e .$$

Let

$$p_\sigma = p - p_0 - q_0 , \quad \rho_\sigma = \rho - \rho_e , \quad \vartheta_0 = \theta - \theta_e , \quad \vartheta = \theta - \theta_{\Omega_t} , \tag{4.1}$$

$$\bar\rho_{\Omega_t} = \rho - \rho_{\Omega_t} ,$$

where $q_0 = \frac{2\sigma}{R_e}$, $\theta_{\Omega_t} = \frac{1}{|\Omega_t|}\int_{\Omega_t} \theta \, dx$, $\rho_{\Omega_t} = \rho_{\Omega_t}(t)$ is a solution of the problem

$$p(\rho_{\Omega_t}, \theta_{\Omega_t}) = p_0 , \quad \rho_{\Omega_t}|_{t=0} = \rho_e .$$

We impose the above initial condition on ρ_{Ω_t} because $\theta_{\Omega_t}|_{t=0} = \theta_e$ and $p(\rho_e, \theta_e) = p_e$.

Notice that using the Taylor formula p_σ can be written as (see [24])

$$p_\sigma = p_1\rho_\sigma + p_2\vartheta_0 \tag{4.2}$$

and

$$p_\sigma = p_3\bar\rho_{\Omega_t} + p_4\vartheta , \tag{4.3}$$

where

$$p_1 = p_1(\rho,\theta) = \int_0^1 p_\rho\,(\rho_e + s(\rho-\rho_e),\theta)\,ds\,,$$

$$p_2 = p_2(\rho,\theta) = \int_0^1 p_\theta\,(\rho_e,\,\theta_e + s(\theta-\theta_e))\,ds\,,$$

$$p_3 = p_3(\rho_{\Omega_t},\rho,\theta) = \int_0^1 p_\rho\,(\rho_{\Omega_t} + s(\rho-\rho_{\Omega_t}),\theta)\,ds\,,$$

$$p_4 = p_4(\rho_{\Omega_t},\theta_{\Omega_t},\theta) = \int_0^1 p_\theta\,(\rho_{\Omega_t},\,\theta_{\Omega_t} + s(\theta-\theta_{\Omega_t}))\,ds\,.$$

By (4.1) problem (1.1) takes the form

$$\begin{aligned}
&\rho\,[v_t + (v\cdot\nabla)\,v] - \operatorname{div}\mathbf{T}(v,p_\sigma) = \rho f && \text{in}\,\Omega_t\,,\ t\in[0,T]\,,\\
&\rho_t + \operatorname{div}(\rho v) = 0 && \text{in}\,\Omega_t\,, t\in[0,T]\,,\\
&\rho c_v(\rho,\theta)(\vartheta_{0t} + v\cdot\nabla\vartheta_0) + \theta p_\theta(\rho,\theta)\ \operatorname{div}\,v\ + \\
&-\kappa\triangle\vartheta_0 - \frac{\mu}{2}\sum_{i,j=1}^3(\partial_{x_i}v_j\ + \partial_{x_j}v_i)^2\ + \\
&-(\nu-\mu)(\operatorname{div}\,v)^2 = \rho r && \text{in}\,\Omega_t\,,\ t\in[0,T]\,,\\
&\mathbf{T}(v,p_\sigma)\bar n = \sigma\triangle_{S_t}x\cdot\bar n\bar n + q_0\bar n && \text{on}\,S_t\,,\ t\in[0,T]\,,\\
&\frac{\partial\vartheta_0}{\partial n} = \theta_1 && \text{on}\,S_t\,,\ t\in[0,T]\,,
\end{aligned}\tag{4.4}$$

where $\mathbf{T}(v,p_\sigma) = \{\mu\,(\partial_{x_i}v_j + \partial_{x_j}v_i\,) + (\nu-\mu)\delta_{ij}\ \operatorname{div}\,v - p_\sigma\delta_{ij}\}$ and T is the time of the local existence.

Denote

$$\phi(t) = \int_{\Omega_t}\rho\sum_{0\le|\alpha|+i\le 3}|D_x^\alpha\partial_t^i v|^2\,dx + \int_{\Omega_t}\left(\frac{p_1}{\rho}\rho_\sigma^2 + \bar\rho_{\Omega_t}^2 + \frac{p_2\rho c_v}{p_\theta\theta}\vartheta_0^2\right)dx +$$

$$+\int_{\Omega_t}\frac{p_{\sigma\rho}}{\rho}\sum_{1\le|\alpha|+i\le 3}|D_x^\alpha\partial_t^i\rho_\sigma|^2\,dx + \int_{\Omega_t}\frac{\rho c_v}{\theta}\sum_{1\le|\alpha|+i\le 3}\left|D_x^\alpha\partial_t^i\vartheta_0\right|^2dx +$$

$$+\frac{\sigma}{2}\int_{S_t}\tilde{\delta}^{\alpha\beta}\sum_{|k|\le 2}\bar{n}\cdot\int_0^t\partial_p^k v_{p^\gamma p^\alpha}\,dt'\bar{n}\cdot\int_0^t\partial_p^k v_{p^\gamma p^\beta}\,dt'\,ds\ + \tag{4.5}$$

$$+\frac{\sigma}{2}\int_{S_t}\sum_{|k|\le 2}|\bar{n}\cdot\int_0^t\partial_p^k v_{p^1p^2}\,dt'|^2\,ds\ +$$

$$+\frac{\sigma}{2}\int_{S_t}\sum_{|k|\le 2}\sum_{i=1}^{2}\left(\frac{1}{2}\bar{n}\cdot\int_0^t\partial_p^k v_{p^ip^i}\,dt'\ +2\partial_p^k\left(H(\cdot,0)+\frac{2}{R_e}\right)\right)^2 ds\ +$$

$$+\frac{\sigma}{2}\int_{S_t}g^{\alpha\beta}\left(\sum_{1\le|k|\le 2}D_p^k v_{p^\alpha}\cdot\bar{n}D_p^k v_{p^\beta}\cdot\bar{n}\ +\bar{n}\cdot v_{tpp^\alpha}\bar{n}\cdot v_{tpp^\beta}\right)ds +$$

$$+\frac{\sigma}{2}\int_{S_t}g^{\alpha\beta}\ \left(\bar{n}\cdot\int_0^t v_{s^\alpha}\,dt'\,\bar{n}\cdot\int_0^t v_{s^\beta}\,dt'+\right.$$

$$\left.+\sum_{|k|\le 2}D_t^k v_{s^\alpha}\cdot\bar{n}D_t^k v_{s^\beta}\cdot\bar{n}\ \right)ds\,,$$

$$\Phi(t)=|v|^2_{4,1,\Omega_t}+|\rho_\sigma|^2_{3,0,\Omega_t}-\parallel\rho_\sigma\parallel^2_{0,\Omega_t}+\parallel\bar{\rho}_{\Omega_t}\parallel^2_{0,\Omega_t}+$$

$$+\ |\vartheta_0|^2_{4,1,\Omega_t}-\parallel\vartheta_0\parallel^2_{0,\Omega_t}+\parallel\vartheta\parallel^2_{0,\Omega_t}\ ,$$

$$\psi(t)=\parallel v\parallel^2_{0,\Omega_t}+\parallel p_\sigma\parallel^2_{0,\Omega_t}+\parallel R(\cdot,t)-R(\cdot,0)\parallel^2_{0,S^1}\ ,$$

$$F(t)=\parallel f_{ttt}\parallel^2_{0,\Omega_t}+|f|^2_{2,0,\Omega_t}+\parallel r_{ttt}\parallel^2_{0,\Omega_t}+|r|^2_{2,0,\Omega_t}+$$

$$+\ \parallel r\parallel_{0,\Omega_t}+|\theta_1|^2_{4,1,\Omega_t}+\parallel\theta_1\parallel_{1,\Omega_t}\ ,$$

where the summation over the repeated indices $(\alpha,\beta,\gamma=1,2)$ and coordinates x is assumed, $\tilde{\delta}^{\alpha\beta}$ on each part of boundary $\sum_t S_t\cap\{\zeta(x)\neq 0\}$ (ζ belongs to a partition of unity of Ω_t) is of the form $\tilde{\delta}^{\alpha\beta}=\delta^{\alpha\beta}+2\varepsilon^{\alpha\beta}$, $\varepsilon^{\alpha\beta}=-\bar{F}_{p^\alpha}\bar{F}_{p^\beta}(1+\bar{F}^2_{p^1}+\bar{F}^2_{p^2})^{-1}$, $\bar{F}$ is the function such that in the local coordinates $\{y\}$, $\sum_t$ is described by the formula : $y_i=p^i$, $i=1,2$, $y_3=\bar{F}(p^1,p^2,t)$ and assuming that supp$\{\zeta\}$ is sufficiently small we have that $|\bar{F}_p|\le\frac{1}{2}$. In the last two terms on the right – hand side of $\phi(t)$ boundary S_t is determined by (1.4).

Now, we formulate the following theorem (see [24], Theorem 3.13).

