

# Synthesis of Fixed-Architecture, Robust $H_2$ and $H_\infty$ Controllers<sup>†</sup>

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This paper discusses and compares the synthesis of fixed-architecture controllers that guarantee either robust  $H_2$  or  $H_\infty$  performance. The synthesis is accomplished by solving a Riccati equation feasibility problem resulting from mixed structured singular value theory with Popov multipliers. Whereas the algorithm for robust  $H_2$  performance had been previously implemented, a major contribution described in this paper is the implementation of the much more complex algorithm for robust  $H_\infty$  performance. Both robust  $H_2$  and  $H_\infty$  controllers are designed for a benchmark problem and a comparison is made between the resulting controllers and control algorithms. It is found that the numerical algorithm for robust  $H_\infty$  performance is much more computationally intensive than that for robust  $H_2$  performance. Both controllers are found to have smaller bandwidth, lower control authority and to be less conservative than controllers obtained using complex structured singular value synthesis.

**Keywords:** Controller synthesis; Homotopy algorithms; Popov multiplier; Fixed-architecture; Robust  $H_2$  performance; Robust  $H_\infty$  performance

## 1 INTRODUCTION

This paper considers the design of robust controllers using the state space Popov analysis criterion which is based on the Popov stability multiplier  $W(s) = H^2 + Ns$ . This is a special case of mixed structured singular value synthesis (Haddad *et al.*, 1994; How and Hall, 1993). Algorithms for both robust  $H_2$  and  $H_\infty$  performance are described

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and compared. The formulations which closely follow those presented in Collins *et al.* (1996, 1997) require the minimization of a cost functional subject to a Riccati equation constraint. These formulations have several advantages. First, compensator order and architecture can be specified *a priori*. In addition, both the controller and multiplier parameters can be optimized simultaneously which avoids  $M-K$  (i.e., multiplier-controller) iteration, potentially leading to better performing robust controllers. For robust  $H_2$  performance the cost function that is minimized is an upper bound on the  $H_2$  performance over the uncertainty set. For  $H_\infty$  performance, an artificial cost function is used.

Because of positive definite constraints on the Riccati equation solution, standard descent techniques cannot be used to solve the resulting optimization problem. Hence, probability-one homotopy algorithms have been formulated (Collins *et al.*, 1996; 1997). These algorithms have desirable properties when applied to controller design. First, they can be initialized with any feasible multiplier and any stabilizing controller. Also, each controller computed as the homotopy curve is traversed is physically meaningful. In particular, for robust  $H_2$  performance each controller along the homotopy path guarantees a specified degree of robust stability while for robust  $H_\infty$  performance problem each controller guarantees a specified degree of both robust stability and robust performance. Collins *et al.* (1996, 1997) describe implementation of the algorithm for  $H_2$  performance. A major contribution described in this paper is the implementation of the algorithm for robust  $H_\infty$  performance and a comparison with the algorithm for robust  $H_2$  performance.

It should be noted that earlier work developed continuation algorithms for designing robust  $H_2$  controllers based on the Popov multiplier (How *et al.*, 1994a,b; 1996; Sparks and Bernstein, 1995). These algorithms have been effectively used, but are harder to initialize than the probability-one homotopy algorithms used here since they cannot be initialized with *any* feasible multiplier and *any* stabilizing controller. Also, it is known that probability-one homotopy algorithms are more reliable and numerically robust than continuation algorithms (Watson, 1987; Watson *et al.*, 1987).

The paper is organized as follows. Section 2 presents Riccati Equation Feasibility Problems (REFPs) for the synthesis of both robust  $H_2$  and  $H_\infty$  controllers using the Popov multiplier. Section 3 discusses the

solution approach via probability-one homotopy algorithms. Section 4 shows the result of both robust  $H_2$  and  $H_\infty$  control design for a standard benchmark problem and compares the results to a controller designed using complex structured singular value synthesis. Conclusions are presented in Section 5.

