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New Results in Subspace-Stabilization Control Theory

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Subspace-stabilization is a generalization of the classical idea of stabilizing motions of a dynamical system to an equilibrium state. The concept of subspace-stabilization and a theory for designing subspace-stabilizing control laws was introduced in a previously published paper. In the present paper, two new alternative methods for designing control laws that achieve subspace-stabilization are presented. These two alternative design methods are based on: (i) a novel application of existing Linear Quadratic Regulator optimal-control theory, and (ii) an algebraic method in which the control-law is expressed as a linear feedback of certain "canonical variables." In some cases, these new design methods may be more effective than existing ones. The results are illustrated by worked examples.

Keywords: Control; Stabilization; Dynamic system; Subspace; LQR; Optimal control

0. PROLOGUE

In the period 1965–70 it was my good fortune to have Bob Skelton as a student in my graduate courses in state-variable, optimal and non-linear control. Bob was an outstanding student then and has since become a world-class researcher and author in the field of control theory. It is an honor and pleasure to participate in this special issue of MPE honoring the 60th birthday of Bob Skelton and his achievements. Since many of Bob's contributions are related to optimal and algebraic aspects of control theory, I have chosen similar topics for this paper and put them in a context that is responsive to an important design issue Bob has recently raised [1].

1. OVERVIEW OF THE SUBSPACE-STABILIZATION CONTROL PROBLEM

One of the important insights associated with the application of modern optimal-control theories and state-space formulations to industrial control problems is the recognition that a wide, diverse variety of those control problems reduce to a common design requirement. Namely, the requirement of first controlling the system-state vector $x(t)$ (or response–error state vector $\varepsilon(t)$) promptly to a certain linear subspace $\mathcal{S}$ in the underlying state-space and thereafter maintaining $x(t)$ (or $\varepsilon(t)$) on or near $\mathcal{S}$ while controlling $\|x(t)\|$ (or $\|\varepsilon(t)\|$) to zero, or keeping it suitably bounded. The state-space geometry of this kind of $x(t)$ motion is illustrated in Fig. 1.

This important feature of the state motions $x(t)$ in a variety of optimal-control problems was apparently first discovered in [2,3] where

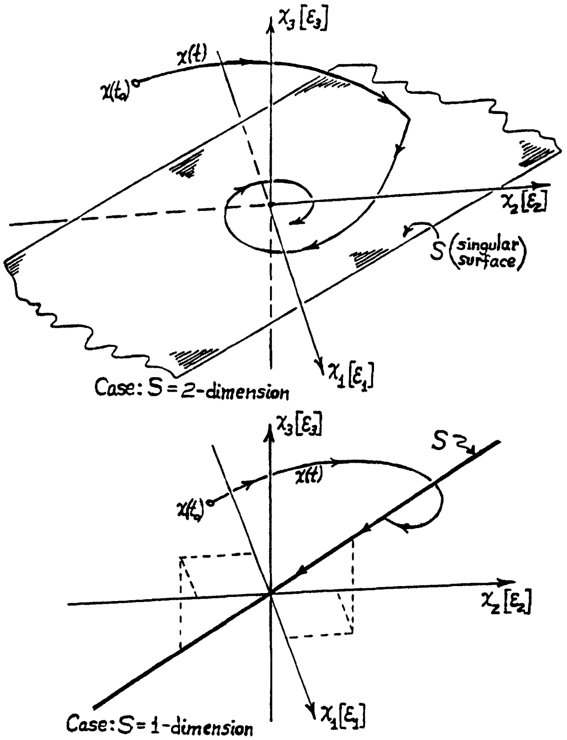

FIGURE 1 State-space geometry of $x(t)$ motions in a variety of optimal control problems.

it was shown that the "singular solutions" in linear-quadratic regulator (LQR)-type optimal-control problems, with "cheap-but-bounded" control, implied the motion of $x(t)$ was invariant on a certain well-defined linear-subspace $\mathcal{S}$; see also [4–7] for additional early results concerning the motion of $x(t)$ on invariant-subspaces.

After the publication of [2–7], it was discovered that this same kind of subspace-stabilization motion $x(t)$ *also* arises as a natural design requirement in a wide variety of stabilization, set-point regulation ("pointing") and servo-tracking control problems, formulated in terms of state-variables, and which do not necessarily involve optimal-control considerations. In the latter cases the subspace-stabilized trajectories $x(t)$ are not required to have the sharp, optimal "corners" illustrated in Fig. 1 and typically have the smoother, asymptotic behavior shown in Fig. 2.

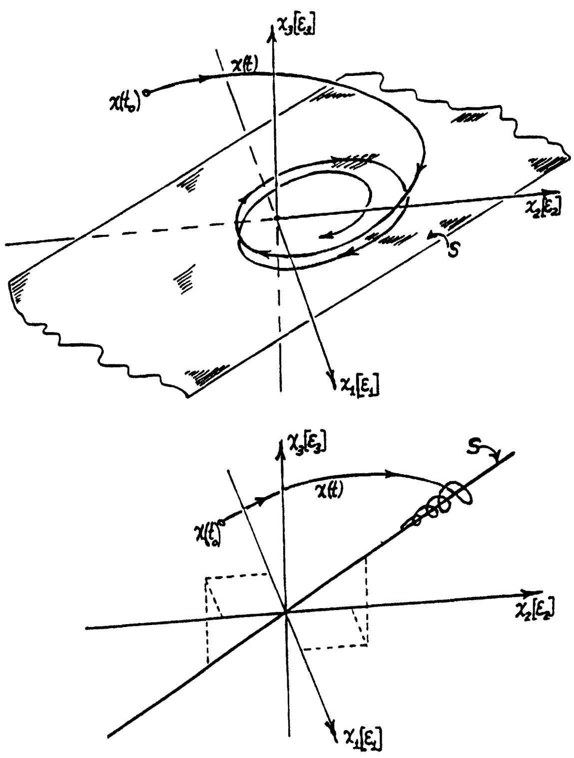

FIGURE 2 State-space geometry of required $x(t)$ motions in many stabilization, set-point regulation (pointing) and servo-tracking control problems.

In recognition of the central importance of achieving and maintaining state-motions $x(t)$ on certain linear subspaces $\mathcal{S}$, in a broad class of practical control applications, that generic control problem was defined precisely, given the name "subspace-stabilization" and was solved rather completely in a 1973 paper [11], for the case of linear systems and control laws. More recently, the subspace-stabilization method has been used to solve a variety of practical control problems in the aerospace field. In the next section we briefly summarize some of the known results concerning theory and applications of the subspace-stabilization method for control design.

