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# Fluid Queues Driven by an $M/M/1/N$ Queue

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In this paper we consider fluid queue models with infinite buffer capacity which receives and releases fluid at variable rates in such a way that the net input rate of fluid into the buffer (which is negative when fluid is flowing out of the buffer) is uniquely determined by the number of customers in an $M/M/1/N$ queue model (that is, the fluid queue is driven by this Markovian queue) with constant arrival and service rates. We use some interesting identities of tridiagonal determinants to find analytically the eigenvalues of the underlying tridiagonal matrix and hence the distribution function of the buffer occupancy. For specific cases, we verify the results available in the literature.

*Keywords and Phrases*: Tridiagonal matrices; Tridiagonal determinants; Eigenvalues; Eigenvectors

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## 1. INTRODUCTION

In the last decade, the literature on queueing theory has paid considerable attention to Markov-modulated fluid models (MMFMs). In these models, a fluid buffer is either filled or depleted, or both, at rates

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which are determined by the current state of a background Markov process. The main reason why the class of MMFMs has attracted so much attention is that they are relevant for modelling certain phenomena in telecommunication networks or otherwise shown to give good approximations for the actual behaviour of network traffic ([1, 6, 7, 10, 11, 14]). In such applications the bursts of data are usually transmitted in many small-sized data packets or cells. Here, the use of fluid models is particularly useful, since the variations on the cell level are almost neglegible compared to those on the more important burst level.

In [2, 13], the authors analyze fluid queue models where the fluid rates are controlled by finite state Markov chains. Fluid queue models where the fluid rates controlled by a state-dependent queueing model $M_n/M_n/N/N$ with linear arrival and service rates have been studied in the literature ([1, 4, 11, 12]). In particular, in [1], the authors find explicit expressions for the eigenvalues and eigenvectors of the underlying matrix.

But there has been no work done in the literature for fluid queues driven by an $M/M/1/N$ queue in providing explicit expressions for the eigenvalues of the matrix and the distribution of the buffer occupancy.

In this paper, we study fluid queues driven by an $M/M/1/N$ queue. In order to achieve closed form analytical expressions for eigenvalues of the underlying tridiagonal matrix by using some identities of tridiagonal determinants, due to Losonczi [9], we restrict our analysis to three particular fluid queues whose net effective input fluid rates differ when the number of customers in the $M/M/1/N$ system is either empty or full. We also give explicit expressions for the distribution functions of the buffer occupancy. In [12], the author gives an explicit expression for the density function of the buffer occupancy in equilibrium for the special case $N=1$. We verify this result by our results.

The rest of the paper is organised as follows: In Section 2, preliminaries and the solution procedure are discussed. Three fluid queue models driven by an $M/M/1/N$ queue are analyzed in Section 3.

Throughout our analysis we use tridiagonal determinants and tridiagonal matrices. Hence we present only the elements in the main diagonal, upper and lower off-diagonals for the convenient sake and other elements are assumed to be zero.

## 2. PRELIMINARIES AND SOLUTION PROCEDURE

Let us denote the $M/M/1/N$ queueing model by $\{X(t),\ t \geq 0\}$ with arrival rate $\lambda$ and service rate $\mu$ where $X(t)$ is the random variable, denoting the number of customers in the system, taking values in $\mathcal{S} = \{0, 1, \ldots, N\}$. Let the generator of the process $\{X(t)\}$ be denoted by $\mathbf{Q}$, that is

$$\mathbf{Q} = \begin{pmatrix} -\lambda & \lambda & & & \\ \mu & -(\lambda+\mu) & \lambda & & \\ & \ddots & \ddots & \ddots & \\ & & & & \lambda \\ & & & \mu & -\mu \end{pmatrix}_{(N+1)\times(N+1)}$$

Let $C(t)$ denote the content of the buffer at time $t$. Whenever $X(t) = j, j \in \mathcal{S}$ and $C(t) \geq 0$, the net input rate of the fluid into the buffer is $r_j$ ($<$ or $>$ 0) with the evident restrictions that at least one $r_i > 0$ (otherwise the buffer will remain empty forever) and the content of the buffer cannot decrease whenever the reservoir is empty. That is,

$$\frac{dC(t)}{dt} = \begin{cases} 0 & \text{if } C(t) = 0 \text{ and } r_{X(t)} < 0. \\ r_{X(t)} & \text{else.} \end{cases}$$

In order that a limit distribution for $C(t)$ exists as $t \to \infty$, the stationary net input rate should be negative, that is,

$$\sum_{i=0}^{N} p_i r_i < 0.$$

where $p_i$, $i \in \mathcal{S}$ be the stationary state probabilities of the background birth–death queue. We assume throughout the analysis that this stability condition is satisfied.

Define

$$F_j(t,u) \equiv \Pr\{X(t) = j, C(t) \leq u\}, \quad j \in \mathcal{S}, \ t, u \geq 0.$$

and

$$F_j(u) \equiv \lim_{t\to\infty} \Pr(X(t) = j, C(t) \leq u), \quad j \in \mathcal{S}, \ u \geq 0$$

That is, $F_j(t,u)$ denotes the probability that the regulating process is in state $j$ and the buffer content does not exceed $u$ at time $t$. Then it can be shown that [3]

$$r_j \frac{dF_j(u)}{du} = \lambda F_{j-1}(u) - (\lambda + \mu)F_j(u) + \mu F_{j+1}(u), \quad u \geq 0,\ j \in \mathcal{S}. \tag{2.1}$$