Theorem 4.1. *Let assumption (A) be satisfied and let* $\nu > \frac{1}{3}\mu$. *Then for a sufficiently smooth solution of problem (4.4) we have*

$$\begin{aligned}
&\frac{d\phi}{dt} + c_0\Phi \le c_1 P(X)X(1+X^3)(X+Y) + c_2 F + \\
&+ c_3\psi + c_4 \| H(\cdot,0) + \frac{2}{R_e} \|^4_{2,S^1} + \varepsilon c_5 \Big(\| H(\cdot,0) + \frac{2}{R_e} \|^2_{2,S^1} + \\
&+ \| R(\cdot,t) - R(\cdot,0) \|^2_{4,S^1} \Big) + \qquad (4.6)\\
&+ c_6 \bigg(\| R(\cdot,t) - R(\cdot,0) \|^2_{4+\frac{1}{2},S^1} \left\| \int_0^t v\,dt' \right\|^2_{3,S_t} + \\
&+ \| R(\cdot,t) - R(\cdot,0) \|^2_{3,S^1} \left\| \int_0^t v\,dt' \right\|^2_{4,S_t} \bigg),
\end{aligned}$$

where $t \in [0,T]$ *(T is the time of the local existence),* $0 < c_0 < 1$ *is a constant depending on* $\rho_*, \rho^*, \theta_*, \theta^*$ ($\rho_* = \inf_{t\in(0,T)} \inf_{\Omega_t} \rho(x,t)$, $\rho^* = \sup_{t\in(0,T)} \sup_{\Omega_t} \rho(x,t)$, $\theta_* = \inf_{t\in(0,T)} \inf_{\Omega_t} \theta(x,t)$, $\theta^* = \sup_{t\in(0,T)} \sup_{\Omega_t} \theta(x,t)$*),* μ, ν *and* κ, c_i $(i = 1,\dots,6)$ *depend on* ρ_*, ρ^*, θ_*, θ^*, T, $\int_0^T \| v \|^2_{3,\Omega_{t'}} dt'$, $\| S \|_{4+\frac{1}{2}}$, *constants from the imbedding lemma and the Korn inequalities (see Section 5 in* [26] *),* ε *is a small parameter and*

$$X = |v|^2_{3,0,\Omega_t} + |\rho_\sigma|^2_{3,0,\Omega_t} + |\vartheta_0|^2_{3,0,\Omega_t} + \| \bar\rho_{\Omega_t} \|^2_{0,\Omega_t} + \int_0^t \| v \|^2_{3,\Omega_{t'}}\, dt',$$

$$\begin{aligned}
Y = {} & |v|^2_{4,1,\Omega_t} + |\rho_{\sigma t}|^2_{2,0,\Omega_t} + \| \rho_{\sigma x} \|^2_{2,\Omega_t} + |\vartheta_{0t}|^2_{3,1,\Omega_t} + \| \vartheta_{0x} \|^2_{3,\Omega_t} + \\
& + \| \vartheta \|^2_{0,\Omega_t} + \| \bar\rho_{\Omega_t} \|^2_{0,\Omega_t} + \int_0^t \| v \|^2_{4,\Omega_{t'}}\, dt'.
\end{aligned}$$

5. Global existence. To prove global existence we assume that

$$f = 0, \quad \theta_1 \ge 0 \qquad (5.1)$$

and

$$\| r_{ttt} \|^2_{0,\Omega_t} + |r|^2_{2,0,\Omega_t} + \| r \|_{0,\Omega_t} + |\theta_1|^2_{4,1,\Omega_t} + \| \theta_1 \|_{1,\Omega_t} \le \eta, \qquad (5.2)$$

where $\eta > 0$ is sufficiently small. Moreover, assume that $\nu > \frac{1}{3}\mu$.

Let $\Phi(t)$ be defined by (4.5) and denote

$$\phi_0(t) = |v|^2_{3,0,\Omega_t} + |\rho_\sigma|^2_{3,0,\Omega_t} + |\vartheta_0|^2_{3,0,\Omega_t} + \| \bar\rho_{\Omega_t} \|^2_{0,\Omega_t}.$$

Introduce the spaces :

$$\mathcal{N}(t) = \{(v, \vartheta_0, \rho_\sigma, \bar\rho_{\Omega_t}) \ : \phi_0(t) < \infty\}\,,$$

$$\mathcal{M}(t) = \{(v, \vartheta_0, \vartheta, \rho_\sigma, \bar\rho_{\Omega_t}) \ : \phi_0(t) + \int_0^t \Phi(t')\, dt' < \infty\}\,.$$

Lemma 5.1. *Let the assumptions of Theorem 3.1 be satisfied. Let the initial data v_0, ρ_0, θ_0, S of problem (1.1) be such that $(v, \vartheta_0, \rho_\sigma, \bar\rho_{\Omega_t}) \in \mathcal{N}(0)$ and $S \in W_2^{4+\frac12}$. Let*

$$\int_\Omega \rho_0 v_0 (a + b \times \xi)\, d\xi = 0\,, \quad \int_\Omega \rho_0 \xi\, d\xi = 0\,,$$

where a, b are constant vectors. Let (A) ((2.3)) be satisfied and let the initial data v_0, ρ_0, θ_0, S and the parameters $p_0, \sigma, d, a_0, \beta, \kappa, M$ of problem (1.1) (d, β and a_0 are defined in Section 2) be such that

$$\phi_0(0) \le \varepsilon_1\,, \quad \chi(0) = \| H(\cdot, 0) + \frac{2}{R_e} \|^2_{2+\frac12, S^1} \le \varepsilon_2\,,$$

$$\omega(t) = \sup_{t' \le t} \| R(\cdot, t') - R_e \|^2_{0, S^1} \le \varepsilon_3\,, \quad t \le T\,,$$

where ε_1, ε_2, ε_3 are sufficiently small , T is the time of the local existence (see Theorem 3.1). Then the local solution of problem (1.1) is such that $(v, \vartheta_0, \vartheta, \rho_\sigma, \bar\rho_{\Omega_t}) \in \mathcal{M}(t)$ for $t \le T$ and

$$\phi_0(t) + \int_0^t \Phi(t')\, dt' \le c(\phi_0(0) + \chi(0) + \omega(t) + \sup_t F(t)) \equiv \tag{5.3}$$

$$\equiv cA \le c(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \eta)\,,$$

where $\eta > 0$ is the sufficiently small constant from assumption (5.2) and F is given by (4.5).

Proof. Take $(v_0, \rho_0, \theta_0) \in W_2^3(\Omega)$, $S \in W_2^{4+\frac12}$ such that assumptions of the lemma hold. Then in view of Theorem 3.1 and Remark 3.2 there exists a local solution of problem (1.1) such that

$$u \in W_2^{4,2}(\Omega^T)\,, \quad \eta_\sigma \in W_2^{3,\frac32}(\Omega^T) \cap C^0(0, T; \Gamma_0^{3,\frac32}(\Omega))\,,$$

$$\vartheta_0 \in W_2^{4,2}(\Omega^T)\,,$$

where T is the time of the local existence. Moreover, Remark 3.2 yields

$$\| u \|^2_{4,\Omega^T} + \| \eta_\sigma \|^2_{3,\Omega^T} + |\eta_\sigma|^2_{3,0,\infty,\Omega^T} + \| \gamma_0 \|^2_{4,\Omega^T} \leq$$

$$\leq c \left(\| v_0 \|^2_{3,\Omega} + \| \rho_{\sigma 0} \|^2_{3,\Omega} + \right. \tag{5.4}$$

$$\left. + \| \vartheta_{00} \|^2_{3,\Omega} + \| H(\cdot,0) + \frac{2}{R_e} \|^2_{2+\frac{1}{2},S^1} \right)$$

$$\leq c\,(\phi_0 + \chi(0)) \leq c(\varepsilon_1 + \varepsilon_2)\,,$$

where $u = v\,(x(\xi,t),t)$, $\eta_\sigma = \rho_\sigma\,(x(\xi,t),t)$, $\gamma_0 = \vartheta_0\,(x(\xi,t),t)$. Using estimate (5.4) for the local solution, Lemma 6.1 of [28] and the following imbeddings (see Lemmas 2.2 and 2.1):

$$\sup_t \left(\| u \|^2_{3,\Omega} + \| u_t \|^2_{1,\Omega} \right) \leq$$

$$\leq c \left(\| u \|^2_{4,\Omega^T} + \| u(0) \|^2_{3,\Omega} + |u(0)|^2_{1,0,\Omega} \right) \leq$$

$$\leq c\,(\phi_0 + \chi(0)) \leq c(\varepsilon_1 + \varepsilon_2)$$

and

$$\int_0^T |u_\xi|_{\infty,\Omega}\,dt' \leq T^{\frac{1}{2}} \| u \|_{4,\Omega^T} \leq cT^{\frac{1}{2}} \phi_0(0)$$

we have

$$N_1 \equiv \sup_t \left(\| \eta_{\sigma tt} \|^2_{0,\Omega} + \| \eta_{\sigma t} \|^2_{2,\Omega} + \| \eta_\sigma \|^2_{3,\Omega} \right) +$$

$$+ \| \eta_{\sigma tt} \|^2_{L_2(0,T;W^1_2(\Omega))} + \| \eta_{\sigma t} \|^2_{L_2(0,T;W^3_2(\Omega))} \leq$$

$$\leq \phi_1\,(T, \phi_0(0) + \chi(0))\,(\phi_0(0) + \chi(0)) \leq$$

$$\leq c\,(\,\varepsilon_1 + \varepsilon_2\,)\,,$$

where ϕ_1 is an increasing continuous function of its arguments.

In the same way as estimate (3.57) from [24] we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\left(\rho v_{xxt}^2+\frac{p_{\sigma\rho}}{\rho}\rho_{\sigma xxt}^2+\frac{\rho c_v}{\theta}\vartheta_{0xxt}^2\right)dx+\\
&+\frac{\sigma}{2}\frac{d}{dt}\int_{S_t}g^{\alpha\beta}\bar{n}\cdot v_{ppp^\alpha}\bar{n}\cdot v_{ppp^\beta}\,ds\ +\\
&+c_0\left(\| v_{xxt}\|_{1,\Omega_t}^2+\|\rho_{\sigma xxt}\|_{0,\Omega_t}^2+\|\vartheta_{0xxt}\|_{1,\Omega_t}^2\right)\le\\
&\le\ (\varepsilon_1'+cN)\ \left(\| v_{xttt}\|_{0,\Omega_t}^2+\| v_{xxxt}\|_{0,\Omega_t}^2+\| v_{xxtt}\|_{0,\Omega_t}^2+\right.\\
&+\left.\|\vartheta_{0xttt}\|_{0,\Omega_t}^2+\|\vartheta_{0xxxt}\|_{0,\Omega_t}^2+\|\vartheta_{0xxtt}\|_{0,\Omega_t}^2\right)\ +\\
&+c\left(N,\int_0^T M(t')dt'\right)\ M+cF(t),
\end{aligned}\tag{5.5}$$

where $N=N_1+N_2$, $N_2=\sup_t(\| u\|_{3,\Omega}^2+\| u_t\|_{1,\Omega}^2+\|\gamma_0\|_{3,\Omega}^2+\|\gamma_{0t}\|_{1,\Omega}^2)$ and M is such that

$$\int_0^T Mdt'\le c\,(\phi_0(0)+\chi(0))$$

holds in, virtue of the estimates for the local solution.