2. BRIEF SUMMARY OF PREVIOUS RESULTS IN SUBSPACE-STABILIZATION

In [2–5] it was shown that the optimal control $u^{\mathrm{o}}(t)$ that minimizes a quadratic performance index (with "cheap-but-bounded" control) of the form

$$J = \int_0^{\infty} x^{\mathrm{T}}(t) Q x(t)\,\mathrm{d}t, \quad Q = Q^{\mathrm{T}} > 0, \tag{1}$$

for a scalar-controlled, constant linear dynamical system of the form

$$\begin{aligned} \dot{x} = Ax + bu(t), \quad x = (x_1, \ldots, x_n),\ u = \textit{scalar} \\ (A, b) = \textit{constant and completely controllable} \end{aligned} \tag{2a}$$

subject to the bounded control constraint

$$|u(t)| \leq 1 \tag{2b}$$

and the null-state regulation specification

$$\lim_{t \to \infty} x(t) = 0, \tag{2c}$$

leads to a "dual-mode" optimal controller. That dual-mode controller consists of first a "bang-bang" mode (with a certain non-linear switching surface) that promptly controls $x(t)$ to some point on a well-defined linear subspace (singular surface) $\mathcal{S} \in E^n$, $E^n = n$-dimensional

Euclidean state-space, in a *finite* time-interval. Then, a "singular" linear-control mode is used to *maintain* $x(t) \in \mathcal{S}$ thereafter, while simultaneously controlling $x(t) \rightarrow 0$ as illustrated in Fig. 1. As shown in [3; Eqs. (14)–(27)] (see also Refs. [12–14]) the subspace $\mathcal{S}$ is related to Q in (1) and to the pair (A, b) in (2a)–(2c). It was shown in [2–5] that, owing to the control constraint (2b), for some systems (2a) the achievement and maintenance of $x(t) \in \mathcal{S}$ is not possible for all $x(t_0)$ and consequently $x(t)$ can be stabilized to only a subset $\tilde{\mathcal{S}} \subset \mathcal{S}$ in those cases. Some topological properties of $\tilde{\mathcal{S}}$ are described in [3].

Remark The reader's attention is called to a long-known technical inaccuracy that appears in [3]. Namely, the scalar constants $\{q_i\}$ in [3; Eq. (7)] do *not* necessarily turn-out to be *all* non-negative, in general; see [12] for a technical explanation, and further discussion, of this point and see [13,14] for some important consequences in general LQR theory.

In [6,7] the purely algebraic and state-space geometric aspects of stabilizing motions $x(t)$ to invariant linear subspaces $\mathcal{S}$ (hyperplanes) was studied and some important eigenvalue–eigenvector and controllability/observability characterizations were derived.

As mentioned in Section 1, the formal definition of "stability with respect to a subspace" was introduced in [11] where the subspace-stabilization control problem for a general class of linear dynamical systems, with (vector) linear state-feedback control laws, was defined and solved. In particular, some fundamental existence conditions, an expression which identifies the set of all linear control laws $u = Kx$ that can achieve subspace-stabilization, and a variety of related results for various special cases were derived in [11].

In [15] a broad class of MIMO "output" stabilization, set-point regulation ("pointing") and servo-tracking control problems, for linear dynamical systems with uncertain, time-varying external *disturbance*-inputs, was formulated and solved as a subspace-stabilization problem, posed in a novel n-dimensional "servo-state" state-space, using the subspace-stabilization control theory developed in [11].

In [16,17] the theory of Disturbance-Accommodation [18] was used to develop a new approach to model-reference adaptive control (MRAC), called "Linear Adaptive Control". In that approach the prompt stabilization of state motions $x(t)$ to a certain p-dimensional linear subspace $\mathcal{M}$ is used as a state-space interpretation of the adaptive-control requirement that the closed-loop motions of $x(t)$ should consistently mimic the behavior of a specified, reduced-order, linear dynamical ideal-model.

More recently, the theory of subspace-stabilization has been used to develop a variety of new control algorithms for high-performance guidance and control system applications, including engine control systems, in the aerospace field [8–10,20–22].

3. PROBLEM FORMULATION

In this paper we consider again the general subspace-stabilization control problem for linear dynamical systems as posed in [11]. Namely, we assume one is given the general MIMO plant model

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u, \quad x = (x_1, \dots, x_n), \; u = (u_1, \dots, u_r), \\ y &= H(t)x \quad y = (y_1, \dots, y_m), \end{aligned} \tag{3}$$

where $(A(t), B(t), H(t))$ are known, real-valued and well-behaved for all t. To keep things simple, it is hereafter assumed the system (3) is uniformly completely controllable and observable in the sense of Kalman. In [11] the problem was to find a linear, state-feedback control-law

$$u = K(t)x \tag{4}$$

such that when (4) is substituted into (3) all solutions $x(t)$ of (3) will be "stabilized" to a given, or specified, p-dimension, $0 \leq p < n$, linear subspace $\mathcal{S}(t)$ defined by

$$\mathcal{S}(t) = \{x \mid C(t)x = 0; \; x \in E^n\}, \tag{5a}$$

where $C(t)$ is a given (designer-chosen), real-valued, $(n-p) \times n$ matrix having constant rank $(n-p)$. A more precise, technical definition of the term subspace-stabilization, in the context of (3) and (5a), is given in (8) and (9) below. It follows from (5a) that $\mathcal{S}(t)$ is the p-dimension *null-space* of the constant-rank matrix $C(t)$.

In *optimal*-control problems such as considered in [2–5], the subspace $\mathcal{S}$ is not specified directly, but rather is implicitly "defined" by the parameter-values in the plant-model (3) and in the optimization criterion J. Consequently, in those cases $\mathcal{S}$ is not visible, *a priori*, and if one wants to identify $\mathcal{S}$ it is necessary to perform certain preliminary transformations and other calculations to obtain an explicit description (5a) for $\mathcal{S}$; see [3, (Section 5), 12]. More generally, when optimization is not a primary concern in the control design, $\mathcal{S}$ is visible, *a priori*, because the explicit description (5a) of $\mathcal{S}$ is given at the outset in terms of a specified *set* of closed-loop performance requirements. Each of those performance requirements is equivalent to motion of $x(t)$ (or $\varepsilon(t)$) on a certain (different) linear subspace $\mathcal{S}_j$ defined by an expression of the form:

$$\mathcal{S}_j = \{x \mid c_{i1}x_1 + c_{i2}x_2 + \cdots + c_{in}x_n = 0;\ i = 1, 2, \ldots, q_j\};$$
$$j = 1, 2, \ldots, \eta. \tag{5b}$$