In matrix notation (2.1) can be written as

$$\frac{d\mathbf{F}(u)}{du} = \mathbf{R}^{-1}\mathbf{Q}^T\mathbf{F}(u), \quad u \geq 0 \tag{2.2}$$

where $\mathbf{F}(u) = [F_0(u), F_1(u), \ldots, F_N(u)]^T$ and $\mathbf{R} = \mathrm{diag}(r_0, r_1, \ldots, r_N)$ and hence

$$\mathbf{R}^{-1}\mathbf{Q}^T = \begin{pmatrix} -\frac{\lambda}{r_0} & \frac{\mu}{r_0} & & & \\ \frac{\lambda}{r_1} & -\frac{\lambda+\mu}{r_1} & \frac{\mu}{r_1} & & \\ & \ddots & \ddots & \ddots & \\ & & & & \frac{\mu}{r_{N-1}} \\ & & & \frac{\lambda}{r_N} & -\frac{\mu}{r_N} \end{pmatrix}_{(N+1)\times(N+1)} \tag{2.3}$$

In [8], the authors give the solution of (2.2) as follows:

$$F_j(u) = p_j + \sum_{l=0}^{N_+-1} \eta_{l,j} \exp(\xi_l u), \quad j \in \mathcal{S} \tag{2.4}$$

where $\eta_{l,j}$ has the following two representations (each representation has its own advantage and disadvantage which we will discuss as we proceed further):

$$\eta_{l,j} = \kappa_l \frac{B_j(\xi_l)}{c_{j0}}, \quad l,j \in \mathcal{S} \tag{2.5}$$

or

$$\eta_{l,j} = \frac{c_{Nj} B_j(\xi_l) \sum_{m \in \mathcal{S}^-} d_m c_{mN} B_m(\xi_l)}{B_N(\xi_l) B'_{N+1}(\xi_l)}, \quad l,j \in \mathcal{S}. \tag{2.6}$$

where $\xi_0, \xi_1, \ldots, \xi_N$ are the eigenvalues of the matrix $\mathbf{R}^{-1}\mathbf{Q}^T$, $\kappa_l$, $l \in \mathcal{S}$ are constants,

$$c_{mj} \equiv \begin{cases} \frac{\lambda^{j-m}}{r_{m+1}r_{m+2}\cdots r_j}, & m<j \\ 1, & m=j, \\ \frac{\mu^{j-m}}{r_j r_{j+1}\cdots r_{m-1}}, & m>j \end{cases} \tag{2.7}$$

the polynomials $B_n(s)$ are defined recursively as follows:

$$\begin{aligned} &B_0(s) = 1 \\ &B_1(s) - \left(s + \frac{\lambda}{r_0}\right)B_0(s) = 0 \\ &B_n(s) - \left(s + \frac{\lambda+\mu}{r_{n-1}}\right)B_{n-1}(s) \\ &\qquad + \frac{\lambda\mu}{r_{n-2}r_{n-1}}B_{n-2}(s) = 0, \quad n = 2,\ldots,N \\ &B_{N+1}(s) - \left(s + \frac{\mu}{r_N}\right)B_N(s) + \frac{\lambda\mu}{r_{N-1}r_N}B_{N-1}(s) = 0. \end{aligned} \tag{2.8}$$

and $d_m \equiv F_m(0)$, $m \in \mathcal{S}$.

Mitra [10] and Stern and Elwalid [14] show that $\mathbf{R}^{-1}\mathbf{Q}^T$ has exactly $N_+$ negative eigenvalues, $N_- - 1$ positive eigenvalues and one zero eigenvalue, where $N_+$ is the cardinality of the set $S^+ \equiv \{j \in \mathcal{S} : r_j > 0\}$ and $N_-$ is that of $S^- \equiv \{j \in \mathcal{S} : r_j < 0\}$. That is,

$$\begin{aligned} &\xi_j < 0, \quad j = 0, 1, \ldots, N_+ - 1, \ \xi_{N_+} = 0 \quad \text{and} \\ &\xi_j > 0, \quad j = N_+ + 1, \ldots, N. \end{aligned}$$

The constants $\kappa_l$, $l = 0,1,\ldots,N_+ - 1$ in (2.5) are determined by solving the system of equations

$$p_j + \sum_{l=0}^{N_+-1} \kappa_l \frac{B_j(\xi_l)}{c_{j0}} = 0, \quad j \in S^+ \tag{2.9}$$

whereas the constants $d_m$, $m \in S^-$ in (2.6) are determined by solving the system of equations ([8])

$$\begin{aligned} &\sum_{m \in S^-} d_m c_{mN} B_m(\xi_l) = 0, \quad l = N_+ + 1, \ldots, N. \\ &\sum_{m \in S^-} r_m d_m = p_0 \sum_{j=0}^{N} r_j \pi_j. \end{aligned} \tag{2.10}$$

where

$$\pi_j = \frac{\lambda_0 \cdots \lambda_{j-1}}{\mu_1 \mu_2 \cdots \mu_j}$$

We observe that the number of unknowns $d_m$ in (2.6) is $N_-$ and so we need to solve as many Eqs. (2.10) to find these quantities. The number of unknowns $\kappa_l$ in (2.5) is $N_+$ and thus we need to solve as many Eqs. (2.9) to find these quantities. Therefore, whenever $N_-$ is large, $N_+$ is small, because $N_- + N_+ = N$, and hence it is convenient to use the expression (2.5) because of a less number of unknowns $\kappa_l$ and whenever $N_-$ is small using (2.6) is more convenient because of a less number of unknowns $d_m$.

In next section we consider three particular fluid queue models driven by an $M/M/1/N$ queue.

## 3. FLUID MODELS

In this section we consider three fluid models, driven by an $M/M/1/N$ queue whose net effective input fluid rates are given as follows:

M-1 The net effective input rates are $r_0 = \left(\lambda r/\left(\lambda - \sqrt{\lambda\mu}\right)\right)$, $r_i = r$, $i = 1, 2, \ldots, N$.

M-2 The net effective input rates are $r_0 = \left(\lambda r/\left(\lambda - \sqrt{\lambda\mu}\right)\right)$, $r_i = r$, $i = 1, 2, \ldots, N-1$ and $r_N = \left(\mu r/\left(\mu - \sqrt{\lambda\mu}\right)\right)$.