Similarly, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\left(\rho v_{xtt}^2+\frac{p_{\sigma\rho}}{\rho}\rho_{\sigma xtt}^2+\frac{\rho c_v}{\theta}\vartheta_{0xxt}^2\right)dx+\\
&+\frac{\sigma}{2}\frac{d}{dt}\int_{S_t}g^{\alpha\beta}\bar{n}\cdot v_{tpp^\alpha}\bar{n}\cdot v_{tpp^\beta}\,ds\ +\\
&+c_0\left(\| v_{xtt}\|_{1,\Omega_t}^2+\|\rho_{\sigma xtt}\|_{0,\Omega_t}^2+\|\vartheta_{0xtt}\|_{0,\Omega_t}^2\right)\le\\
&\le\ (\varepsilon_2'+cN)\ \left(\| v_{xttt}\|_{0,\Omega_t}^2+\| v_{xxtt}\|_{0,\Omega_t}^2+\| v_{xxxt}\|_{0,\Omega_t}^2\ +\right.\\
&+\left.\|\vartheta_{0xttt}\|_{0,\Omega_t}^2+\|\vartheta_{0xxxt}\|_{0,\Omega_t}^2+\|\vartheta_{0xxtt}\|_{0,\Omega_t}^2\right)\ +\\
&+c(1+N)^2\left(\| v_{xxt}\|_{1,\Omega_t}^2+\|\vartheta_{0xxt}\|_{1,\Omega_t}^2\right)+
\end{aligned}\tag{5.6}$$

$$+c\left(\| v_{ttt} \|^2_{0,\Omega_t} + \| \vartheta_{0ttt} \|^2_{0,\Omega_t}\right) + c(N, \int_0^T M(t')dt')\, M +$$

$$+cF(t)\,.$$

Next, Lemma 3.12 from [24] implies

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\left(\rho v^2_{ttt} + \frac{p_{\sigma\rho}}{\rho}\rho^2_{\sigma ttt} + \frac{\rho c_v}{\theta}\vartheta^2_{0ttt}\right)dx + \\
&+\frac{\sigma}{2}\frac{d}{dt}\int_{S_t} g^{\alpha\beta}\bar{n}\cdot v_{tts^\alpha}\bar{n}\cdot v_{tts^\beta}\, ds \; + \\
&+c_0\left(\| v_{ttt} \|^2_{1,\Omega_t} + \| \rho_{\sigma ttt} \|^2_{0,\Omega_t} + \| \vartheta_{0ttt} \|^2_{0,\Omega_t}\right) \le \\
&\le c(\varepsilon_4' + N + M)\left(\| v_{ttt} \|^2_{0,\Omega_t} + \| \vartheta_{0ttt} \|^2_{0,\Omega_t}\right) + \\
&+\varepsilon_4'\left(\| v_{xxtt} \|^2_{0,\Omega_t} + \| \vartheta_{0xxtt} \|^2_{0,\Omega_t}\right) + \\
&+cN\left(\| v_{xtt} \|^2_{1,\Omega_t} + \| v_{xttt} \|^2_{0,\Omega_t} + \right. \\
&+ \| v_{xxt} \|^2_{1,\Omega_t} + \| \vartheta_{0xtt} \|^2_{1,\Omega_t} + \| \vartheta_{0xttt} \|^2_{0,\Omega_t} + \| \vartheta_{0xxt} \|^2_{1,\Omega_t} + \\
&\left. + \| \rho_{\sigma ttt} \|^2_{0,\Omega_t}\right) + cM\left(\| v_t \|^2_{2,\Omega_t} + \| \vartheta_{0t} \|^2_{2,\Omega_t}\right) + \\
&+c\left(\| v_{xtt} \|^2_{1,\Omega_t} + \| \vartheta_{0xtt} \|^2_{1,\Omega_t}\right) + c(N)M + cF(t)\,,
\end{aligned}\tag{5.7}$$

where in virtue of the equation of continuity $(4.4)_2$ we have

$$\| \rho_{\sigma ttt} \|^2_{0,\Omega_t} \le c(1+N)\, \| v_{xtt} \|^2_{0,\Omega_t} + c(N)M\,. \tag{5.8}$$

From (5.5)–(5.8) we obtain for sufficiently small ε_1', ε_2', ε_3', N, $\int_0^T Mdt'$ and η from (5.2) that

$$\sup_t\left(\| v_{xxt} \|^2_{0,\Omega_t} + \| v_{xtt} \|^2_{0,\Omega_t} + \| v_{ttt} \|^2_{0,\Omega_t} + \| \vartheta_{0xxt} \|^2_{0,\Omega_t} + \right.$$

$$+ \| \vartheta_{0xtt} \|^2_{0,\Omega_t} + \| \vartheta_{0ttt} \|^2_{0,\Omega_t} + \| \rho_{\sigma xxt} \|^2_{0,\Omega_t} + \| \rho_{\sigma xtt} \|^2_{0,\Omega_t} +$$

$$+ \parallel \rho_{\sigma ttt} \parallel^2_{0,\Omega_t}\Big) + \sup_t \int_{S_t} g^{\alpha\beta} \Big(\bar{n} \cdot v_{ppp^\alpha} \bar{n} \cdot v_{ppp^\beta} + \tag{5.9}$$

$$+ \ \bar{n} \cdot v_{tpp^\alpha} \bar{n} \cdot v_{tpp^\beta} + \bar{n} \cdot v_{tts^\alpha} \bar{n} \cdot v_{tts^\beta}\Big) ds +$$

$$+ \int_0^t \Big(\parallel v_{xxt} \parallel^2_{1,\Omega_t} + \parallel v_{xtt} \parallel^2_{1,\Omega_t} + \parallel v_{ttt} \parallel^2_{1,\Omega_t}\Big) dt \le$$

$$\le c\,(\phi_0(0) + \chi(0)) + c \int_0^t F(t')\, dt' \le c(\varepsilon_1 + \varepsilon_2)\,.$$

Now, to estimate $\parallel \bar{\rho}_{\Omega_t} \parallel^2_{0,\Omega_t} + \parallel \vartheta \parallel^2_{0,\Omega_t}$ rewrite equation

$$p(\rho_{\Omega_t}, \theta_{\Omega_t}) - p(\rho_e, \theta_e) = 0$$

using the Taylor formula as

$$p_\rho(\rho_{\Omega_t} - \rho_e) + p_\theta(\theta_{\Omega_t} - \theta_e) = 0\,.$$

Therfore

$$\parallel \bar{\rho}_{\Omega_t} \parallel^2_{0,\Omega_t} + \parallel \vartheta \parallel^2_{0,\Omega_t} \le \parallel \rho_\sigma \parallel^2_{0,\Omega_t} + \parallel \rho_e - \rho_{\Omega_t} \parallel^2_{0,\Omega_t} +$$

$$+ \parallel \vartheta_0 \parallel^2_{0,\Omega_t} + \parallel \theta_e - \theta_{\Omega_t} \parallel^2_{0,\Omega_t} \le \ c \ \Big(\parallel \rho_\sigma \parallel^2_{0,\Omega_t} + \tag{5.10}$$

$$+ \parallel \vartheta_0 \parallel^2_{0,\Omega_t} + \parallel \theta_e - \theta_{\Omega_t} \parallel^2_{0,\Omega_t}\Big) \le c \Big(\parallel \rho_\sigma \parallel^2_{0,\Omega_t} +$$

$$+ \parallel \vartheta_0 \parallel^2_{0,\Omega_t} + \left\| \frac{1}{|\Omega_t|} \int_{\Omega_t} \vartheta_0 dx \right\|^2_{0,\Omega_t} \Bigg) \le$$

$$\le c\phi_0(0) \le c\varepsilon_1\,,$$

where to estimate $\parallel \rho_\sigma \parallel^2_{0,\Omega_t}$ and $\parallel \vartheta_0 \parallel^2_{0,\Omega_t}$ we have used (5.4).