For instance, a set of subspaces $\{\mathcal{S}_j\}_1^\eta$ like (5b) might embody an explicitly defined "ideal-model" performance requirement and/or an output set-point requirement (with the x_i in (5b) replaced by corresponding "response–error" states $\varepsilon_i(t)$). In such cases, the linear subspace $\mathcal{S}$ in (5a) is the *intersection subspace* defined by

$$\mathcal{S} = \cap\{\mathcal{S}_j\}_1^\eta. \tag{5c}$$

Introduction of the Canonical Variables (z_1, z_2)

The results obtained in [11] were made possible, in part, by invoking a novel, non-singular linear transformation on the state-space of (3) as follows (unlike in [11], here we assume $C(t)$ in (5a), rather than $M(t)$, is the *given* quantity that defines $\mathcal{S}$):

$$(x) = \left[C^{\#}(t) | M(t)\right] \left(\frac{z_1}{z_2}\right),$$
$$C^{\#} = C^{\mathrm{T}}[CC^{\mathrm{T}}]^{-1},\ [\cdot]^{\mathrm{T}} - \text{denotes transpose,} \tag{6a}$$

where z_1, z_2 are sub-vectors of dimension $(n-p)$ and (p), respectively, and $M(t)$ is *any* $n \times p$ matrix, having constant rank p, such that

$$C(t)M(t) \equiv 0. \tag{6b}$$

In other words, the columns of $M(t)$ form a *basis* for $\mathcal{S}(t)$ (= null-space of $C(t)$).

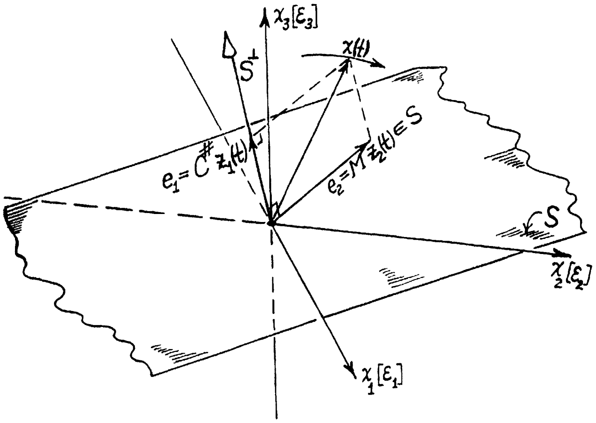

FIGURE 3 Geometric relations between the state-vector x, the projections e_1, e_2 and the associated canonical-variables (sub-vectors) z_1, z_2.

The explicit inverse of (6a) is

$$(z) = \left(\frac{z_1}{z_2}\right) = \left[\frac{C(t)}{M^{\#}(t)}\right](x), \quad M^{\#} = [M^{\mathrm{T}}M]^{-1}M^{\mathrm{T}}. \tag{6c}$$

Referring to Fig. 3, it can be seen that the $(n-p)$-dimension sub-vector $z_1(t)$ is the coordinate-vector (representation) associated with the projection $e_1 = C^{\#}z_1 = C^{\#}Cx$ of $x(t)$ *along* $\mathcal{S}$ *onto* the orthogonal complement $\mathcal{S}^{\perp}$ of $\mathcal{S}$; likewise, the p-dimension sub-vector $z_2(t)$ is the coordinate-vector associated with the projection $e_2 = Mz_2 = MM^{\#}x$ of $x(t)$ along $\mathcal{S}^{\perp}$ onto $\mathcal{S}$. Consequently, the "distance" ∂ of $x(t)$ from $\mathcal{S}$ can be taken as

$$\partial = \|z_1(t)\|, \tag{7}$$

and the motion $e_2(t)$ of $x(t)$ *on* $\mathcal{S}$ is equivalent to the motion of $z_2(t)$ *when* $z_1(t) \equiv 0$. We will refer to the two sub-vectors (z_1, z_2) as the *canonical variables* for the subspace-stabilization problem (3), (5).

The transformation (6) takes the state model (3) into (we occasionally drop the t-argument on matrices to simplify notation)

$$\left(\frac{\dot{z}_1}{\dot{z}_2}\right) = \left[\begin{array}{c|c} \dfrac{C(AC^{\#} - \dot{C}^{\#})}{M^{\#}(AC^{\#} - \dot{C}^{\#})} & \dfrac{C(AM - \dot{M})}{M^{\#}(AM - \dot{M})} \end{array}\right]\left(\frac{z_1}{z_2}\right) + \left[\frac{CB}{M^{\#}B}\right]^{u}. \tag{8}$$

The subspace-stabilization problem considered in this paper can now be precisely stated in terms of (8) as follows.

Definition of the Subspace-Stabilization Problem for (3), (5) inTerms of (8)

Find a feedback control function $u(\cdot)$ such that, when substituted into (8), the following three conditions on the solutions $\{z_1(t), z_2(t)\}$ of (8) are achieved:

$$\text{(i)}\ \lim_{t\to\infty} z_1(t) = 0, \quad \text{for all } \{z_1(t_0), z_2(t_0)\}, \tag{9a}$$

$$\text{(ii)}\ \|z_i(t)\| < \infty; \quad \text{for all } t \geq t_0;\ i = 1, 2, \tag{9b}$$

$$\text{(iii)}\ z_1(t_0) = 0 \Rightarrow z_1(t) \equiv 0, \quad \text{for all } t > t_0. \tag{9c}$$

Remarks Condition (9a) is the requirement that *all* solutions $x(t)$ of (3) are attracted to (asymptotically approach) the subspace $\mathcal{S}$. Condition (9c) is recognized as the requirement for $\mathcal{S}$ to be an *invariant-subspace* (equilibrium manifold) for the closed-loop system (3); i.e., $x(t_0) \in \mathcal{S} \Rightarrow x(t) \in \mathcal{S}$, $\forall t > t_0$. Condition (9b) embodies the important practical requirement that the closed-loop state motion $x(t)$ must always remain bounded. In many practical applications it is prudent or necessary to constrain the quantitative features of the motion $x(t) \in \mathcal{S}$ even more than does (9b). For instance, by replacing the $i = 2$ case of (9b) with a stronger requirement such as *uniform* boundedness, or by an asymptotic stability condition such as

$$\lim_{t\to\infty} z_2(t) = 0, \quad \text{for all } \{z_2(t_0);\ z_1(t) \equiv 0\}. \tag{10}$$

Moreover, engineering requirements in industrial applications typically demand that the convergence (9a) occur *more rapidly* than the convergence (10), as measured by the respective closed-loop "settling-times." It is remarked that conditions (9a)–(9c) are not achievable for every subspace $\mathcal{S}$, in general; see [11] and Eqs. (22)–(25) in this paper. Moreover, in some applications, when (9c) is enforced, the "natural" motion of $x(t) \in \mathcal{S}$ is sufficient to satisfy (9b) or (10).