M-3 The net effective input rates are $r_0 = \left(\lambda r/\left(\lambda - \sqrt{\lambda\mu}\right)\right)$, $r_i = r$, $i = 1, 2, \ldots, N-1$ and $r_N = \left(\mu r/\left(\mu + \sqrt{\lambda\mu}\right)\right)$.

In all the three models, we assume $r > 0$ and $\lambda < \mu$, so that, $r_0 < 0$ and $r_i > 0$, $i = 1, 2, \ldots, N$.

Before finding exact expressions for the eigenvalues and $\eta_{l,j}$ for the above three models, we give an explicit expression for $B_n(s)$ in the following theorem which will be used in analyzing the models under consideration. We use the following notations:

$$\tilde{\lambda} = \frac{\lambda}{r_0}, \quad \tilde{\mu} = \frac{\mu}{r_N}. \tag{3.1}$$

THEOREM 3.1 *For all the three models under consideration the polynomial $B_n(s)$ in (2.8) can be given by, for $n = 1, 2, \ldots, N$,*

$$B_n(s) = \frac{(\lambda\mu)^{(n/2)}}{r^n sin\theta}\left\{sin(n+1)\theta - \left(\frac{\lambda+\mu-r\tilde{\lambda}}{\sqrt{\lambda\mu}}\right)sin\; n\theta + \left(\frac{\lambda - r\tilde{\lambda}}{\lambda}\right)sin(n-1)\theta\right\} \tag{3.2}$$

*where* $s + (\lambda/r) + (\mu/r) = 2(\sqrt{\lambda\mu}/r)\; cos\,\theta.$

*Proof* From (2.8), $B_n(s)$ can be written in determinant form as follows: for $n = 1, 2, \ldots, N$,

$$B_n(s) = \begin{vmatrix} s+\frac{\lambda}{r_0} & \frac{\mu}{r_1} & & & & \\ \frac{\lambda}{r_0} & s+\frac{\lambda+\mu}{r_1} & \frac{\mu}{r_2} & & & \\ & \frac{\lambda}{r_1} & s+\frac{\lambda+\mu}{r_2} & \frac{\mu}{r_3} & & \\ & \ddots & \ddots & \ddots & & \\ & & & & \frac{\mu}{r_N} & \\ & & & \frac{\lambda}{r_{N-1}} & s+\frac{\lambda+\mu}{r_{n-1}} & \end{vmatrix}_{n\times n} \tag{3.3}$$

Substituting $r_0 = r_0$ and $r_i = r$, $i = 1, 2, \ldots, n-1$ and using (3.1) in (3.3) we get,

$$B_n(s) = \begin{vmatrix} s+\tilde{\lambda} & \frac{\mu}{r} & & & \\ \tilde{\lambda} & s+\frac{\lambda}{r}+\frac{\mu}{r} & \frac{\mu}{r} & & \\ & \frac{\lambda}{r} & s+\frac{\lambda}{r}+\frac{\mu}{r} & \frac{\mu}{r} & \\ & \ddots & \ddots & \ddots & \\ & & & & \frac{\mu}{r} \\ & & & \frac{\lambda}{r} & s+\frac{\lambda}{r}+\frac{\mu}{r} \end{vmatrix}_{n\times n}$$

which can be written as

$$B_n(s) = \begin{vmatrix} s+\frac{\lambda}{r}+\frac{\mu}{r} & \frac{\sqrt{\lambda\mu}}{r} & & & \\ \frac{\sqrt{\lambda\mu}}{r} & s+\frac{\lambda}{r}+\frac{\mu}{r} & \frac{\sqrt{\lambda\mu}}{r} & & \\ & \ddots & \ddots & \ddots & \\ & & & & \frac{\sqrt{\lambda\mu}}{r} \\ & & & \frac{\sqrt{\lambda\mu}}{r} & s+\frac{\lambda}{r}+\frac{\mu}{r} \end{vmatrix}_{n\times n}$$

$$+\left(\tilde{\lambda}-\frac{\lambda}{r}-\frac{\mu}{r}\right)\times\begin{vmatrix} s+\frac{\lambda}{r}+\frac{\mu}{r} & \frac{\sqrt{\lambda\mu}}{r} & & & \\ \frac{\sqrt{\lambda\mu}}{r} & s+\frac{\lambda}{r}+\frac{\mu}{r} & \frac{\sqrt{\lambda\mu}}{r} & & \\ & \ddots & \ddots & \ddots & \\ & & & & \frac{\sqrt{\lambda\mu}}{r} \\ & & & \frac{\sqrt{\lambda\mu}}{r} & s+\frac{\lambda}{r}+\frac{\mu}{r} \end{vmatrix}_{(n-1)\times(n-1)}$$

$$-\left(\frac{\lambda}{r}-\tilde{\lambda}\right)\frac{\mu}{r}\times\begin{vmatrix} s+\frac{\lambda}{r}+\frac{\mu}{r} & \frac{\sqrt{\lambda\mu}}{r} & & & \\ \frac{\sqrt{\lambda\mu}}{r} & s+\frac{\lambda}{r}+\frac{\mu}{r} & \frac{\sqrt{\lambda\mu}}{r} & & \\ & \ddots & \ddots & \ddots & \\ & & & & \frac{\sqrt{\lambda\mu}}{r} \\ & & & \frac{\sqrt{\lambda\mu}}{r} & s+\frac{\lambda}{r}+\frac{\mu}{r} \end{vmatrix}_{(n-2)\times(n-2)}$$

The theorem follows by dividing each row by $(\sqrt{\lambda\mu}/r)$, putting $s+(\lambda/r)+(\mu/r)=2(\sqrt{\lambda\mu}/r)\cos\theta$ and using the following identity:

$$\begin{vmatrix} 2\cos\theta & 1 & & & \\ 1 & 2\cos\theta & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & 1 \\ & & & 1 & 2\cos\theta \end{vmatrix}_{n\times n} = \frac{\sin(n+1)\theta}{\sin\theta}, \quad n\geq 1.$$