### Notation and Definitions

$\mathcal{R}, \mathcal{C}$	real, complex numbers
$\mathcal{D}^{r \times r}$	$r \times r$ real diagonal matrices
$\text{tr}, (\cdot)^*$	trace, complex conjugate transpose
$0_r, I_r$	$r \times r$ zero matrix, $r \times r$ identity
$Z_2 > Z_1$	$Z_2 - Z_1$ positive definite
$Z_2 \geq Z_1$	$Z_2 - Z_1$ nonnegative definite
$\ G(s)\ _2$	$[(1/2\pi) \int_{-\infty}^{\infty} \text{tr}(G(j\omega)^* G(j\omega)) d\omega]^{1/2}$
$\ G(s)\ _\infty$	$\sup_\omega \bar{\sigma}(G(j\omega))$
$\mathcal{F}_u(M, \Delta)$	$M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}$
$\mu$	structured singular value

## 2 RICCATI EQUATION APPROACHES TO ROBUST CONTROLLER SYNTHESIS USING THE POPOV MULTIPLIER

Consider the uncertainty feedback system shown in Fig. 1, where  $G(s)$  has the  $n$ th order, stabilizable and detectable realization

$$G(s) \sim \left[ \begin{array}{c|ccc} A & B_0 & D_1 & B \\ \hline C_0 & 0 & 0 & 0 \\ E_1 & 0 & 0 & 0 \\ C & 0 & D_2 & 0 \end{array} \right], \quad (2.1)$$

$K(s)$  has a realization of order  $n_c \leq n$  given by

$$K(s) \sim \left[ \begin{array}{c|c} A_c & B_c \\ \hline -C_c & 0 \end{array} \right], \quad (2.2)$$

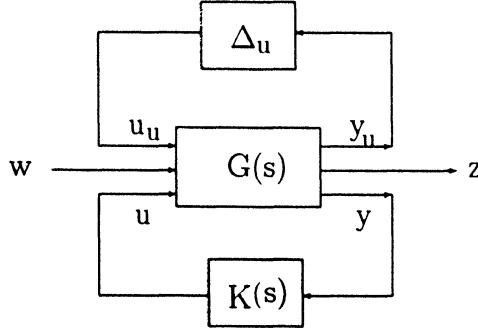


FIGURE 1 Uncertain feedback system.

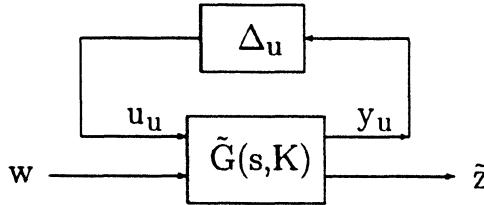


FIGURE 2 Closed-loop representation of uncertain system.

and  $\Delta_u \in \mathcal{U}$ , where for  $M_1$  and  $M_2$  in  $\mathcal{D}^{m_0 \times m_0}$  with  $M_2 - M_1 > 0$ ,  $\mathcal{U}$  is the real parametric uncertainty set

$$\mathcal{U} \triangleq \{\Delta_u \in \mathcal{R}^{m_0 \times m_0}: M_1 < \Delta_u < M_2\}. \quad (2.3)$$

Let

$$\tilde{z} = [z^T \ (E_2 u)^T]^T \quad (2.4)$$

and let  $\theta$  be a vector representation of the controller state space matrices, for example

$$\theta = [\text{vec}(A_c)^T \ \text{vec}(B_c)^T \ \text{vec}(C_c)^T]^T. \quad (2.5)$$

Then Fig. 1 is equivalent to Fig. 2, where

$$\tilde{G}(s, K) \sim \left[ \begin{array}{c|cc} \tilde{A}(\theta) & \tilde{B}_0 & \tilde{D}(\theta) \\ \hline \tilde{C}_0 & 0 & 0 \\ \tilde{E}(\theta) & 0 & 0 \end{array} \right], \quad (2.6)$$

where

$$\tilde{A}(\theta) = \begin{bmatrix} A & -BC_c \\ B_c C & A_c \end{bmatrix}, \quad (2.7)$$

$$\tilde{B}_0 = \begin{bmatrix} B_0 \\ 0 \end{bmatrix}, \quad \tilde{D}(\theta) = \begin{bmatrix} D_1 \\ B_c D_2 \end{bmatrix}, \quad (2.8)$$

$$\tilde{C}_0 = [C_0 \quad 0], \quad \tilde{E}(\theta) = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 C_c \end{bmatrix}. \quad (2.9)$$