As in conventional stabilization problems, conditions (9) and (10) can be achieved, in general, by a variety of feedback control functions (control-laws) $u(\cdot)$ having different mathematical structures, such as linear, non-linear, bang–bang, digital, etc., as demonstrated in [3]. In [11] attention was focused on the design of *linear* control laws (4) using purely algebraic methods. In the next section we will show how subspace-stabilization control laws of the linear form (4) can be systematically designed by a novel application of existing LQR optimal-control techniques. In Section 6 of this paper an alternative algebraic method is proposed for designing subspace-stabilization control laws in the "canonical-variables" form $u = K_1 z_1 + K_2 z_2$, which, in some cases, may have computational advantages over the methods used in [11,15]. Due to space limitations, we will confine our attention in Section 4 to the *time-invariant* case of (3)–(5).

4. FORMULATION AND SOLUTION OF THE (LINEAR) SUBSPACE-STABILIZATION CONTROL PROBLEM AS A SPECIAL TYPE OF LINEAR-QUADRATIC PROBLEM IN OPTIMAL CONTROL

The traditional, infinite-time LQR optimal-control problem [19] for the *time-invariant* case of the system (3), with no control constraints, is concerned with finding the/a control $u(t)$ that *minimizes* the quadratic functional (performance index or criterion)

$$J[u] = \int_0^\infty [x^T Q x + u^T R u]\,dt; \quad x = x(t) = \text{sol. of (3)} \tag{11a}$$

subject to the null-state regulation specification (boundary condition)

$$\lim_{t\to\infty} x(t) = 0, \tag{11b}$$

and the assumptions $R = R^T > 0$, $Q = Q^T > 0$; the latter sometimes being replaced by the weaker assumption

$$Q = Q^T \geq 0, \tag{11c}$$

subject to an additional technical restriction. Although we will not pursue the matter here, it is perhaps worth mentioning that the semi-definite assumption (11c) has been shown to be overly restrictive (not necessary), in general, and may cause the LQR design process to rule-out some highly desirable sets of closed-loop poles in the left-half plane, such as ITAE poles, etc.; see [13,14]. The designer's choice of the element-values q_{ij} of Q and the element-values r_{ij} of R determines the qualitative and quantitative features of the optimal control $u^o(t)$ and corresponding optimal response $x^o(t)$, as determined by (3). It is well-known [19] that the LQR optimal-control law $u^o(x)$ associated with (3) and (11) always exists and has the linear form (4) – provided certain controllability/observability conditions involving $(A, B, Q^{1/2})$ are satisfied.

To design an LQR optimal control $u^o(t)$ that satisfies the subspace-stabilization requirements (9) and (10) in a limiting or approximate sense, for a subspace $\mathcal{S}$ explicitly given or specified in the form (5), one can use the technique of [3; Eq. (27)], [12] and the transformation (6) to "design" a special form of Q in the performance functional (11) as follows. First, observe that, with respect to (8), the approximate satisfaction of conditions (9a) and (9b) can be approached by choosing $u(t)$ to minimize the special "sum of state-quadratics" functional

$$J[u] = \int_0^\infty [z_1^T Q_1 z_1 + z_2^T Q_2 z_2 + u^T R u]\,dt, \tag{12}$$

where it will be assumed that $Q_1 = Q_1^T > 0$, $Q_2 = Q_2^T > 0$, for simplicity [see remarks below (11c)] and that $R = R^T > 0$, where $\|R\|$ is suitably *small*. To encourage the optimal control $u^o(t)$ to *also* satisfy (9c) one should choose $\|Q_1\| \gg \|Q_2\|$ in (12). When $z_1(0) \approx 0$, the latter choice will encourage the optimal-control $u^o(t)$ to maintain $z_1(t) \approx 0$ thereby tending to make $\mathcal{S}$ become an *approximate* invariant-subspace – *provided* $\|R\|$ is chosen sufficiently small to "discount" the cost of control effort $\int u^T(t) R u(t)\,dt$ required to maintain the "invariant" motion $z_1(t) \approx 0$. The choice $\|Q_1\| \gg \|Q_2\|$ will also tend to encourage $z_1(t) \to 0$ "faster than" $z_2(t) \to 0$, as mentioned below (10). The weaker boundedness condition (9b) for $z_2(t)$ can be realized by allowing $\|Q_2\|$ to become arbitrarily small, > 0, in some appropriate sense. If the "natural" motion of $x(t) \in \mathcal{S}$ is satisfactory, in the sense of (9b) or (10), one can let $Q_2 \to 0$ in (12).

Next, the relation (6c) is used to convert (12) to the normal LQR form (11) by substituting the inverse relations

$$z_1 = Cx, \quad z_2 = M^{\#} x \tag{13}$$

into (12) to obtain the conventional expression

$$J[u] = \int_0^{\infty} [x^{\mathrm{T}} \bar{Q} x + u^{\mathrm{T}} R u] \, \mathrm{d}t, \tag{14a}$$

where $\bar{Q}$ has the special structure

$$\bar{Q} = C^{\mathrm{T}} Q_1 C + M^{\#\mathrm{T}} Q_2 M^{\#}, \quad (C, M^{\#}) \text{ – defined in (5) and (6).} \tag{14b}$$

The existing theory [19] and widely available computational algorithms for the LQR problem can now be applied to (3), (11) and (14) to easily obtain the LQR optimal-control $u^{\mathrm{o}}(t)$ in the form

$$u^{\mathrm{o}}(t) = Kx(t), \quad K = -R^{-1} B^{\mathrm{T}} P, \tag{15a}$$

where the $n \times n$ matrix P is the symmetric, positive-definite solution to the matric, steady-state (algebraic) Riccati equation

$$PA + A^{\mathrm{T}} P - PBR^{-1} B^{\mathrm{T}} P + \bar{Q} = 0. \tag{15b}$$

If the subspace-stabilization control problem (5), (9), (10) has a solution, the LQR optimal-control law (15) will constitute an approximate solution, provided the numerical values of (Q_1, Q_2, R) in (12) are chosen appropriately as indicated below (12). As in most LQR applications, determining those appropriate numerical values may require a trial-and-error design procedure, using simulation "exercises", to achieve the desired qualitative and quantitative characteristics of conditions (9) and (10) *and of* $u^{\mathrm{o}}(x(t))$. Practically speaking, this latter feature constitutes the real merit of this LQR method for designing a subspace-stabilizing control-law because the matric parameters $\{Q_1, Q_2, R\}$ in (12) provide a notably simple and convenient means for the control designer to "manage", and explore "trade-offs between", the levels of effort/energy required of the control $u^{\mathrm{o}}(t)$ and the degree to which the corresponding "optimal" motions $z_1(t), z_2(t)$ satisfy conditions (9) and (10). For practical implementation purposes the control-law (15a) would be written as $u^{\mathrm{o}}(t) = K\hat{x}(t)$, where $\hat{x}(t)$ is a real-time estimate of $x(t)$ generated by an observer or Kalman filter [19].