■

We note that $B_{N+1}(s)$, defined recursively by (2.8), can be given in a tridiagonal determinant form as follows:

$$B_{N+1}(s)=\begin{vmatrix} s+\frac{\lambda}{r_0} & -\frac{\mu}{r_0} & & & \\ -\frac{\lambda}{r_1} & s+\frac{\lambda+\mu}{r_1} & -\frac{\mu}{r_1} & & \\ & -\frac{\lambda}{r_2} & s+\frac{\lambda+\mu}{r_2} & -\frac{\mu}{r_2} & \\ & & & \ddots & \\ & & & & s+\frac{\mu}{r_N} \end{vmatrix}_{(N+1)\times(N+1)}$$

and hence $B_{N+1}(s)$ is the characteristic polynomial of $\mathbf{R}^{-1}\mathbf{Q}^T$ in $s$ of degree $N+1$. The above expression can be written as follows [8, Theorem 4.1]:

$$B_{N+1}(s) = s \times \begin{vmatrix} s+\frac{\lambda}{r_0}+\frac{\mu}{r_1} & \frac{\mu}{r_1} & & & \\ \frac{\lambda}{r_1} & s+\frac{\lambda}{r_1}+\frac{\mu}{r_2} & \frac{\mu}{r_2} & & \\ & \ddots & \ddots & \ddots & \\ & & & & \frac{\mu}{r_{N-1}} \\ & & & \frac{\lambda}{r_{N-1}} & s+\frac{\lambda}{r_{N-1}}+\frac{\mu}{r_N} \end{vmatrix}_{N\times N} \tag{3.4}$$

Substituting $r_0=r_0$, $r_N=r_N$ and $r_i=r$, $i=1,2,\dots,N-1$ and using (3.1) in (3.4) we get

$$B_{N+1}(s) = s \times \begin{vmatrix} s+\tilde{\lambda}+\frac{\mu}{r} & \frac{\mu}{r} & & & \\ \frac{\lambda}{r} & s+\frac{\lambda}{r}+\frac{\mu}{r} & \frac{\mu}{r} & & \\ & \ddots & \ddots & \ddots & \\ & & & & \frac{\mu}{r} \\ & & & \frac{\lambda}{r} & s+\frac{\lambda}{r}+\tilde{\mu} \end{vmatrix}_{N\times N} \tag{3.5}$$

Since in all the three models $r_0<0$ and $r_i>0$, $i=1,2,\dots,N$, $S^-=\{0\}$ and $S^+=\{1,2,\dots,N\}$. Hence $\mathbf{R}^{-1}\mathbf{Q}^T$ will have one zero eigenvalue $(=\xi_0$, say) and $N$ negative eigenvalues ($\xi_l$, $l=1,2,\dots,N$, say). Moreover, $N_-=1$ we use (2.6) to compute $\eta_{l,j}$, which can be rewritten as

$$\eta_{l,j} = \frac{c_{Nj}B_j(\xi_l)d_0c_{0N}}{B_N(\xi_l)\xi_l\prod_{i=1,i\neq l}^{N}(\xi_l-\xi_i)}. \tag{3.6}$$

Since $S^-=\{0\}$ and $\pi_j=\rho^j$, $j\in\mathcal{S}$, where $\rho=(\lambda/\mu)$, the constant $d_0$ in (2.10) can be given by

$$d_0 = \frac{1-\rho}{r_0(1-\rho^{N+1})}\sum_{j=0}^{N} r_j\rho^j. \tag{3.7}$$

Also, for all the three models we have

$$p_j = \frac{(1-\rho)\rho^j}{1-\rho^{N+1}}, \quad j\in\mathcal{S}.$$

Now we are ready to find exact expressions for $F_j(u)$ for the three fluid queue models under consideration. We present these results in the following three theorems. We give the proof for all the three theorems for we use different trigonometric identities in each of these theorems.

THEOREM 3.2 *For the fluid queue model M-1 the eigenvalues $\xi_l$, $l=0, 1,\ldots,N$ of the matrix $\boldsymbol{R}^{-1}\boldsymbol{Q}^T$ are given by*

$$\xi_0 = 0,\ \xi_l = -\frac{\lambda}{r} - \frac{\mu}{r} + 2\frac{\sqrt{\lambda\mu}}{r} cos\frac{(2l-1)\pi}{2N+1}, \quad l = 1,2,\ldots,N \tag{3.8}$$

*and for $j \in \mathcal{S}$,*

$$F_j(u) = \frac{(1-\rho)\rho^j}{1-\rho^{N+1}} + \frac{4d_0\sqrt{\rho}^{j+2}}{2N+1} \sum_{l=1}^{N} \frac{cos(y_l/2)[cos(2j+1)(y_l/2) - \rho^{-1/2}cos(2j-1)(y_l/2)]}{[1+\rho^{-2}\sqrt{\rho}\,cos\,y_l]} exp(\xi_l u) \tag{3.9}$$

*where*

$$d_0 = 1 - \sqrt{\rho}\frac{1-\rho^N}{1-\rho^{N+1}} \quad and \quad y_l = \frac{(2l-1)\pi}{2N+1}.$$

*Proof* We use the following identity of [9] to find the eigenvalues of the matrix $\mathbf{R}^{-1}\mathbf{Q}^T$ related to this model.