It is desired to determine  $K(s)$  or equivalently  $\theta$  such that for all  $\Delta_u \in \mathcal{U}$ , the system of Fig. 2 is asymptotically stable and either  $\|\mathcal{F}_u(\tilde{G}, \Delta_u)\|_2$  or  $\|\mathcal{F}_u(\tilde{G}, \Delta_u)\|_\infty$  satisfies some prespecified bounds.

Define

$$\tilde{R}(\theta) \triangleq \tilde{E}(\theta)^T \tilde{E}(\theta), \quad \tilde{V}(\theta) \triangleq \tilde{D}(\theta) \tilde{D}(\theta)^T. \quad (2.10)$$

The next theorem formulates a synthesis problem for robust  $H_2$  performance in terms of the Popov multiplier  $W(s) = H^2 + Ns$ .

**THEOREM 1** *Suppose  $\tilde{G}(s, K)$  is asymptotically stable. If there exist  $\theta$ ,  $H \in \mathcal{D}^{m_0 \times m_0}$ ,  $N \in \mathcal{D}^{m_0 \times m_0}$ ,  $P > 0$ , and  $\epsilon > 0$  such that*

$$Y = [2H^2(M_2 - M_1)^{-1} + N\tilde{C}_0\tilde{B}_0 + \tilde{B}_0^T\tilde{C}_0^T N] > 0 \quad (2.11)$$

and

$$\begin{aligned} 0 = & (\tilde{A}(\theta) - \tilde{B}_0 M_1 \tilde{C}_0)^T P + P(\tilde{A}(\theta) - \tilde{B}_0 M_1 \tilde{C}_0) \\ & + [\tilde{B}_0^T P - H^2 \tilde{C}_0 - N\tilde{C}_0(\tilde{A}(\theta) - \tilde{B}_0 M_1 \tilde{C}_0)]^T \cdot Y^{-1} \\ & \times [\tilde{B}_0^T P - H^2 \tilde{C}_0 - N\tilde{C}_0(\tilde{A}(\theta) - \tilde{B}_0 M_1 \tilde{C}_0)] + \epsilon \tilde{R}(\theta), \end{aligned} \quad (2.12)$$

then the uncertain system of Fig. 2 is asymptotically stable for each  $\Delta_u \in \mathcal{U}$ . In addition,

$$\begin{aligned} \max_{\Delta_u \in \mathcal{U}} \|\mathcal{F}_u(\tilde{G}, \Delta_u)\|_2 & \leq J(\epsilon, \theta, H, N, P) \\ & \triangleq \frac{1}{\epsilon} \text{tr}[P + \tilde{C}_0^T(M_2 - M_1)N\tilde{C}_0]\tilde{V}(\theta). \end{aligned} \quad (2.13)$$

*Proof* See Haddad *et al.* (1994).

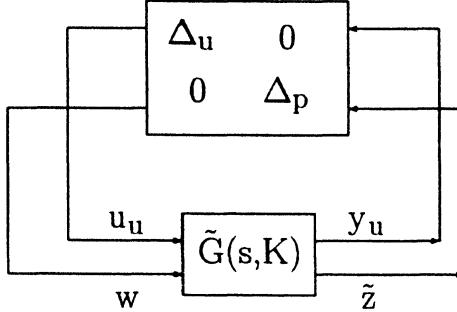


FIGURE 3 Closed-loop uncertain system with 'performance block'.