The more-general time-varying cases of (3), (5) and (11) can be handled by introducing the same LQR performance criterion (12), and employing a solution procedure similar to that used here, at the expense of more tedious technical details and calculations involving asymptotic behavior of solutions $P(t)$ to the matric Riccati *differential* equation [19] associated with (15b).

If the subspace-stabilization control problem (5), (9), (10) *does not* have a solution in the *strict* sense, the LQR method (12)–(15) can be used as a basis for deriving various forms of "optimally-approximate" subspace-stabilization control laws [23].

5. SOLUTION OF AN EXAMPLE USING THE LQR METHOD FOR SUBSPACE-STABILIZATION CONTROL DESIGN

To illustrate the new LQR methodology for subspace-stabilization control design, as proposed in Section 4 of this paper, consider the general, time-invariant, scalar-controlled 2nd-order system modeled by (3) with

$$A = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix}, \; B = b = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{16}$$

and assume the subspace $\mathcal{S}$ to be stabilized is 1-dimensional and defined by (5a) with

$$C \to c = (c_1, c_2) \quad (= \text{constant, non-zero } 1 \times 2 \text{ matrix}). \tag{17a}$$

Following (6) the matrix $C^{\#}$, and associated (non-unique) matrices $M, M^{\#}$, are computed/chosen to be

$$C^{\#} = \|c\|^{-2}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad M = \begin{pmatrix} c_2 \\ -c_1 \end{pmatrix}, \quad M^{\#} = \|c\|^{-2}(c_2, -c_1), \tag{17b}$$

where $\|c\|^2 = (c_1^2 + c_2^2) > 0$. For this example, the corresponding LQR functional (14) that, when minimized, will lead to a subspace-stabilizing optimal control $u^{o}(t)$, for appropriate choices of $Q_1 \to q_1 = scalar \gg 0$,

$Q_2 \rightarrow q_2 = scalar > 0$, $R \rightarrow r = scalar > 0$, is given by

$$J = \int_0^\infty [x^T \bar{Q} x + r u^2]\,\mathrm{d}t, \; x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{18a}$$

In the case of (17), $\bar{Q}$ in (14b), (18a) reduces to the 2×2 matrix

$$\bar{Q} = \left[q_1 \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}^{(c_1, c_2)} + q_2 \|c\|^{-4} \begin{pmatrix} c_2 \\ -c_1 \end{pmatrix}^{(c_2, -c_1)} \right]. \tag{18b}$$

The problem data (16) and (18) can now be substituted into (15) to determine the subspace-stabilizing, LQR optimal-control law $u^{\circ}(t) = Kx(t)$ – *assuming the existence conditions in* [11] *are satisfied*; alternatively, see the more specific existence conditions for this particular example as given in Eqs. (28)–(32) presented later in this paper.

As explained below (12) and (15), it may be necessary to experimentally adjust the values of the positive LQR parameters $\{q_1, q_2, r\}$ in (18), in a trial-and-error manner, to determine a set of values that yield the desired settling-times and other quantitative properties of the subspace-stabilization conditions (9) and (10), as well as acceptable behavior of $u^{\circ}(x(t))$. In particular, as the value of $r > 0$ decreases in (18a) (increasingly "cheap-control") conditions (9) and (10) should occur in a near-ideal manner, for appropriate choices of $q_1 \gg q_2 > 0$ – *assuming* the existence conditions referred-to below (18b) are satisfied for the chosen subspace $\mathcal{S}$. The limiting case $r \rightarrow 0$ has little practical interest because it typically leads to unbounded behavior of $\|K\|$, and consequently of $\|u(t)\|$, due to the term R^{-1} in (15a); see [3] for a detailed study of the scalar control case with $r = 0$ and with the hard constraint (2b) imposed on $u(t)$.

To demonstrate the considerations in the preceding paragraph, a specific case of the Example (16)–(18) will now be considered and simulation results presented.

A Specific Case of the Example, Using the LQR Method

As a specific case of the preceding Example (16)–(18), illustrating the LQR design method, consider the system (16) with $a_1 = a_2 = 0$ (the "double-integrator" system) and suppose $c_1 = c_2 = 1$ in the definitions (5a) and (17) of the subspace $\mathcal{S}$. In that case the matrix $\bar{Q}$ in (18) becomes

$$\bar{Q} = \begin{bmatrix} (q_1 + q_2/4) & (q_1 - q_2/4) \\ (q_1 - q_2/4) & (q_1 + q_2/4) \end{bmatrix}. \tag{19a}$$

For this specific case the elements (p_{11}, p_{12}, p_{22}), of the corresponding 2×2 solution $P = P^{\mathrm{T}} > 0$ of (15b), can be easily computed analytically and are found to be

$$\begin{aligned} p_{11} &= -(q_1 - q_2/4) + \sqrt{2\sqrt{r(q_1 + q_2/4)^3} + (q_1 + q_2/4)^2}, \\ p_{12} &= \sqrt{r(q_1 + q_2/4)}, \\ p_{22} &= \sqrt{r[2\sqrt{r(q_1 + q_2/4)} + (q_1 + q_2/4)]}. \end{aligned} \tag{19b}$$

Finally from (15a) the LQR optimal, subspace-stabilizing control $u^{\mathrm{o}}(\cdot)$ is given as

$$u^{\mathrm{o}}(x) = -r^{-1} b^{\mathrm{T}} P x = k_1 x_1 + k_2 x_2, \tag{19c}$$

where

$$\begin{aligned} k_1 &= -\sqrt{r^{-1}(q_1 + q_2/4)}, \\ k_2 &= -\sqrt{2\sqrt{r^{-1}(q_1 + q_2/4)} + r^{-1}(q_1 + q_2/4)}. \end{aligned} \tag{19d}$$

Some plots of the closed-loop, optimal state-trajectories $x^0(t)$ for the two initial-conditions $x(0) = (5, -1)$ and $x(0) = (4, -5)$, with the *fixed*-values $q_1 = 10$, $q_2 = 0.1$ and for decreasing values of $r > 0$, are shown in Fig. 4. It can be seen in Fig. 4 that as r decreases in value the LQR optimal-control (19c) and (19d) becomes increasingly effective at quickly stabilizing $x^0(t)$ to the subspace $\mathcal{S}$.