Next, by the same argument as in [29] we obtain the following inequalities (see inequalities (4.104) and (4.154) from [29]) :

$$\left\| \int_0^t v dt' \right\|^2_{3,S_t} \le \varepsilon \Big(\parallel v \parallel^2_{3,\Omega_t} + \parallel \rho_{\sigma x} \parallel^2_{1,\Omega_t} + \parallel \vartheta_{0x} \parallel^2_{1,\Omega_t}\Big) +$$

$$+ C_1 \left(\left\| \int_0^t v dt' \right\|^2_{0,\Omega_t} + \parallel v \parallel^2_{0,\Omega_t} + \parallel p_\sigma \parallel^2_{0,\Omega_t} + \right. \tag{5.11}$$

$$+ \parallel H(\cdot,0) + \frac{2}{R_e} \parallel^2_{1,S^1} + \parallel R(\cdot,t) - R(\cdot,0) \parallel^2_{3,S^1}\Big) +$$

$$+ C_2 \Big(\parallel v \parallel^2_{3,\Omega_t} + \parallel \rho_\sigma \parallel^2_{2,\Omega_t} + \parallel \vartheta_0 \parallel^2_{2,\Omega_t} + \parallel \rho_{\sigma x} \parallel^4_{1,\Omega_t} +$$

$$+ \parallel \vartheta_{0x} \parallel^4_{1,\Omega_t}\Big) \int_0^t \parallel v \parallel^2_{4,\Omega_{t'}} dt'$$

and

$$\left\| \int_0^t v dt' \right\|^2_{4,S_t} \le \varepsilon \Big(\parallel v_{xxxx} \parallel^2_{0,\Omega_t} + \parallel \rho_{\sigma xxx} \parallel^2_{0,\Omega_t} +$$

$$+ \parallel \vartheta_{0xxx} \parallel^2_{0,\Omega_t}\Big) + C_1 \left(\left\| \int_0^t v dt' \right\|^2_{0,\Omega_t} + \parallel v \parallel^2_{0,\Omega_t} + \parallel p_\sigma \parallel^2_{0,\Omega_t} + \right. \quad (5.12)$$

$$\left. + \parallel H(\cdot,0) + \frac{2}{R_e} \parallel^2_{0,S^1} + \parallel R(\cdot,t) - R(\cdot,0) \parallel^2_{4,S^1}\right) +$$

$$+ C_2 \Big(\parallel v \parallel^2_{3,\Omega_t} + \parallel \rho_\sigma \parallel^2_{2,\Omega_t} + \parallel \vartheta_0 \parallel^2_{2,\Omega_t} + \parallel \rho_\sigma \parallel^4_{2,\Omega_t} +$$

$$+ \parallel \vartheta_0 \parallel^4_{2,\Omega_t}\Big) \left\| \int_0^t v\, dt' \right\|^2_{4,\Omega_t} \cdot \left(1 + \left\| \int_0^t v\, dt' \right\|^2_{3,\Omega_t} \right) .$$

Using (5.11), (5.12) and (4.3) in (4.6) we have

$$\frac{d\phi}{dt} + c_0 \Phi \le c_1 P(X) X (1 + X^3)(X + Y) + c_2 F +$$

$$+ c_3 \psi + c_4 \parallel H(\cdot,0) + \frac{2}{R_e} \parallel^4_{2,S^1} + \varepsilon c_5 \Big(\parallel H(\cdot,0) + \frac{2}{R_e} \parallel^2_{2,S^1} + \quad (5.13)$$

$$+ \parallel R(\cdot,t) - R(\cdot,0) \parallel^2_{4,S^1}\Big) +$$

$$+ c_6 \Big\{ \parallel R(\cdot,t) - R(\cdot,0) \parallel^2_{4+\frac{1}{2},S^1} \Big[\varepsilon \Big(\parallel v \parallel^2_{3,\Omega_t} + \parallel \rho_{\sigma x} \parallel^2_{1,\Omega_t} +$$

$$+ \parallel \vartheta_{0x} \parallel^2_{1,\Omega_t}\Big) + C_1 \left(\left\| \int_0^t v\, dt' \right\|^2_{0,\Omega_t} + \parallel v \parallel^2_{0,\Omega_t} + \right.$$

$$+ \parallel \bar\rho_{\Omega_t} \parallel^2_{0,\Omega_t} + \parallel \vartheta \parallel^2_{0,\Omega_t} + \parallel H(\cdot,0) + \frac{2}{R_e} \parallel^2_{1,S^1} +$$

$$+ \parallel R(\cdot,t) - R(\cdot,0) \parallel^2_{3,S^1} \Big) + C_2 \Big(\parallel v \parallel^2_{3,\Omega_t} + \parallel \rho_\sigma \parallel^2_{2,\Omega_t} +$$

$$+ \parallel \vartheta_0 \parallel^2_{2,\Omega_t} + \parallel \rho_{\sigma x} \parallel^4_{1,\Omega_t} + \parallel \vartheta_{0x} \parallel^4_{1,\Omega_t} \Big) \int_0^t \parallel v \parallel^2_{4,\Omega_{t'}} dt' \Big] +$$

$$+ \parallel R(\cdot,t) - R(\cdot,0) \parallel^2_{3,S^1} \Big[\varepsilon \Big(\parallel v_{xxxx} \parallel^2_{0,\Omega_t} + \parallel \rho_{\sigma xxx} \parallel^2_{0,\Omega_t} +$$

$$+ \parallel \vartheta_{0xxx} \parallel^2_{0,\Omega_t} \Big) + C_1 \Big(\parallel R(\cdot,t) - R(\cdot,0) \parallel^2_{4,S^1} +$$

$$+ \parallel H(\cdot,0) + \frac{2}{R_e} \parallel^2_{0,S^1} + \left\| \int_0^t v\, dt' \right\|^2_{0,\Omega_t} + \parallel v \parallel^2_{0,\Omega_t} +$$

$$+ \parallel \bar\rho_{\Omega_t} \parallel^2_{0,\Omega_t} + \parallel \vartheta \parallel^2_{0,\Omega_t} \Big) + C_2 \Big(\parallel v \parallel^2_{3,\Omega_t} + \parallel \rho_\sigma \parallel^2_{2,\Omega_t} + \parallel \vartheta_0 \parallel^2_{2,\Omega_t} +$$

$$+ \parallel \rho_\sigma \parallel^4_{2,\Omega_t} + \parallel \vartheta_0 \parallel^4_{2,\Omega_t} \Big) \left\| \int_0^t v\, dt' \right\|^2_{4,\Omega_t} \cdot \left(1 + \left\| \int_0^t v\, dt' \right\|^2_{3,\Omega_t} \right) \Big] \Big\} .$$

By Theorem 2.11 and boundary condition $(4.4)_1$ we have

$$\parallel R(\cdot,t) - R(\cdot,0) \parallel^2_{4+\frac12,S^1} \le \parallel R(\cdot,t) - R_0 \parallel^2_{4+\frac12,S^1} +$$

$$+ \parallel R(\cdot,0) - R_0 \parallel^2_{4+\frac12,S^1} + |S^1| \Big(|R_0 - R_e|^2 + |R_t - R_e|^2 \Big) \le \tag{5.14}$$

$$\le c \Big(\parallel v \parallel^2_{4,\Omega_t} + \parallel \rho_\sigma \parallel^2_{3,\Omega_t} + \parallel \vartheta_0 \parallel^2_{3,\Omega_t} + \parallel H(\cdot,0) + \frac{2}{R_e} \parallel^2_{2+\frac12,S^1} +$$

$$+ \parallel R(\cdot,t) - R_e \parallel^2_{0,S^1} + \parallel R(\cdot,0) - R_e \parallel^2_{0,S^1} + |R_0 - R_e|^2 +$$

$$+ |R_t - R_e|^2 \Big) .$$

Next, using Lemmas 2.2 and 2.1 and estimate (5.4) we obtain

$$|R_t - R_e|^2 \le c| \; |\Omega_t|^{\frac13} - |\Omega_e|^{\frac13} |^2 =$$

$$= c \left| \left(\frac{M}{\rho} \right)^{\frac13} - \left(\frac{M}{\rho_e} \right)^{\frac13} \right|^2 \le c |\rho_\sigma|_{\infty,\Omega_t} \le \tag{5.15}$$

$$\le c(\phi_0(0) + \chi(0))$$

and

$$|R_0 - R_e|^2 \le c|\rho_{\sigma 0}|_{\infty,\Omega} \le c\left(\phi_0(0) + \chi(0)\right) . \tag{5.16}$$

Therefore, inequality (5.13), estimates (5.4),(5.9),(5.10) (5.14)–(5.16), the definitions of X and Y and the assumptions of the lemma yield

$$\begin{aligned} &\phi_0(t) + \int_0^t \Phi(t')dt' \le \\ &\le c\int_0^t \psi(t')dt' + c\int_0^t F(t')dt' + c\Bigg(\| H(\cdot,0) + \frac{2}{R_e} \|^2_{2+\frac{1}{2},S^1} + \\ &+ \sup_{t'\le t} \| R(\cdot,t') - R_e \|^2_{0,S^1} + \phi_0(0)\Bigg) \le c(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \end{aligned} \tag{5.17}$$

if $\eta > 0$ from assumption (5.2) is sufficiently small.

Hence, we have proved that $(v, \vartheta_0, \vartheta, \rho_\sigma, \rho_{\Omega_t}) \in \mathcal{M}(t)$ for $t < T$ and (5.3) holds. We have to underline that to prove the above result the standard technique of mollifiers or differences should be used. This concludes the proof. □

Lemma 5.2. *Assume that there exists a local solution to problem (1.1) which belongs to $\mathcal{M}(t)$ for $t \le T$, i.e. let the assumptions of Lemma 5.1 be satisfied. Let assumptions of Lemma 2.5 be satisfied. Moreover, assume*

$$\int_\Omega |\rho_0 - \rho_e| d\xi \le \delta_0 .$$

Then there exist $\delta_1, \delta_2 \in (0,1)$ sufficiently small such that

$$\| p_\sigma \|^2_{0,\Omega_t} \le \delta_1 , \tag{5.18}$$

$$\| \vartheta_0 \|^2_{0,\Omega_t} + \| \rho_\sigma \|^2_{0,\Omega_t} \le \delta_2 , \tag{5.19}$$

where $\delta_1 = c(\varepsilon_1 + \varepsilon_2)\delta' + c(\delta')(\varepsilon_0 + \tilde{\delta})$, $\delta_2 = c(\varepsilon_1 + \varepsilon_2)\delta' + c(\delta')(\varepsilon_0 + \delta_0 + \tilde{\delta})$, $\delta' \in (0,1)$ and $c(\delta')$ is a decreasing function of δ', ε_0 and $\tilde{\delta}$ are taken from Lemma 2.5.

Proof. Estimate (5.18) can be obtained by using Lemma 2.5 exactly in the same way as estimate (5.16) in [29], while to prove (5.19) the same argument as in the proof of Lemma 5.2 from [25] can be applied. □

Lemma 5.3. *Let* $(v, \vartheta_0, \vartheta, \rho_\sigma, \bar\rho_{\Omega_t}) \in \mathcal{M}(t)$, $t \le T$, *be a solution of problem (1.1). Then* $u(\xi,t) = v(x(\xi,t),t) \in C^0(t_0+\lambda, T; W_2^4(\Omega))$ *and the estimate holds*

$$\sup_{t_0+\lambda<t\le T} \| v \|^2_{4,\Omega_t} \le c(A)A\,, \tag{5.20}$$

where $t_0 > 0$, $\lambda > 0$, $t_0+\lambda < T$, $A \equiv \phi_0(t) + \int_0^t \Phi(t')dt'$, $c(A)$ *is a nondecreasing continuous function of* A.