*Identity*

$$\begin{vmatrix} v-\sqrt{pq} & p & & & \\ q & v & p & & \\ & \ddots & \ddots & \ddots & \\ & & & v & p \\ & & & q & v \end{vmatrix}_{N\times N} = \prod_{l=1}^{N}\left(v - 2\sqrt{pq}\cos\frac{(2l-1)\pi}{2N+1}\right).$$

Since $r_0 = r\lambda/(\lambda - \sqrt{\lambda\mu})$ and $r_N = r$ we have from (3.1),

$$\tilde{\lambda} = \frac{\lambda}{r} - \frac{\sqrt{\lambda\mu}}{r} \quad \text{and} \quad \tilde{\mu} = \frac{\mu}{r}.$$

Substituting for $\tilde{\lambda}$ and $\tilde{\mu}$ in (3.5) we get

$$B_{N+1}(s) = s \times \begin{vmatrix} s+\frac{\lambda}{r}+\frac{\mu}{r}-\frac{\sqrt{\lambda\mu}}{r} & \frac{\mu}{r} & & & \\ \frac{\lambda}{r} & s+\frac{\lambda}{r}+\frac{\mu}{r} & \frac{\mu}{r} & & \\ & \ddots & \ddots & \ddots & \\ & & & & \frac{\mu}{r} \\ & & & \frac{\mu}{r} & s+\frac{\lambda}{r}+\frac{\mu}{r} \end{vmatrix}_{N\times N} \tag{3.10}$$

Hence by the above identity we have

$$B_{N+1}(s) = s\prod_{l=1}^{N}\left(s+\frac{\lambda}{r}+\frac{\mu}{r}-2\frac{\sqrt{\lambda\mu}}{r}\cos\frac{(2l-1)\pi}{2N+1}\right).$$

Therefore the eigenvalues of $\mathbf{R}^{-1}\mathbf{Q}^T$ are

$$\xi_0 = 0,\ \xi_l = -\frac{\lambda}{r}-\frac{\mu}{r}+2\frac{\sqrt{\lambda\mu}}{r}\cos\frac{(2l-1)\pi}{2N+1}, \quad l = 1,2,\dots,N. \tag{3.11}$$

Now, we find an expression for $B_n(\xi_l)$ for this model. Substituting $\tilde{\lambda} = (\lambda/r) - (\sqrt{\lambda\mu}/r)$ and $\xi_l$ in (3.2) and using the following identity

$$\frac{\sin(n+1)\theta - \sin n\theta}{\sin\theta} = \frac{\cos(2n+1)\theta/2}{\cos\theta/2} \tag{3.12}$$

we get

$$B_n(\xi_l) = \frac{(\sqrt{\lambda\mu})^n}{r^n\cos(y_l/2)}\left[\cos(2n+1)\frac{y_l}{2} - \rho^{-1/2}\cos(2n-1)\frac{y_l}{2}\right], \quad n\in\mathcal{S}. \tag{3.13}$$

where $y_l = (2l-1)\pi/(2N+1)$. In particular,

$$B_N(\xi_l) = \frac{-\rho^{-1/2}(\sqrt{\lambda\mu})^N}{r^N\cos(y_l/2)}\cos(2N-1)\frac{y_l}{2}. \tag{3.14}$$

Now we find an expression for $\xi_l\prod_{i=1,i\neq l}^{N}(\xi_l-\xi_i)$.

Substituting for $\xi_l$ in the above expression and using the following identity [5]:

$$\prod_{i=1, i\neq l}^{N} [\cos y_l - \cos y_i] = \frac{2N+1}{2^{N+1}\cos(2N-1)(y_l/2)\cos(y_l/2)} \tag{3.15}$$

(where $y_l = ((2l-1)\pi)/(2N+1)$, $l = 1, 2, \ldots, N$) we get

$$\xi_l \prod_{i=1, i\neq l}^{N} (\xi_l - \xi_i) = -\frac{(2N+1)\rho^{-1/2}\left(\sqrt{\lambda\mu}\right)^N}{4r^N} \frac{[1+\rho-2\rho^{1/2}\cos y_l]}{\cos(2N-1)(y_l/2)\cos(y_l/2)}. \tag{3.16}$$

Also, from (2.7) we have

$$c_{Nj} = \begin{cases} \frac{\mu^{N-j}}{r^{N-j}}, & j = 1, 2, \ldots, N \\ \left(1 - \sqrt{\rho}^{-1}\right)\frac{\mu^{N-j}}{r^{N-j}}, & j = 0 \end{cases} \tag{3.17}$$

and

$$c_{0N} = \frac{\lambda^N}{r^N}. \tag{3.18}$$

Putting $r_0 = \lambda r/\left(\lambda - \sqrt{\lambda\mu}\right)$ and $r_i = r$, $i = 1, 2, \ldots, N$ in (3.7) we get after some calculation

$$d_0 = 1 - \sqrt{\rho}\frac{1-\rho^N}{1-\rho^{N+1}}.$$

Substituting (3.13), (3.14), (3.16), (3.17) and (3.18) in (3.6) we get after considerable simplification

$$\eta_{l,j} = \frac{4d_0\sqrt{\rho}^{j+2}}{2N+1} \times \frac{\cos(y_l/2)[\cos(2j+1)(y_l/2) - \rho^{-1/2}\cos(2j-1)(y_l/2)]}{[1+\rho-2\sqrt{\rho}\cos y_l]}, \quad l, j \in \mathcal{S}.$$

Theorem follows by substituting for $\eta_{l,j}$ in (2.4). ■

*Special Case* $N = 1$.

The eigenvalues are

$$\xi_0 = 0 \quad \text{and} \quad \xi_1 = -\frac{\lambda}{r} - \frac{\mu}{r} + \frac{\sqrt{\lambda\mu}}{r} = -\frac{\mu}{r}\left(1 + \rho - \sqrt{\rho}\right).$$

The equilibrium distributions of the buffer occupancy are given by

$$F_0(u) = \frac{1}{1+\rho} - \frac{\rho(1-\sqrt{\rho})}{1+\rho}\exp(\xi_1 u), \quad \text{and}$$
$$F_1(u) = \frac{\rho}{1+\rho} - \frac{\rho}{1+\rho}\exp(\xi_1 u).$$

Hence, the marginal distribution of the buffer occupancy is given by

$$P(C \leq u) = F_0(u) + F_1(u) = 1 - \frac{\sqrt{\rho}}{1+\rho}\exp(\xi_1 u).$$

Therefore, the probability density function $b(u)$ of having $u$ amount of fluid in the buffer is given by

$$b(u) = \frac{\mu}{r}\frac{\sqrt{\rho}(1+\rho-\sqrt{\rho})}{1+\rho}\exp(\xi_1 u)$$

with point mass $d_0$ at $x=0$:

$$d_0 = 1 - \frac{\sqrt{\rho}}{1+\rho}$$

which verifies the result for the case

$$S = 1, c = 0, \delta = r \quad \text{and} \quad \eta = \frac{r}{1-\sqrt{\rho}}$$

in [12, p. 122].