This same Example (16)–(18) will be considered again in Section 7 of this paper, where it will be used to illustrate an alternative, purely algebraic method for designing subspace-stabilization control-laws.

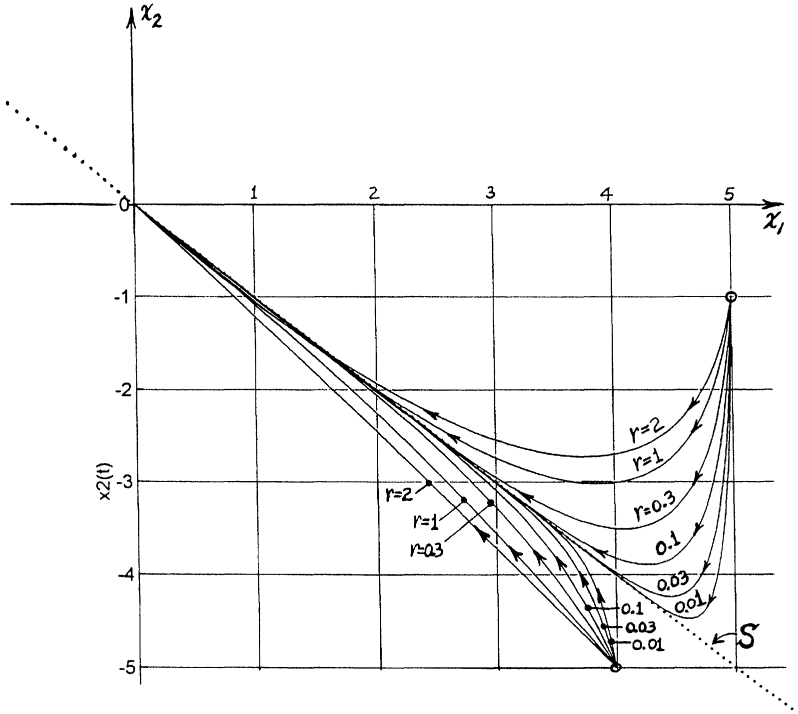

FIGURE 4 Some closed-loop, optimal state-trajectories for the Example specific case: $a_1 = a_2 = 0$; $c_1 = c_2 = 1$ in (16) and (17), with $q_1 = 10$, $q_2 = 0.1$ and $2 \geq r \geq 0.01$.

6. FORMULATION AND SOLUTION OF THE (LINEAR) SUBSPACE-STABILIZATION CONTROL PROBLEM USING AN ALGEBRAIC METHOD AND LINEAR FEEDBACK OF THE CANONICAL VARIABLES (z_1, z_2)

The widespread availability of computer-aided design programs for LQR-type problems makes the LQR methodology for designing subspace-stabilizing control-laws, as presented in Section 4 of this paper, an effective design tool for complex, high-order problems. On the other hand, the introduction of a possibly artificial or contrived optimization performance criterion J, whose parameters must be "tuned" by trial-and-error, to solve what is basically a linear-algebraic problem in stabilization theory is viewed by some as an *indirect* design procedure that may obscure important mathematical features of the problem and its solution. Thus, there is an interest in developing more direct methodologies for subspace-stabilization control design that are purely algebraic in nature and reveal important scientific aspects of the problem, such as underlying necessary and sufficient conditions for existence of a solution, etc.

The algebraic design methodology developed in [11] and refined in [15] was an attempt to satisfy that interest. However, the design methodology in [11,15] can sometimes become rather involved and unwieldy. In this section, we introduce an alternative algebraic methodology for designing subspace-stabilizing controllers that may offer computational advantages, in some cases.

To begin, we return to expression (8) and agree to write the sought control-law $u = Kx$ equivalently in terms of the canonical variables (z_1, z_2) as follows:

$$u = Kx = K[C^{\#} \mid M]\left(\frac{z_1}{z_2}\right) = K_1 z_1 + K_2 z_2, \tag{20a}$$

where

$$K_1 = KC^{\#}; \quad K_2 = KM. \tag{20b}$$

Our assertion here is that, in some problems, it may be easier to *begin* with the form (20a), where (K_1, K_2) are arbitrary, and design the *two* matrices (K_1, K_2) rather than to begin with (4) and design the *one* matrix K, as was considered in [11].

Substituting (20a) into (8) yields the "closed-loop" plant state-model in canonical variables as

$$\left(\frac{\dot z_1}{\dot z_2}\right) = \left[\begin{array}{c|c} C(AC^{\#} - \dot C^{\#}) + CBK_1 & C(AM - \dot M) + CBK_2 \\ \hline M^{\#}(AC^{\#} - \dot C^{\#}) + M^{\#}BK_1 & M^{\#}(AM - \dot M) + M^{\#}BK_2 \end{array}\right]\left(\frac{z_1}{z_2}\right). \tag{21}$$

Design Requirements for the Canonical Gains (K_1, K_2)

It was shown in [11] that the three requirements for achieving subspace-stabilization, cited here in Eq. (9), imply the following conditions in (21).

(9a) $\Rightarrow$ The matrix K_1 must be chosen such that all solutions of the decoupled sub-system

$$\dot{z}_1 = [C(AC^{\#} - \dot{C}^{\#}) + CBK_1]z_1 \qquad (22)$$

are asymptotically stable to $z_1 = 0$, with sufficiently short setting-time, for all $z_1(t_0)$.

(9b) $\Rightarrow$ K_1 and K_2 must be chosen such that all solutions $z_1(t)$ of (22), and all solutions $z_2(t)$ of the truncated sub-system

$$\dot{z}_2 = [M^{\#}(AM - \dot{M}) + M^{\#}BK_2]z_2, \qquad (23)$$

remain suitably bounded for all $t \geq t_0$ (or, in the case of $z_2(t)$, suitably stable/asymptotically stable with respect to $z_2 = 0$, as dictated by application requirements).

(9c) $\Rightarrow$ K_2 must be chosen such that the upper-right coupling block in (21) is (ideally) zero. That is,

$$[C(AM - \dot{M}) + CBK_2] \equiv 0. \qquad (24)$$

It should be noted that the matrix K_2 must be designed to simultaneously satisfy *both* (23) and (24). To systematically identify candidate choices for such a K_2, it is convenient to proceed as in [11] and first identify the set of all K_2 that will satisfy (24). The necessary and sufficient condition for existence of a matrix K_2 that achieves (24) can be written as [11; p. 180]

$$rank[CB \mid C(AM - \dot{M})] = rank[CB] \qquad (25)$$

in which case it is necessarily true that

$$[C(AM - \dot{M})] = CB\Gamma \qquad (26a)$$

for some (possibly non-unique) matrix Γ. Any such Γ can be used to compute a K_2 satisfying (24) by setting

$$K_2 = -\Gamma; \quad \Gamma \text{ satisfies (26a).} \qquad (26b)$$

A parametric expression for the set $\{\Gamma\}$ of *all* Γ satisfying (26a), assuming (25) is satisfied, can now be developed by using the appropriate

Moore–Penrose generalized-inverse expression, as was done in [11; p. 180]. Then, one can proceed to determine if there is any $K_2 = -\Gamma$, $\Gamma \in \{\Gamma\}$, that produces the required boundedness (or stability) of solutions $z_2(t)$ of (23); see [11, Section 6].