Proof. We use the argument from [13] (see Theorem 6). Let $\zeta_\lambda(t) \in C^\infty$ be a function such that $\zeta_\lambda(t) = 1$ for $t \ge t_0+\lambda$, $\zeta_\lambda(t) = 0$ for $t \le t_0 + \frac{\lambda}{2}$, $0 \le \zeta_\lambda(t) \le 1$, $|\dot\zeta_\lambda(t)| \le c/\lambda$, where $\dot\zeta_\lambda = \frac{d\zeta_\lambda}{dt}$. Let $u_\lambda = u\zeta_\lambda$, $\eta_{\sigma\lambda} = \eta_\sigma\zeta_\lambda$, $\gamma_{0\lambda} = \gamma_0\zeta_\lambda$. Then $u_\lambda, \eta_{\sigma\lambda}, \gamma_{0\lambda}$ satisfy the problem (see (4.4) and (5.1)) :

$$\eta u_{\lambda t} - \mu\nabla_u^2 u_\lambda - \nu\nabla_u\nabla_u\cdot u_\lambda =$$

$$= p_\eta\nabla_u\eta_{\sigma\lambda} + p_\Gamma\nabla_u\gamma_{0\lambda} + \eta u\dot\zeta_\lambda \qquad \text{in } \Omega^T\,, \tag{5.21}$$

$$\Pi_0\Pi\mathbf{D}_u(u_\lambda)\bar n = 0 \qquad \text{on } S^T\,,$$

$$\bar n_0\mathbf{D}_u(u_\lambda)\bar n - \sigma\bar n_0\int_0^t \triangle_{S_\tau}(\tau)u_\lambda(\tau)d\tau =$$

$$= \int_0^t \Big[\dot\zeta_\lambda\bar n_0\mathbf{T}(u,p_\sigma)\bar n + \sigma\bar n_0\zeta_\lambda\dot\triangle_{S_\tau}(\tau)\cdot$$

$$\cdot\left(\xi + \int_0^\tau u(\tau')d\tau'\right) + \zeta_\lambda\partial_\tau(q_0\bar n_0\cdot\bar n)\Big]d\tau\; +$$

$$+\;(p_1\eta_{\sigma\lambda} + p_2\gamma_{0\lambda})\,\bar n_0\cdot\bar n \equiv$$

$$\equiv \int_0^t B(\tau)d\tau + (p_1\eta_{\sigma\lambda} + p_2\gamma_{0\lambda})\,\bar n_0\cdot\bar n \qquad \text{on } S^T\,,$$

$$u_\lambda|_{t=0} = 0 \qquad \text{in } \Omega\,,$$

where $\eta_{\sigma\lambda}$ and $\gamma_{0\lambda}$ are treated as given functions, $\Pi_0 g = g - \bar n_0(\bar n_0\cdot g)$, $\Pi g = g - \bar n(\bar n\cdot g)$.

The second boundary condition $(5.21)_3$ follows from the integration by parts :

$$0 = \int_0^t \zeta_\lambda(\tau)\partial_\tau \left[\bar{n}_0 \mathbf{T}(u,p_\sigma)\bar{n} - \sigma\bar{n}_0 \triangle_{S_\tau}(\tau)\cdot\right.$$

$$\left.\cdot\left(\xi + \int_0^T u(\tau')d\tau'\right) - q_0\bar{n}_0\cdot\bar{n}\right] d\tau =$$

$$= \int_0^t \partial_\tau\left[\zeta_\lambda \bar{n}_0 \mathbf{T}(u,p_\sigma)\bar{n}\right]d\tau - \int_0^t \left[\dot{\zeta}_\lambda \bar{n}_0\mathbf{T}(u,p_\sigma)\bar{n} +\right.$$

$$\left. +\ \sigma\bar{n}_0\zeta_\lambda\dot{\triangle}_{S_\tau}(\tau)\left(\xi + \int_0^\tau u(\tau')d\tau'\right) + q_0\zeta_\lambda\partial_\tau(\bar{n}_0\cdot\bar{n})\right]d\tau\ +$$

$$-\sigma\bar{n}_0\cdot\int_0^t \triangle_{S_\tau}(\tau)u_\lambda(\tau)d\tau\,.$$

Next, we introduce the differences :

$$u^{(s)}(\xi,t) = u_\lambda(\xi,t) - u'_\lambda(\xi,t)\,,$$

$$\eta_\sigma^{(s)}(\xi,t) = \eta_{\sigma\lambda}(\xi,t) - \eta'_{\sigma\lambda}(\xi,t)\,,$$

$$\gamma_0^{(s)}(\xi,t) = \gamma_{0\lambda}(\xi,t) - \gamma'_{0\lambda}(\xi,t)\,,$$

where $w'(\xi,t) = w(\xi,t-s)$, $0 < s < t_0$. Therefore, we obtain the following equations

$$\eta u_t^{(s)} - \mu\nabla_u^2 u^{(s)} - \nu\nabla_u\nabla_u\cdot u^{(s)} = p_\eta\nabla_u\eta_\sigma^{(s)}\ +$$

$$+\,p_\Gamma\nabla_u\gamma_0^{(s)}\ - (\eta-\eta')u'_{\lambda t} + \mu(\nabla_u^2 - \nabla_{u'}^2)u'_\lambda\ +$$

$$+\,\nu(\nabla_u\nabla_u - \nabla_{u'}\nabla_{u'})\cdot u'_\lambda + p_\eta(\nabla_u - \nabla_{u'})\eta'_{\sigma\lambda} + p_\Gamma(\nabla_u - \nabla_{u'})\gamma'_{0\lambda}\ +$$

$$+\,(p_\eta - p_{\eta'})\nabla_{u'}\eta'_{\sigma\lambda} + (p_\Gamma - p_{\Gamma'})\nabla_{u'}\gamma'_{0\lambda} + (\eta-\eta')u\dot{\zeta}_\lambda\ +$$

$$+\,\eta'u'(\dot{\zeta}_\lambda - \dot{\zeta}'_\lambda) \equiv E \qquad \text{in } \Omega^T\,,$$

$$\Pi_0 \Pi \mathbf{D}_u(u^{(s)})\bar{n} =$$

$$= \Pi_0 \left(\Pi \mathbf{D}_u(u'_\lambda)\bar{n} - \Pi \mathbf{D}_{u'}(u'_\lambda)\bar{n}' \right) \equiv F \qquad \text{on } S^T,$$

$$\bar{n}_0 \mathbf{D}_u(u^{(s)})\bar{n} - \sigma \bar{n}_0 \int_0^t \triangle_{S_\tau}(\tau) u^{(s)}(\tau) d\tau =$$

$$= p_1(\eta, \Gamma)\eta^{(s)} \bar{n}_0 \cdot \bar{n} + p_2(\eta, \Gamma)\gamma_0^{(s)} \bar{n}_0 \cdot \bar{n} +$$

$$+ \bar{n}_0 \left(\mathbf{D}_u(u'_\lambda)\bar{n} - \mathbf{D}_{u'}(u'_\lambda)\bar{n}' \right) +$$

$$- \sigma \bar{n}_0 \int_0^t \left(\triangle_{S_\tau}(\tau) - \triangle'_{S_\tau}(\tau) \right) u'_\tau d\tau +$$

$$+ \sigma \int_0^t \left(B(\tau) - B'(\tau') \right) d\tau + p_1(\eta, \Gamma)\eta'_{\sigma\lambda} \bar{n}_0 \cdot (\bar{n} - \bar{n}') +$$

$$+ p_2(\eta, \Gamma)\gamma'_{0\lambda} \bar{n}_0 \cdot (\bar{n} - \bar{n}') + \left[p_1(\eta, \Gamma) - p_1(\eta', \Gamma') \right] \eta'_{\sigma\lambda} \bar{n}_0 \cdot \bar{n}' +$$

$$+ \left[p_2(\eta, \Gamma) - p_2(\eta', \Gamma') \right] \gamma'_{0\lambda} \bar{n}_0 \cdot \bar{n}' \equiv$$

$$\equiv G_1 + \int_0^t G_2(\tau) d\tau \qquad \text{on } S^T,$$

$$u^{(s)}|_{t=0} = 0 \qquad \text{in } \Omega.$$

Let $\lambda \in (0,1)$, $t_0 + \lambda < T$, $Q_\lambda = \Omega \times (t_0 + \lambda, T)$, $G_\lambda = S \times (t_0 + \lambda, T)$. Then for sufficiently small $\phi_0(0)$ Lemma 5.2 from [28] yields

$$\| u^{(s)} \|_{4,Q_\lambda} \leq c \Big(\| E \|_{2,Q_{\frac{\lambda}{2}}} + \| F \|_{3-\frac{1}{2},G_{\frac{\lambda}{2}}} + \\ + \| G_1 \|_{3-\frac{1}{2},G_{\frac{\lambda}{2}}} + \| G_2 \|_{2-\frac{1}{2},G_{\frac{\lambda}{2}}} \Big).$$

Now, using the explicite forms of E, F, G_1, G_2 in the same way as in [30] we get

$$\| u^{(s)} \|^2_{4,Q_\lambda} \leq c(A)As^2 .$$

Hence

$$\| \| u(\cdot) \|_{4,\Omega} \|^2_{B^1_{2,\infty}(t_0+\lambda,T)} =$$

$$= \sup_s \int_{t_0+\lambda}^T \frac{| \| u(t) \|_{4,\Omega} - \| u(t-s) \|_{4,\Omega} |^2}{s^2} dt + \tag{5.22}$$

$$+ \int_{t_0+\lambda}^T \| u(t) \|^2_{4,\Omega} \, dt \le$$

$$\le \sup_s \int_{t_0+\lambda}^T \frac{\| u(t) - u(t-s) \|^2_{4,\Omega}}{s^2} dt + \int_{t_0+\lambda}^T \| u(t) \|^2_{4,\Omega} \, dt \le$$

$$\le c(A)A \, .$$

By the imbedding theorems for the Besov and Nikolskii spaces we have (see [10], Ch. 6.1)

$$B^\beta_{2,\infty}(0,T) \subset L_\infty(0,T) \quad \text{for } \beta > \frac{1}{2} \, .$$

Therefore (5.22) yields (5.20). This concludes the proof. □

Now, we prove a lemma which guarantees a prolongation of the local solution. The lemma is analogous to Lemma 5.4 from [28].