THEOREM 3.3 *For the fluid queue model M-2 the eigenvalues $\xi_l$, $l=0, 1, \ldots, N$ of the matrix $\boldsymbol{R}^{-1}\boldsymbol{Q}^T$ are given by*

$$\xi_0 = 0,\ \xi_l = -\frac{\lambda}{r} - \frac{\mu}{r} + 2\frac{\sqrt{\lambda\mu}}{r}\cos\frac{(l-1)\pi}{N}, \quad l = 1, 2, \ldots, N \quad (3.19)$$

*and for $j \in S$,*

$$F_j(u) = \frac{(1-\rho)\rho^j}{1-\rho^{N+1}} + d_0\sqrt{\rho}^{j+2}$$
$$\sum_{l=1}^{N} \frac{\sin y_l(\cos y_l + 1)\left[\cos(2j+1)(y_l/2) - \rho^{-1/2}\cos(2j-1)(y_l/2)\right]}{\cos(2N-1)(y_l/2)[N\cos Ny_l \sin y_l + \sin Ny_l][1+\rho-2\sqrt{\rho}\cos y_l]}\exp(\xi_l u)$$
$$(3.20)$$

*where*

$$d_0 = \frac{(1-\rho^N)(1-\sqrt{\rho})}{1-\rho^{N+1}} \text{ and } y_l = \frac{(l-1)\pi}{N}.$$

*Proof* We use the following identity of [9] to find the eigenvalues of the matrix $\mathbf{R}^{-1}\mathbf{Q}^T$ related to this model.

*Identity*

$$\begin{vmatrix} v-\sqrt{pq} & p & & & \\ q & v & p & & \\ & \ddots & \ddots & \ddots & \\ & & & v & p \\ & & & q & v-\sqrt{pq} \end{vmatrix}_{N\times N} = \prod_{l=1}^{N}\left(v - 2\sqrt{pq}\cos\frac{(l-1)\pi}{N}\right).$$

Since $r_0 = r\lambda/(\lambda-\sqrt{\lambda\mu})$ and $r_N = r\mu/(\mu-\sqrt{\lambda\mu})$ we have from (3.1),

$$\tilde{\lambda} = \frac{\lambda}{r} - \frac{\sqrt{\lambda\mu}}{r} \quad \text{and} \quad \tilde{\mu} = \frac{\mu}{r} - \frac{\sqrt{\lambda\mu}}{r}.$$

Substituting for $\tilde{\lambda}$ and $\tilde{\mu}$ in (3.5) we get

$$B_{N+1}(s) = s \times \begin{vmatrix} s+\frac{\lambda}{r}+\frac{\mu}{r}-\frac{\sqrt{\lambda\mu}}{r} & \frac{\mu}{r} & & & \\ \frac{\lambda}{r} & s+\frac{\lambda}{r}+\frac{\mu}{r} & \frac{\mu}{r} & & \\ & \ddots & \ddots & \ddots & \\ & & & & \frac{\mu}{r} \\ & & & \frac{\mu}{r} & s+\frac{\lambda}{r}+\frac{\mu}{r}-\frac{\sqrt{\lambda\mu}}{r} \end{vmatrix}_{N\times N} \tag{3.21}$$

Hence by the above identity we have

$$B_{N+1}(s) = s\prod_{l=1}^{N}\left(s+\frac{\lambda}{r}+\frac{\mu}{r}-2\frac{\sqrt{\lambda\mu}}{r}\cos\frac{(l-1)\pi}{N}\right).$$

Therefore the eigenvalues of $\mathbf{R}^{-1}\mathbf{Q}^T$ are

$$\xi_0 = 0,\ \xi_l = -\frac{\lambda}{r} - \frac{\mu}{r} + 2\frac{\sqrt{\lambda\mu}}{r}\cos\frac{(l-1)\pi}{N}, \quad l = 1,2,\ldots,N. \quad (3.22)$$

Now, we find an expression for $B_n(\xi_l)$ for this model. Substituting $\tilde{\lambda} = (\lambda/r) - (\sqrt{\lambda\mu}/r)$ and $\xi_l$ in (3.2) and using the identity (3.12) we get

$$B_n(\xi_l) = \frac{\left(\sqrt{\lambda\mu}\right)^n}{r^n\cos(y_l/2)}\left[\cos(2n+1)\frac{y_1}{2} - \rho^{-1/2}\cos(2n-1)\frac{y_l}{2}\right], \quad n\in\mathcal{S} \quad (3.23)$$

where $y_l = (l-1)\pi/N$. In particular,

$$B_N(\xi_l) = \frac{(1-\rho^{-1/2})\left(\sqrt{\lambda\mu}\right)^N}{r^N\cos(y_l/2)}\cos(2N-1)\frac{y_l}{2}. \quad (3.24)$$

Now we find an expression for $\xi_l\prod_{i=1,i\neq l}^N(\xi_l - \xi_i)$.