As indicated in [11], the necessary and sufficient conditions for existence of a K_1 that achieves stabilization of (22), in the most general *time-varying* case, are not available. A noteworthy sufficient condition for that general case, due to Kalman, and also the known, necessary *and* sufficient existence conditions for the *time-invariant*, and other special cases of (22), are given in [11]. As previously mentioned, any K_2 used in (23) *must* also satisfy (24) and therefore must be chosen from among the *set* $\{K_2\}$ of all K_2 that satisfy (25) and (26). If the latter set is empty, the (strict) subspace-stabilization control problem does not have a solution. A similar conclusion obtains if every K_2 satisfying (25) and (26) yields an unbounded or unacceptable behavior of one or more solutions $z_2(t)$ in (23). In such cases, to avoid an impasse in the design procedure, it is necessary to revise the subspace-stabilization problem-data by, for instance, introducing an alternative or compromised choice of S in (5a) and/or allowing *more* control-inputs (components) u_i in the plant model (3). Alternatively, when the essential condition (25) is not satisfied, one can use the same Moore–Penrose generalized-inverse expression employed in [11] to identify a set of K_2 that minimize $\|C(AM - \dot{M}) + CBK_2\|$ in (24); cf. [18, pp. 456–7]. In that way, one can employ algebraic methods to explore the possibility of achieving "best-approximation" type solutions to subspace-stabilization control problems [23]. In any case, after a suitable pair (K_1, K_2) has been designed, the control-law (20a) can be expressed in terms of the system's natural-state x by using (13) to write $u = [K_1 C + K_2 M^{\#}]x$, where x would be replaced by an observer-generated estimate $\hat{x}$ for practical implementations.

7. SOLUTION OF AN EXAMPLE USING LINEAR FEEDBACK OF THE CANONICAL VARIABLES

To illustrate the alternative algebraic methodology for subspace-stabilization control design, using linear feedback of the canonical variables, we will consider the *same* general 2nd-order, scalar-input plant model (16), and 1-dimensional subspace $\mathcal{S}$, used previously in Section 5 to illustrate the LQR design methodology. Namely, we will assume

$$A = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix}, \quad B \to b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{27a}$$

$$\mathcal{S} = \{x \mid c_1 x_1 + c_2 x_2 = 0\}, \quad c = (c_1, c_2) \neq 0, \tag{27b}$$

where the (a_i, c_i) are real-valued constants. For this case, $K_1 \to k_1 =$ *scalar*, $K_2 \to k_2 =$ *scalar*, and $\{C^{\#}, M, M^{\#}\}$ are computed/chosen just as in (17b).

Substituting (17b) into (22)–(24), and assuming (24) is not satisfied by the trivial solution $k_2 = 0$, it is found that a non-zero k_2 satisfying the invariant-subspace condition (24) exists if, and only if, $c_2 \neq 0$, in which case the k_2 satisfying (24) for this particular example is *uniquely* defined as

$$k_2\Big|_{(24)} = \left(\frac{(c_2^2 a_1 + c_1^2 - c_1 c_2 a_2)}{c_2} \right); \quad c_2 \neq 0. \tag{28}$$

Note that the unique k_2-value defined by (28) eliminates all design options in selecting a k_2-value that also suitably bounds the $z_2(t)$ motions in (23). This unusual outcome is *not* representative of higher-order examples where $u \neq$ scalar, but it does illustrate the kind of useful technical insights made possible by a purely algebraic approach to subspace-stabilization control design.

If one now substitutes (28) into (23) the following surprizingly simple result obtains:

$$\dot{z}_2 = [-(c_1/c_2)] z_2. \tag{29}$$

It follows from (29) that, for this example, the value (28) of the unique k_2 satisfying (24) *also* makes all solutions $z_2(t)$ of (23) become stable/asymptotically stable to $z_2 = 0$ if, and only if, (29) is "naturally" stable, i.e., the (c_1, c_2) in (27b) satisfy

$$(c_1/c_2) \geq 0, \quad c_2 \neq 0, \tag{30}$$

where the equality $(c_1/c_2) = 0$ in (30) corresponds to motions $z_2(t)$ that are stable with respect to $z_2 = 0$, but not asymptotically so.

Finally, substituting (17b) into (22) yields

$$\dot{z}_1 = \|c\|^{-2}[c_1 c_2(1 - a_1) - c_2^2 a_2 + c_2 k_1]z_1. \tag{31}$$

Thus, all solutions $z_1(t)$ of (31) will be forced to become asymptotically stable to $z_1 = 0$ if, and only if, k_1 is chosen to achieve

$$c_2(k_1 - c_2 a_2 + c_1(1 - a_1)) < 0, \quad c_2 \neq 0. \tag{32}$$

As mentioned previously, practical applications typically carry the additional requirement that the settling-time for $z_1(t) \to 0$ in (31) should be somewhat *smaller* than that for $z_2(t) \to 0$ in (29). Under that requirement, the inequality in (32) may require strengthening, compared to (29) and (30).

In summary, the algebraically designed subspace-stabilizing control law for this simple example, expressed in canonical variables, is

$$u = k_1 z_1 + k_2 z_2, \quad z_1, z_2 = scalars, \tag{33}$$

where k_2 is given by (28) and k_1 satisfies (32) – *provided* (30) is satisfied. To convert the control-law (33) to the natural-state form (4) one should use the corresponding relations (6c) and (13) to replace the z_i-terms in (33) by x_i-terms. Note that the precise existence conditions and other important technical results associated with (28)–(32) tend to be obscured when the LQR method is used to solve this same Example, as in Section 5.

This illustrative example has been chosen simple to enable easy visualization. Some more realistic and complicated aircraft, aerospace and engine control problems, formulated as subspace-stabilization problems and solved using algebraic-type subspace-stabilization control-design methodologies, are presented in [8–10], [20–22].

8. CONSIDERATION OF OTHER STRUCTURAL FORMS OF SUBSPACE-STABILIZATION CONTROL LAWS

As explained in Section 2, the earliest appearance of the subspace-stabilization idea in optimal control was in connection with the infinite-time, LQR Problem (1) and (2) with "cheap-but-bounded" control.