Lemma 5.4. *Assume that there exists a local solution of (1.1) in $\mathcal{M}(t)$, $0 \le t \le T$, i.e. assume that the assumptions of Lemma 5.1 be satisfied. Let the assumptions of Lemma 5.2 be satisfied. Moreover, assume*

$$\phi_0(0) \le \gamma \, , \quad \| H(\cdot,0) + \frac{2}{R_e} \|^2_{2,S^1} \le d \, . \tag{5.23}$$

Then for sufficiently small $\gamma, d, \eta, \delta_1, \delta_2$ and δ_0 (where δ_0 is a constant from Lemma 2.4, δ_i $(i = 1,2)$ are constants from Lemma 5.2 and η is a constant from (5.2)) we have

$$\phi_0(t) \le \gamma \quad \text{for } t \le T \, . \tag{5.24}$$

Proof. At first add $\| \int_0^t v dt' \|^2_{4,S_t}$ to the both sides of inequality (4.6). Using (5.12) we get

$$\frac{d\phi}{dt} + \Phi_0 \le cP(X)X(1+X^3)(X+Y) + cF +$$

$$+ c\psi_1 + c\left(\| H(\cdot,0) + \frac{2}{R_e} \|^2_{0,S^1} + \| R(\cdot,t) - R(\cdot,0) \|^2_{4,S^1}\right) +$$

$$+ c \| H(\cdot,0) + \frac{2}{R_e} \|^4_{2,S^1} + c \| R(\cdot,t) - R(\cdot,0) \|^2_{4+\frac{1}{2},S^1} \cdot$$

$$\cdot \left\| \int_0^t v\, dt' \right\|^2_{3,S_t} + c \| R(\cdot,t) - R(\cdot,0) \|^2_{3,S^1} \left\| \int_0^t v\, dt' \right\|^2_{4,S_t}$$

where

$$\Phi_0 = c_0\Phi + \left\| \int_0^t v\, dt' \right\|^2_{4,S_t} ,$$

$$\psi_1 = \| v \|^2_{0,\Omega_t} + \| \rho_\sigma \|^2_{0,\Omega_t} + \int_0^t \| v \|^2_{0,\Omega_{t'}} \ dt' +$$

$$+ \| R(\cdot,t) - R_e \|^2_{0,S^1} + \| R(\cdot,0) - R_e \|^2_{0,S^1} .$$

Now, by estimates (5.14)–(5.16), assumptions (5.23) and Lemmas 5.1 and 5.3 we have

$$\| R(\cdot,t) - R(\cdot,0) \|^2_{4+\frac{1}{2},S^1} \le c\,(\varepsilon_3 + \gamma + \varepsilon_2 + \eta) . \tag{5.25}$$

Using (5.25) with sufficiently small ε_3, γ, ε_2 and η we obtain

$$\frac{d\phi}{dt} + c_1\Phi_0 \le cP(X)X(1+X^3)(X+Y) + cF +$$

$$+ c\psi_1 + c\left(\| R(\cdot,t) - R(\cdot,0) \|^2_{4,S^1} + \right. \tag{5.26}$$

$$\left. + \| H(\cdot,0) + \frac{2}{R_e} \|^2_{2,S^1}\right) ,$$

where we have used that $\| H(\cdot,0)+\frac{2}{R_e} \|^2_{2,S^1}$ is also small. Applying the inequality

$$\| R(\cdot,t)-R(\cdot,0) \|^2_{4,S^1} \le \| R(\cdot,t)-R_t \|^2_{4,S^1} +$$

$$+ \| R(\cdot,0)-R_0 \|^2_{4,S^1} + |S^1| \left(|R_0-R_e|^2 + |R_t-R_e|^2 \right) \le$$

$$\le \varepsilon \left(\| v \|^2_{4,\Omega_t} + \| \rho_\sigma \|^2_{3,\Omega_t} + \| \vartheta_0 \|^2_{3,\Omega_t} \right) + c \| H(\cdot,0) + \frac{2}{R_e} \|^2_{2,S^1} +$$

$$+ c \left(|R_0-R_e|^2 + |R_t-R_e|^2 \right) + c\psi_1(t),$$

where $\varepsilon > 0$ is a sufficiently small constant, and introducing the new quantity

$$\Phi_{00} = c_1 \Phi_0 + c \| H(\cdot,0) + \frac{2}{R_e} \|^2_{2,S^1}$$

instead of (5.26) we get

$$\frac{d\Phi}{dt} + \Phi_{00} \le cP(X)X(1+X^3)(X+Y) +$$

$$+ cF(t) + c\psi_1(t) + c \| H(\cdot,0) + \frac{2}{R_e} \|^2_{2,S^1} + \tag{5.27}$$

$$+ c \left(|R_0-R_e|^2 + |R_t-R_e|^2 \right) .$$

By Lemma 5.2 we have

$$\phi_0 \le C\phi \le c \left(\Phi_{00} + \| \vartheta_0 \|^2_{0,\Omega_t} + \| \rho_\sigma \|^2_{0,\Omega_t} \right) \le \tag{5.28}$$

$$\le c \left(\Phi_{00} + \delta_2 \right) .$$

Then, using Lemma 5.1, (5.27), (5.28) and the inequalities

$$X \le c_2 \left(\phi_0(t) + \int_0^t \Phi(t')dt' \right) \tag{5.29}$$

and

$$Y \le \Phi(t) + \int_0^t \Phi(t')dt' \tag{5.30}$$

we obtain

$$\int_0^t \Phi(t')dt' \leq$$

$$\leq c\delta_2 + c\tilde{\psi}(t) + c \parallel H(\cdot,0) + \frac{2}{R_e} \parallel^2_{2,S^1} + \tag{5.31}$$

$$+ c\phi(0) + c\int_0^t F(t')dt' +$$

$$+ c(|R_0 - R_e|^2 + |R_t - R_e|^2)\,,$$

where $\tilde{\psi}(t) = \sup_{t' \leq t} \psi_1(t')$, $c > 0$ is a constant and for simplicity we do not distinguish different c's.

Next, inequalities (5.28) – (5.30) yield

$$P(X)X(1+X^3)(X+Y) \leq c\phi_0\left(1+\phi_0^3\right)\Phi_{00} +$$

$$+ c\phi_0\left(1+\phi_0^3\right)\delta_2 + c\gamma\Phi_{00} + c\gamma\delta_2 + c\left(\int_0^t \Phi dt'\right)^2 .$$

Hence, using (5.31), instead of (5.27) we get

$$\frac{d\phi}{dt} + \Phi_{00} \leq c\phi_0(1+\phi_0^3)\Phi_{00} + c\phi_0(1+\phi_0^3)\delta_2 +$$

$$+ c\gamma\Phi_{00} + c\gamma\delta_2 + c\delta_2^2 + c\tilde{\psi}^2(t) + \tag{5.32}$$

$$+ c \parallel H(\cdot,0) + \frac{2}{R_e} \parallel^4_{2,S^1} + c\phi^2(0) + c\left(\int_0^t F(t')dt'\right)^2 +$$

$$+ c\left(|R_0 - R_e|^4 + |R_t - R_e|^4\right) + cF(t) + c\tilde{\psi}(t) +$$

$$+ c \parallel H(\cdot,0) + \frac{2}{R_e} \parallel^2_{2,S^1} + c\left(|R_0 - R_e|^2 + |R_t - R_e|^2\right) .$$

Next, using Lemma 2.4 and the relation

$$|\Omega| - |\Omega_e| = \frac{1}{\rho_e}\int_\Omega (\rho_e - \rho_0)\, d\xi$$

we obtain

$$|R_t - R_e| + |R_0 - R_e| \le \tag{5.33}$$

$$\le c\,(|\,|\Omega_t| - |\Omega_e|\,| + |\,|\Omega| - |\Omega_e|\,|) \le \delta_3\,,$$

where $\delta_3 = \delta_3(\delta_0)$ and $\delta_3(\delta_0) \to 0$ as $\delta_0 \to 0$ (δ_0 is a constant from Lemmas 2.4 and 5.2).

Taking into account (5.32), (5.33), (5.23), (5.1)–(5.2) and Lemmas 5.1, 5.2 and 2.5 we have for sufficiently small γ

$$\frac{d\phi}{dt} + \frac{\Phi_{00}}{2} \le c\phi_0(1+\phi_0^3)\Phi_{00} + c\delta_2 + c\gamma\delta_2 + c\delta_2^2 +$$

$$+ c(\delta_1^2 + \tilde{\delta}^2 + \varepsilon_0^2) + cd^2 + c\gamma^2 + c\eta^2 + c\delta_3^4 + c\eta + \tag{5.34}$$

$$+ c(\delta_1 + \tilde{\delta} + \varepsilon_0) + cd + c\delta_3^2\,,$$

(where $\tilde{\delta}$ and ε_0 are constants from Lemma 2.5 and $\varepsilon_0 \to 0$ as $\tilde{\delta} \to 0$).