Substituting for $\xi_l$ in the above expression and using the following identity [5]:

$$\prod_{i=1,i\neq l}^{N}[\cos y_l - \cos y_i] = \frac{N\cos Ny_l\sin y_l + \sin Ny_l}{2^{N-1}\sin y_l(\cos y_l + 1)} \quad (3.25)$$

(where $y_l = (l-1)\pi/N$, $l = 1,2,\ldots,N$) we get after some simplification

$$\xi_l\prod_{i=1,i\neq l}^{N}(\xi_l - \xi_i) = -\frac{\rho^{1/2}\left(\sqrt{\lambda\mu}\right)^N}{r^N} \frac{N\cos Ny_l\sin y_l + \sin Ny_l}{\sin y_l(\cos y_l + 1)}[1 + \rho - 2\rho^{1/2}\cos y_l] \quad (3.26)$$

Also, from (2.7) we have

$$c_{Nj} = \begin{cases} \frac{\mu^{N-j}}{r^{N-j}}, & j = 1,2,\ldots,N \\ \left(1-\sqrt{\rho}^{-1}\right)\frac{\mu^{N-j}}{r^{N-j}}, & j = 0 \end{cases} \quad (3.27)$$

and

$$c_{0N} = \left(1-\sqrt{\rho}\right)\frac{\lambda^N}{r^N}. \quad (3.28)$$

Putting $r_0 = \lambda r/(\mu - \sqrt{\lambda\mu})$, $r_N = \mu r/(\lambda - \sqrt{\lambda\mu})$ and $r_i = r$, $i = 1, 2, \ldots, N-1$ in (3.7) we get after some calculation

$$d_0 = \frac{(1-\rho^N)(1-\sqrt{\rho})}{1-\rho^{N+1}}.$$

Substituting (3.23), (3.24), (3.26), (3.27) and (3.28) in (3.6) we get after considerable simplification

$$\eta_{l,j} = d_0\sqrt{\rho}^{\,j+2} \\ \frac{\sin y_l(\cos y_l+1)\left[\cos(2j+1)(y_l/2)-\rho^{-1/2}\cos(2j-1)(y_l/2)\right]}{\cos(2N-1)(y_l/2)[N\cos Ny_l\sin y_l+\sin Ny_l][1+\rho-2\sqrt{\rho}\cos y_l]},\quad l,j,S.$$

Theorem follows by substituting for $\eta_{l,j}$ in (2.4). ∎

*Special Case* $N=1$.

The eigenvalues are

$$\xi_0 = 0 \quad \text{and} \quad \xi_1 = -\frac{\lambda}{r} - \frac{\mu}{r} + 2\frac{\sqrt{\lambda\mu}}{r} = -\frac{\mu}{r}(1-\sqrt{\rho})^2.$$

The equilibrium distributions of the buffer occupancy are given by

$$F_0(u) = \frac{1}{1+\rho} - \frac{\sqrt{\rho}}{1+\rho}\exp(\xi_1 u), \quad \text{and}$$
$$F_1(u) = \frac{\rho}{1+\rho} - \frac{\rho}{1+\rho}\exp(\xi_1 u).$$

Hence, the marginal distribution of the buffer occupancy is given by

$$P(C \le u) = 1 - \frac{\sqrt{\rho}(1+\sqrt{\rho})}{1+\rho}\exp(\xi_1 u).$$

Therefore, the probability density function $b(u)$ of having $u$ amount of fluid in the buffer is given by

$$b(u) = \frac{\mu}{r}\frac{\sqrt{\rho}(1-\sqrt{\rho})(1-\rho)}{1+\rho}\exp(\xi_1 u)$$

with point mass $d_0$ at $x=0$:

$$d_0 = \frac{1-\sqrt{\rho}}{1+\rho}$$

which verifies the result for the case

$$S = 1, c = 0, \delta = \frac{r}{1-\sqrt{\rho}} \quad \text{and} \quad \eta = \frac{r(1-\rho)}{(1-\sqrt{\rho})^2}.$$

in [12, p. 122].

THEOREM 3.4 *For the fluid queue model M-3 the eigenvalues $\xi_l$, $l=0, 1,\ldots,N$ of the matrix $\boldsymbol{R}^{-1}\boldsymbol{Q}^T$ are given by*

$$\xi_0 = 0, \xi_l = -\frac{\lambda}{r} - \frac{\mu}{r} + 2\frac{\sqrt{\lambda\mu}}{r} cos\frac{(2l-1)\pi}{2N}, \quad l = 1,2,\ldots,N \tag{3.29}$$

*and for $j \in S$,*

$$F_j(u) = \frac{(1-\rho)\rho^j}{1-\rho^{N+1}} - \frac{d_0\sqrt{\rho}^{j+1}(1+\sqrt{\rho})}{N} \sum_{l=1}^{N} \frac{sin\ y_l\left[cos(2j+1)(y_l/2) - \rho^{-1/2}cos(2j-1)(y_l/2)\right]}{\Theta_1 sin\ Ny_l[1+\rho-2\sqrt{\rho}\ cos\ y_l]} exp \tag{3.30}$$

*where*

$$d_0 = \frac{(1+\rho^N)(1-\sqrt{\rho})}{1-\rho^{N+1}} \quad and \quad y_l = \frac{(2l-1)\pi}{2N}$$

*and*

$$\Theta_l = \left[cos(2N+1)\frac{y_l}{2} - \rho^{-1/2}cos(2N-1)\frac{y_l}{2}\right].$$

*Proof* We use the following identity of [9] to find the eigenvalues of the matrix $\mathbf{R}^{-1}\mathbf{Q}^T$ related to this model.

*Identity*

$$\begin{vmatrix} v-\sqrt{pq} & p & & & \\ q & v & p & & \\ & \ddots & \ddots & \ddots & \\ & & & v & p \\ & & & q & v+\sqrt{pq} \end{vmatrix}_{N\times N} = \prod_{l=1}^{N}\left(v-2\sqrt{pq}\cos\frac{(2l-1)\pi}{2N}\right).$$

Since $r_0 = r\lambda/(\lambda-\sqrt{\lambda\mu})$ and $r_N = r\mu/(\mu+\sqrt{\lambda\mu})$ we have from (3.1),

$$\tilde{\lambda} = \frac{\lambda}{r} - \frac{\sqrt{\lambda\mu}}{r} \quad \text{and} \quad \tilde{\mu} = \frac{\mu}{r} + \frac{\sqrt{\lambda\mu}}{r}.$$