For that problem the optimal control turned-out [3] to be a "dual-mode" subspace-stabilizing controller consisting of a strategic, non-linear bang–bang control law, which "optimally" controls $x(t) \rightarrow \mathcal{S}$ in some *finite* time, and a linear control law which "optimally" maintains $x(t) \in \mathcal{S}$ thereafter, while regulating $x(t) \rightarrow 0$ as shown in Fig. 1.

Thus, it is important for control designers to recognize that the *linear* state-feedback control laws (4), as considered here and in [11,15], etc., represent only one of many possible options for the mathematical structure of a subspace-stabilizing control law for linear plants (3). In this section we will describe a general framework in which one can systematically explore other structural options in subspace-stabilization control designs for linear plants (3). In addition, the consideration of non-linear plants and manifolds $\mathcal{S}$ that are not linear subspaces is addressed briefly.

A Design "Separation Principle" for Subspace-Stabilization Control

As explained below (5a), in many practical applications the dimension and orientation of the subspace $\mathcal{S}$ is *explicitly* defined, in the form (5a), by a given set of closed-loop performance specifications, such as requirements for "ideal-model" response of tracking-errors, etc. Thus, assuming $\mathcal{S}$ has been identified explicitly by such a set of specifications, as in (5b) and (5c), one can consider the overall subspace-stabilization control-design problem as separated into the following two, essentially decoupled, sequential phases: Phase 1 – the control of $x(t) \rightarrow \mathcal{S}$, and Phase 2 – the maintenance and appropriate control of $x(t) \in \mathcal{S}$, as dictated by (9b) or (10) and the application requirements.

Design Options for the Phase 1 Control Problem

The design of a control-law to solve the Phase 1 problem can be approached by either optimal-control techniques or algebraic stabilization techniques. In the case of optimal-control techniques, the "given" p-dimension subspace $\mathcal{S}$ (or an appropriate $\tilde{\mathcal{S}} \subset \mathcal{S}$, in the case of bounded control) should be defined as the mathematical "terminal manifold" for the optimization problem, cf. the procedure used in

[3, p. 109], and one should then choose the optimization criterion J (J need not be restricted to LQR/LQG type [19]) to reflect the designer's performance requirements for the Phase 1 motions $x(t) \rightarrow \mathcal{S}(\tilde{\mathcal{S}})$. A Mayer-term (terminal-state penalty term) can be appended to J to reflect the "preferred" regions of $\mathcal{S}$ (if any) for $x(t)$ to "terminate" on; i.e., the preferred values of z_2 when $x(t)$ first contacts $\mathcal{S}$. Then, the standard tools of optimal-control theory can, in principle, be used to derive the corresponding (optimal) control-law $u^o(x(t))$ that solves the Phase 1 problem $x(t) \rightarrow \mathcal{S}$. In this way, one can incorporate a wide variety of performance criteria J that produce Phase 1 control-laws which achieve practical engineering requirements such as minimal-time, or minimal-energy, or minimal-stress, etc., as $x(t) \rightarrow \mathcal{S}$.

The rich variety of algebraic-type linear and non-linear stabilization control-law design techniques available in the existing literature (Liapunov, Feedback-Linearization, Bang–Bang, Discrete-Time, etc.), can also be used to solve the Phase 1 problem by first proceeding as in DAC theory [18] and splitting $u(\cdot)$ in (8) into two terms as $u = u_1 + u_2$, where $u_2 = K_2 z_2$ and K_2 is designed as in (26b), assuming the essential condition (25) is satisfied. This forces the subspace $\mathcal{S}$ to become a generalized "equilibrium-manifold" for the stabilization problem, i.e., achieves condition (9c). Then, the design of the other (not necessarily linear) control term $u_1 = f(z_1)$ to stabilize $x(t) \rightarrow \mathcal{S}$ ($z_1(t) \rightarrow 0$) can proceed using the decoupled sub-system model $\dot{z}_1 = [C(AC^{\#} - \dot{C}^{\#})]z_1 + CBu_1$, $u_1 = f(z_1)$.

Design Options for the Phase 2 Control Problem

The design of a control-law to solve the Phase 2 problem (= maintenance and appropriate control of $x(t) \in \mathcal{S}$), generally does not offer the same extent of structural options as the Phase 1 problem. In particular, if the plant is linear as in (3), the choices for the Phase 2 control may be limited-to those control laws (4), (20a) that are *linear* in $x(t)$ [$z_2(t)$] (or, in some averaged sense, are essentially *equivalent to* such a linear control-law). In this regard, the possible non-uniqueness of the linear control "gain-matrix" $K[K_2]$ in solving the Phase 2 problem is discussed in [11; Section 6] and in the discussion of "candidate" K_2 options associated with Eqs. (24)–(26) in the present paper.

Consideration of Non-linear Plants and/or Manifolds $\mathcal{S}$

In principle, the (linear) subspace-stabilization ideas and control-design techniques presented here, and in [11,15], can be generalized to accommodate dynamical systems modeled by *non-linear* equations (3), and/or manifolds $\mathcal{S}$ that are *not* linear subspaces. However, general control design procedures for non-linear systems (3) and precise existence conditions for solutions, like those presented here and in [11,15], are rather difficult to develop. On the other hand, the results obtained in [24], using linear-adaptive control, demonstrate that the technique of *active-linearization* can effectively "force" some non-linear systems to behave like a designer-chosen linear system. That novel approach allows control-designers to use the *linear* system and subspace-stabilization results presented here, on some dynamical systems that are *not* linear [25].

9. SUMMARY AND CONCLUSIONS

In this paper two new methods for solving the (linear) subspace-stabilization control problem have been introduced and illustrated by worked examples. One method utilizes the existing theory and well-developed computational algorithms of LQR optimal-control theory to systematically design "optimal" linear control-laws $u^{\mathrm{o}} = Kx$ that solve the subspace-stabilization control problem for a linear system (3) and a specified linear subspace $\mathcal{S}$ defined by (5). The other method, like that in [11], is purely algebraic in nature but differs from [11] in that the control-law is expressed as a linear feedback of the canonical variables z_1, z_2. In some cases these new methodologies for designing linear subspace-stabilizing control laws may be more effective, computationally, than the methods originally presented in [11,15]. A general methodology for extending the solution procedures used here, to accommodate cases of non-linear, bang–bang, discrete-time and other forms of control-laws, has also been presented.

The results presented here and in [11,15] are intended for continuous-time (analog) control designs $u = u(t)$. The (linear) subspace-stabilization control problem for *discrete-time* controls $u = u(kT)$ is considered in [26,27].

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