Now, assume that $t_* = \inf\{t \in [0,T] : \phi(t) > \gamma\}$. Consider (5.34) in $[0,t_*]$ and assume that γ, d, η, δ_1, δ_2 and δ_0 are so small that

$$c(\delta_2 + \gamma\delta_2 + \delta_2^2 + \delta_1^2 + \tilde{\delta}^2 + \varepsilon^2 + d^2 + \gamma^2 + \eta^2 + \delta_3^4 + \eta +$$

$$+ \delta_1 + \tilde{\delta} + \varepsilon_0 + d + \delta_3^2) \le \frac{\gamma\bar{c}C}{16}\,,$$

where $\bar{c}$ is a constant from the inequality

$$\Phi_{00} \ge \bar{c}(\phi - \delta_2) \tag{5.35}$$

and C is the constant form (5.28).

Then, since $\phi_0(t_*) \le C\phi(t_*) = C\gamma$ we obtain

$$\frac{d\phi}{dt}(t_*) + \frac{\Phi_{00}}{2} \le c\gamma(1+\gamma^3)\Phi_{00} + \frac{\gamma\bar{c}C}{16}\,.$$

Hence

$$\frac{d\phi}{dt}(t_*) \le -\Phi_{00}[\frac{1}{2} - c(\gamma+\gamma^4)] + \frac{\gamma\bar{c}C}{16}\,.$$

Thus, using (5.35) we have

$$\frac{d\phi}{dt}(t_*) \le -\bar{c}C\gamma\left[\frac{1}{2} - c(\gamma+\gamma^4)\right] + c\delta_2\left[\frac{1}{2} - c(\gamma+\gamma^4)\right] + \\ + \frac{\gamma\bar{c}C}{16}.$$

Assuming that γ is so small that $c(\gamma+\gamma^4) < \frac{1}{4}$ and δ_2 is so small that $c\delta_2 < \gamma\bar{c}C/8$ we get

$$\frac{d\phi}{dt}(t_*) \le \frac{-C\bar{c}\gamma}{4} + \frac{C\bar{c}\gamma}{8} - c^2\delta_2(\gamma+\gamma^4) < 0\,.$$

Hence $\phi_t(t_*) < 0$, so contradiction.

This concludes the proof. □

Finally, we prove the main result of the paper.

Theorem 5.5. *Let $\nu > \frac{1}{3}\mu$. Let (5.1) – (5.2) and assumptions of Theorem 3.1 with $r, \theta_1 \in C_B^{2,1}(I\!R^3 \times [0,\infty))$ be satisfied, let $(v, \vartheta_0, \rho_\sigma, \bar{\rho}_{\Omega_t}) \in \mathcal{N}(0)$ and*

$$\phi_0(0) \le \alpha_1\,, \quad \| v_0 \|^2_{4,\Omega} \le \alpha_2\,, \tag{5.36}$$

where $\alpha_1, \alpha_2 > 0$ are sufficiently small. Let the following compatibility conditions be satisfied :

$$D_\xi^\alpha \partial_t^i\left(\mathbf{T}\bar{n} - \sigma H\bar{n} + p_0\bar{n}\right)|_{t=0,S} = 0\,, \quad |\alpha| + i \le 2\,,$$

$$D_\xi^\alpha \partial_t^i\left(\bar{n}\cdot\nabla\theta - \theta_1\right)|_{t=0,S} = 0\,, \quad |\alpha| + i \le 2\,.$$

Assume that the internal energy per unit mass $\varepsilon = \varepsilon(\rho,\theta)$ has the form (2.5), conditions (2.6)–(2.7) hold and the parameters μ_0, ν_0, p_0,β, a_0, M satisfy one of the relations

$$\nu_0 \in I_i\,, \quad 0 < \Phi_i\left(\mu_0, \phi_i, p_0, \beta, a_0, M\right) \le \delta_0\,, \tag*{(5.37)$_i$}$$

where $i = 1,2,3$, I_i are defined in (2.8) – (2.10), Φ_i are given by (2.11) and $\delta_0 > 0$ is a sufficiently small constant. Moreover, assume that the following relations hold :

$$\frac{1}{2}\int_\Omega \rho_0 v_0^2\,d\xi + a_0\left(\int_\Omega \rho_0^\beta\,d\xi - \frac{M^\beta}{(c_2\delta+Q_i)^{\beta-1}}\right) + \\ + \int_\Omega \rho_0\left(h(\rho_0,\theta_0) - h_*\right)\,d\xi + \sigma\left[|S| - \tilde{c}(Q_i - c_2\delta)^{\frac{2}{3}}\right] + \tag{5.38}$$

$$+\rho_0\,[|\Omega| - (Q_i - c_2\delta)] + \kappa \sup_t \int_0^t dt' \int_{S_{t'}} \theta_1(s,t')\,ds \le \tilde{\delta}\,,$$

for $\nu_0 \in I_i \quad (i = 1,2,3)$,

$$\int_\Omega |\rho_0 - \rho_e| d\xi \le \tilde{\delta}\,, \tag{5.39}$$

$$\int_\Omega \rho_0 v_0 (a + b \times \xi) d\xi = 0\,, \tag{5.40}$$

$$\int_\Omega \rho_0 \xi d\xi = 0\,, \quad \int_\Omega \rho_0 d\xi = M\,,$$

where Q_i *(*$i = 1,2,3$*),* $\tilde{c} = (36\pi)^{\frac{1}{3}}$, c_2 *and* δ *are defined in Lemma 2.4,* $\tilde{\sigma} > 0$ *is a sufficiently small constant,* a *and* b *are arbitrary constant vectors.*

Next, assume that Ω *is diffeomorphic to a ball and let* S *be described by* $|\xi| = \tilde{R}(\omega)$, $\omega \in S^1$ *(*S^1 *is the unit sphere), where* $\tilde{R}$ *satisfies*

$$\sup_{S^1} |\nabla \tilde{R}|^2 + \| \tilde{R} - R_e \|^2_{0,S^1} \le \alpha_3 \tag{5.41}$$

($\alpha_2 > 0$ *is sufficiently small). Finally, assume that* $S \in W_2^{4+\frac{1}{2}}$ *and it is very close to a sphere, so*

$$\| H(\cdot,0) + \frac{2}{R_e} \|^2_{2,S^1} \le \alpha_4\,, \tag{5.42}$$

where $\alpha_4 > 0$ *is sufficiently small. Then there exists a global solution of problem (1.1) such that* $(v, \vartheta_0, \vartheta, \rho_\sigma, \bar{\rho}_{\Omega_t}) \in \mathcal{M}(t)$ *for* $t \in \mathbb{R}^1_+$, $S_t \in W_2^{4+\frac{1}{2}}$ *and* $\phi_0(t) \le \alpha_1$, $\| H(\cdot,t) + \frac{2}{R_e} \|^2_{2,S^1} \le \alpha_4$.

Proof. At first notice that assumption (5.36) and boundary condition $(4.4)_4$ for $S_t = S$ yield

$$\| H(\cdot,0) + \frac{2}{R_e} \|^2_{2+\frac{1}{2},S^1} \le c_1 \alpha_1\,, \tag{5.43}$$

for α_2 sufficiently small, where c_1 depends on μ, ν, α_1 and the constants from the imbeddings theorems (which depend on $|\Omega_t|$ and the shape of Ω_t, so they also depend on δ_0 and α_1).

Since Ω is diffeomorphic to a ball and $x = \xi + \int_0^t u dt'$ (where $t \le T$, T is the time of the local existence), we obtain that Ω_t $(t \le T)$ is diffeomorphic to a ball and S_t can be described by $|x| = R(\omega,t)$ $(t \le T)$, where $R(\omega,0) = \tilde{R}(\omega)$.

Hence, in view of (5.4), (5.33), (5.36), (5.41) – (5.42) for $t \le T$ we get

$$|R(\cdot,t)-R_t|^2 \le |R(\cdot,0)-R_e|^2 + |R_t-R_e|^2 + \\ + |R(\cdot,t)-R(\cdot,0)|^2 \le \varepsilon \parallel H(\cdot,0)+\frac{2}{R_e} \parallel^2_{0,S^1} + \\ + c \parallel R(\cdot,0)-R_e \parallel^2_{0,S^1} + |R_t-R_e|^2 + \\ + c\left|\int_0^t u dt'\right|^2 \left(1+\left|\int_0^t u dt'\right|\right)^2 \le c_2\,(\alpha_1+\alpha_3+\delta_3)\,, \tag{5.44}$$

where we have used that $|R(\cdot,t)-R(\cdot,0)| = |\,|x|-|\xi|\,| = \frac{|x^2-\xi^2|}{|x|+|\xi|}$ and $x = \xi + \int_0^t u dt'$, $\varepsilon > 0$ is sufficiently small.

Moreover, (5.4), (5.36) and (5.41) imply

$$|\nabla R(\omega,t)|^2 \le \\ \le c\left(|\nabla R(\omega,0)|^2 + \left|\int_0^t u dt'\right|^2 + \left|\int_0^t \nabla u dt'\right|^2\right) \le \\ \le c_3(\alpha_1+\alpha_3)\,. \tag{5.45}$$

Assume that $\alpha_1+\alpha_2$ is so small that (2.4) holds with sufficiently small $\hat{\delta}$. Then Lemma 2.5 implies

$$\parallel R(\cdot,t)-R_t \parallel^2_{1,S^1} \le \varepsilon_0\,,$$

where ε_0 depends on $\tilde{\delta}$ and does not depend on $\hat{\delta}$.

Hence, by (5.33) we have

$$\parallel R(\cdot,t)-R_e \parallel^2_{0,S^1} \le \varepsilon_0 + c_4\delta_3 \equiv \alpha_4\,, \tag{5.46}$$

where α_4 depends on $\tilde{\delta}$ and δ_3, $0 \le t \le T$.