Substituting for $\tilde{\lambda}$ and $\tilde{\mu}$ in (3.5) we get

$$B_{N+1}(s) = s \times \begin{vmatrix} s+\frac{\lambda}{r}+\frac{\mu}{r}-\frac{\sqrt{\lambda\mu}}{r} & \frac{\mu}{r} & & & \\ \frac{\lambda}{r} & s+\frac{\lambda}{r}+\frac{\mu}{r} & \frac{\mu}{r} & & \\ & \ddots & \ddots & \ddots & \\ & & & & \frac{\mu}{r} \\ & & & \frac{\mu}{r} & s+\frac{\lambda}{r}+\frac{\mu}{r}+\frac{\sqrt{\lambda\mu}}{r} \end{vmatrix}_{N\times N} \tag{3.31}$$

Hence by the above identity we have

$$B_{N+1}(s) = s\prod_{l=1}^{N}\left(s+\frac{\lambda}{r}+\frac{\mu}{r}-2\frac{\sqrt{\lambda\mu}}{r}\cos\frac{(2l-1)\pi}{2N}\right).$$

Therefore the eigenvalues of $\mathbf{R}^{-1}\mathbf{Q}^T$ are

$$\xi_0 = 0,\ \xi_l = -\frac{\lambda}{r}-\frac{\mu}{r}+2\frac{\sqrt{\lambda\mu}}{r}\cos\frac{(2l-1)\pi}{2N}, \quad l = 1,2,\ldots,N. \tag{3.32}$$

Now, we find an expression for $B_n(\xi_l)$ for this model. Substituting $\tilde{\lambda} = (\lambda/r) - (\sqrt{\lambda\mu}/r)$ and $\xi_l$ in (3.2) and using the identity (3.12) we get

$$B_n(\xi_l) = \frac{(\sqrt{\lambda\mu})^n}{r^n \cos(y_l/2)} \left[\cos(2n+1)\frac{y_l}{2} - \rho^{-1/2}\cos(2n-1)\frac{y_l}{2}\right], \quad n \in S \tag{3.33}$$

where $y_l = (2l-1)\pi/2N$. In particular,

$$B_N(\xi_l) = \frac{(\sqrt{\lambda\mu})^N}{r^N \cos(y_l/2)} \left[\cos(2N+1)\frac{y_l}{2} - \rho^{-1/2}\cos(2N-1)\frac{y_l}{2}\right]. \tag{3.34}$$

Now we find an expression for $\xi_l \prod_{i=1, i\neq l}^{N} (\xi_l - \xi_i)$.

Substituting for $\xi_l$ in the above expression and using the following identity [5]:

$$\prod_{i=1, i\neq l}^{N} [\cos y_l - \cos y_i] = \frac{N \sin N y_l}{2^{N-1} \sin y_l} \tag{3.35}$$

(where $y_l = (2l-1)\pi/2N$, $l = 1, 2, \ldots, N$) we get after some simplification

$$\xi_l \prod_{i=1, i\neq l}^{N} (\xi_l - \xi_i) = -\frac{N\rho^{-1/2}(\sqrt{\lambda\mu})^N \sin N y_l}{r^N} \frac{[1 + \rho - 2\rho^{1/2}\cos y_l]}{\sin y_l} \tag{3.36}$$

Also, from (2.7) we have

$$c_{Nj} = \begin{cases} \frac{\mu^{N-j}}{r^{N-j}}, & j = 1, 2, \ldots, N \\ (1 - \sqrt{\rho}^{-1}) \frac{\mu^{N-j}}{r^{N-j}}, & j = 0 \end{cases} \tag{3.37}$$

and

$$c_{0N} = (1 + \sqrt{\rho}) \frac{\lambda^N}{r^N}. \tag{3.38}$$

Putting $r_0 = \lambda r/(\lambda - \sqrt{\lambda\mu})$, $r_N = \mu r/(\mu + \sqrt{\lambda\mu})$ and $r_i = r$, $i = 1, 2, \ldots, N-1$ in (3.7) we get after some calculation

$$d_0 = \frac{(1+\rho^N)(1-\sqrt{\rho})}{1-\rho^{N+1}}.$$

Substituting (3.33), (3.34), (3.36), (3.37) and (3.38) in (3.6) we get after considerable simplification

$$\eta_{l,j} = -\frac{d_0\sqrt{\rho}^{j+1}(1+\sqrt{\rho})}{N} \frac{\sin y_l[\cos(2j+1)(y_l/2) - \rho^{-1/2}\cos(2j-1)(y_l/2)]}{\Theta_l \sin Ny_l[1+\rho-2\sqrt{\rho}\cos y_l]}, \quad l,j \in \mathcal{S}.$$

Theorem follows by substituting for $\eta_{l,j}$ in (2.4). ■■

*Special Case* $N = 1$.

The eigenvalues are

$$\xi_0 = 0 \text{ and } \xi_1 = -\frac{\lambda}{r} - \frac{\mu}{r} = -\frac{\mu}{r}(1+\rho).$$

The equilibrium distributions of the buffer occupancy are given by

$$F_0(u) = \frac{1}{1+\rho} - \frac{\sqrt{\rho}(1-\sqrt{\rho})}{(1+\rho)(1+\sqrt{\rho})}\exp(\xi_1 u), \text{ and}$$
$$F_1(u) = \frac{\rho}{1+\rho} - \frac{\rho}{1+\rho}\exp(\xi_1 u).$$

Hence, the marginal distribution of the buffer occupancy is given by

$$P(C \le u) = 1 - \frac{\sqrt{\rho}}{1+\sqrt{\rho}}\exp(\xi_1 u).$$

Therefore, the probability density function $b(u)$ of having $u$ amount of fluid in the buffer is given by

$$b(u) = \frac{\mu}{r}\frac{\sqrt{\rho}(1+\rho)}{1+\sqrt{\rho}}\exp(\xi_1 u)$$

with point mass $d_0$ at $x=0$:

$$d_0 = \frac{1}{1+\sqrt{\rho}}$$

which verifies the result for the case

$$S=1,\ c=0,\ \delta = \frac{r}{1+\sqrt{\rho}} \text{ and } \eta = \frac{r(1+\rho)}{1-\rho}.$$

in [12, p. 122].

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