From the assumptions of the theorem and from estimates (5.43) – (5.46) it follows that the assumptions of Lemmas 5.1 and 5.2 are satisfied. Therefore, Lemma 5.4 yields

$$\phi_0(t) \le \alpha_1 \quad \text{for} \quad t \le T\,. \tag{5.47}$$

Now, integrating inequality (5.32) from Lemma 5.4 and using (5.1), (5.2), (5.33), (5.39), (5.41), (5.42), (5.46) and Lemma 5.2 we get

$$\phi + \frac{1}{2}\int_0^t \Phi_{00}\,dt' \le c\alpha_1 + c\alpha_1(1+\alpha_1^3)\int_0^t \Phi_{00}\,dt' +$$

$$+ c\alpha_1(1+\alpha_1^3)\delta_2 + c\alpha_1\int_0^t \Phi_{00}\,dt' + c(\alpha_1\delta_2 + \delta_2^2 +$$

$$+ \delta_1^2 + \tilde{\delta}^2 + \varepsilon_0^2 + d^2 + \alpha_1^2 + \eta^2 + \delta_3^4 + \eta + \delta_1 + \tilde{\delta} + \varepsilon_0 + d + \delta_3^2)\,.$$

Assuming that α_1, δ_1, δ_2, δ_0, $\tilde{\delta}$, η and d are so small that

$$c\alpha_1(1+\alpha_1^3) + c\alpha_1 < \frac{1}{2}$$

and

$$c[\alpha_1(1+\alpha_1^3)\delta_2 + \alpha_1\delta_2 + \delta_2^2 + \delta_1^2 + \tilde{\delta}^2 + \varepsilon_0^2 + d^2 +$$

$$+ \alpha_1^2 + \eta^2 + \delta_3^4 + \eta + \delta_1 + \tilde{\delta} + \varepsilon_0 + d + \delta_3^2] < \alpha_1$$

we get

$$\phi + \int_0^t \Phi_{00}\,dt' \le c_5\alpha_1\,.$$

Therefore, Lemma 5.3 implies

$$\| v \|_{4,\Omega_t}^2 \le c_6\alpha_1\,. \tag{5.48}$$

Thus, by (5.47) and (5.48) we have

$$\| v \|_{4,\Omega_t}^2 + \| \rho_\sigma \|_{3,\Omega_t}^2 + \| \vartheta_0 \|_{3,\Omega_t}^2 \le c_7\alpha_1 \quad \text{for} \quad t \le T\,. \tag{5.49}$$

Therefore, from boundary condition $(4.4)_4$ we obtain that $S_t \in W_2^{4+\frac{1}{2}}$. Moreover, using boundary condition $(4.4)_4$, interpolation inequality (2.2), Lemmas 2.5 and 5.2 we get

$$\begin{aligned} &\| H(\cdot,T) + \frac{2}{R_e} \|_{2,S^1}^2 \le c\left(\| v \|_{3,S^T}^2 + \| \rho_\sigma \|_{2,S^T}^2\right) \le \\ &\le \varepsilon\left(\| v \|_{4,\Omega^T}^2 + \| \rho_\sigma \|_{3,\Omega^T}^2 + \| \vartheta_0 \|_{3,\Omega^T}^2\right) + \\ &+ c(\varepsilon)\left(\| v \|_{0,\Omega^T}^2 + \| \rho_\sigma \|_{0,\Omega^T}^2 + \| \vartheta_0 \|_{0,\Omega^T}^2\right) \le \\ &\le \varepsilon c_7\alpha_1 + c(\varepsilon)c_8(\tilde{\delta} + \delta_2) \le \alpha_4 \end{aligned} \tag{5.50}$$

if ε, δ_2 and $\tilde{\delta}$ are sufficiently small. Finally, boundary condition $(4.4)_4$ and (5.49) imply

$$\| H(\cdot,T) + \frac{2}{R_e} \|^2_{2+\frac{1}{2},S^1} \le c_9\alpha_1 \,, \tag{5.51}$$

where c_9 depends on the same constants as c_1 from (5.43) and in view of Lemma 2.4 and (5.47) we have $|c_1 - c_9| < \delta(\delta_0, \alpha_1)$ and $\delta(\delta_0,\alpha_1)$ is as small as we need if δ_0 and α_1 are sufficiently small.

Now, we are in a position to extend the consideration for interval $[T, 2T]$. By (5.47) and (5.51) we obtain the local existence of solution for $t \in [T, 2T]$ which satisfies

$$\| u \|_{4,\Omega\times(T,2T)} + \| \eta_\sigma \|_{3,\Omega\times(T,2T)} +$$

$$+ |\eta_\sigma|_{3,0,\infty,\Omega\times(T,2T)} + \| \gamma_0 \|_{4,\Omega\times(T,2T)} \le c_9\alpha_1 \,,$$

where on the right-hand side we have the same bound as for $t \in [0,T]$. Therefore,

$$\left| \int_T^{2T} u dt' \right| \le c_{10}\alpha_1 \,,$$

so the change of the shape of Ω_t is as small as for interval $[0,T]$. Hence, the Korn ineqaulities and the imbedding theorems necessary in the proof of (4.6) can be applied with the same constants. Therefore, the same inequality (4.6) holds for $[T, 2T]$.

Now, using (5.44), (5.45), Lemma 2.5 and (5.33) we obtain (5.46) for $t \in [T, 2T]$. Thus, estimates (5.47), (5.50) and (5.46) for $t \in [T, 2T]$ imply that

$$\phi_0(2T) \le \alpha_1 \,.$$

Continuing in the same way considerations we prove the global existence. This completes the proof of the theorem. □

In the case $p_0 = 0$ the following theorem holds.

Theorem 5.6. *Let $p_0 = 0$ and $\nu > \frac{1}{3}\mu$. Let (5.1)–(5.2) and assumptions of Theorem 3.1 with r, $\theta_1 \in C_B^{2,1}(\mathbb{R}^3 \times [0,\infty))$ be satisfied, let $(v, \theta_0, \rho_\sigma, \bar{\rho}_{\Omega_t}) \in \mathcal{N}(0)$ and*

$$\phi_0(0) \le \alpha_1 \,, \quad \| v_0 \|^2_{4,\Omega} \le \alpha_2 \,,$$

where α_1, $\alpha_2 > 0$ *are sufficiently small. Let the following compatibility conditions be satisfied:*

$$D_\xi^\alpha \partial_t^i \left(\mathbf{T}\bar{n} - \sigma H \bar{n} + p_0 \bar{n}\right)|_{t=0,S} = 0\,, \quad |\alpha| + i \le 2\,,$$

$$D_\xi^\alpha \partial_t^i \left(\bar{n} \cdot \nabla\theta - \theta_1\right)|_{t=0,S} = 0\,, \quad |\alpha| + i \le 2\,.$$

Assume that the internal energy per unit mass $\varepsilon = \varepsilon(\rho, \theta)$ *has the form (2.5), conditions (2.6)–(2.7) hold and the parameters* μ_0, ν_0, p_0, β, a_0, M *satisfy the following relations:*

$$\int_\Omega \rho_0 \frac{v_0^2}{2}\, d\xi + \int_\Omega \rho_0 \left(h(\rho_0, \theta_0) - h_*\right)\, d\xi +$$

$$+\kappa \sup_t \int_0^t dt' \int_{S_{t'}} \theta_1(s, t')\, ds \le \delta_0\,,$$

$$\int_\Omega |\rho_e - \rho_0|\, d\xi \le \delta_0\,,$$

$$||S| - |S_e|| \le \delta_0\,,$$

$$0 < \left[\frac{2}{3}(\beta - 1)^{\frac{3(\beta-1)}{2}} \left(\beta - \frac{1}{3}\right)^{\frac{-(3\beta-1)}{2}} (\tilde{c}\sigma)^{\frac{-3(\beta-1)}{2}} \cdot\right.$$

$$\left.\cdot \left(a_0 \rho_e^\beta + \tilde{c}\sigma |\Omega_e|^{-\frac{1}{3}}\right)^{\frac{3\beta-1}{2}} |\Omega_e|^{\frac{\beta-1}{2}} - a_0 \rho_e^\beta\right] |\Omega_e|^\beta \le \delta_0\,,$$

$$\int_\Omega \rho_0 v_0 (a + b \times \xi) d\xi = 0\,,$$

$$\int_\Omega \rho_0 \xi d\xi = 0\,, \quad \int_\Omega \rho_0 d\xi = M\,,$$

$$\frac{1}{2}\int_\Omega \rho_0 v_0^2\, d\xi + a_0 \left(\int_\Omega \rho_0^\beta\, d\xi - \frac{M^\beta}{(c_2\delta + Q_i)^{\beta-1}}\right) +$$

$$+\int_\Omega \rho_0 \left(h(\rho_0, \theta_0) - h_*\right)\, d\xi + \sigma\left[|S| - \tilde{c}(Q_i - c_2\delta)^{\frac{2}{3}}\right] +$$

$$+\kappa \sup_t \int_0^t dt' \int_{S_{t'}} \theta_1(s, t')\, ds \le \tilde{\delta}\,,$$

where $\delta_0 > 0$ *and* $\tilde{\delta} > 0$ *are a sufficiently small constants,* a *and* b *are arbitrary constant vectors.*

Next, assume that Ω *is diffeomorphic to a ball and* S *is described by* $|\xi| = \tilde{R}(\omega)$, $\omega \in S^1$ *(*S^1 *is the unit sphere), where* $\tilde{R}$ *satisfies (5.41). Finally, assume that* $S \in W_2^{4+\frac{1}{2}}$ *and that condition (5.42) is satisfied. Then there exists a global solution of problem (1.1) such that* $(v, \vartheta_0, \vartheta, \rho_\sigma, \bar{\rho}_{\Omega_t}) \in \mathcal{M}(t)$ *for* $t \in \mathbb{R}^1_+$, $S_t \in W_2^{4+\frac{1}{2}}$ *and* $\phi_0(t) \leq \alpha_1$, $\| H(\cdot, t) + \frac{2}{R_e} \|^2_{2,S^1} \leq \alpha_4$.

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