To consider  $H_\infty$  performance, a fictitious complex uncertainty block  $\Delta_p$  is inserted into Fig. 2 (Doyle *et al.*, 1982; Packard and Doyle, 1993) as shown in Fig. 3. It is assumed that  $\sigma_{\max}(\Delta_p) < \gamma$ . For ease of presentation assume that  $\dim(\tilde{z}) = \dim(w) = q$ , such that  $\Delta_p \in \mathcal{C}^{q \times q}$ . Define

$$\bar{M}_1 \triangleq \text{block-diag}\{M_1, -\gamma I_q\}, \quad \bar{M}_2 \triangleq \text{block-diag}\{M_2, \gamma I_q\}, \quad (2.14)$$

$$\tilde{B}(\theta) \triangleq [\tilde{B}_0 \quad \tilde{D}(\theta)], \quad \tilde{C}(\theta) \triangleq \begin{bmatrix} \tilde{C}_0 \\ \tilde{E}(\theta) \end{bmatrix}. \quad (2.15)$$

The next theorem formulates a synthesis problem for robust  $H_\infty$  performance in terms of the Popov multiplier  $W(s) = H^2 + Ns$ .

**THEOREM 2** *Suppose  $\tilde{G}(s, K)$  is asymptotically stable. If there exist  $\theta$ ,  $H = \text{block-diag}\{H_1, H_2\}$ , where  $H_1 \in \mathcal{D}^{m_0 \times m_0}$  and  $H_2 \in \mathcal{R}^{q \times q}$  satisfies  $H_2 \Delta_p = \Delta_p H_2$ ,  $N = \text{block-diag}\{N_1, 0_q\}$ , where  $N_1 \in \mathcal{D}^{m_0 \times m_0}$ ,  $P > 0$  and  $\epsilon > 0$  such that*

$$\tilde{Y} = [2H^2(\bar{M}_2 - \bar{M}_1)^{-1} + N\tilde{C}\tilde{B} + \tilde{B}^T\tilde{C}^T N] > 0 \quad (2.16)$$

and

$$\begin{aligned} 0 &= (\tilde{A}(\theta) - \tilde{B}\bar{M}_1\tilde{C})^T P + P(\tilde{A}(\theta) - \tilde{B}\bar{M}_1\tilde{C}) \\ &\quad + [\tilde{B}^T P - H^2\tilde{C} - N\tilde{C}(\tilde{A}(\theta) - \tilde{B}\bar{M}_1\tilde{C})]^T \cdot \tilde{Y}^{-1} \\ &\quad \times [\tilde{B}^T P - H^2\tilde{C} - N\tilde{C}(\tilde{A}(\theta) - \tilde{B}\bar{M}_1\tilde{C})] + \epsilon I, \end{aligned} \quad (2.17)$$

then the uncertain system of Fig. 3 is asymptotically stable for each  $\Delta_u \in \mathcal{U}$ . In addition,

$$\max_{\Delta_u \in \mathcal{U}} \|\mathcal{F}_u(\tilde{G}, \Delta_u)\|_\infty < \frac{1}{\gamma}. \quad (2.18)$$

*Proof* Follows from results in Haddad *et al.* (1994, 1995) and a straightforward variant of the main loop theorem (Packard and Doyle, 1993).

### 3 ALGORITHMS FOR ROBUST CONTROLLER SYNTHESIS

Both Theorems 1 and 2 pose robust controller synthesis as a Riccati Equation Feasibility Problem (REFP) (Collins *et al.*, 1996; 1997). As discussed in Collins *et al.* (1996, 1997) an approach to solving the REFP of either Theorem 1 or 2 can be based on solving an optimization problem

$$\min_{\epsilon, \theta, H, N, P} J(\epsilon, \theta, H, N, P) \text{ subject to (2.12) or (2.17)}, \quad (3.1)$$

where  $J(\cdot)$  denotes an appropriate cost functional.

For control design for robust  $H_2$  performance  $J(\cdot)$  is given by (2.13). For robust  $H_\infty$  performance  $J(\cdot)$  can be chosen to minimize the artificial cost function

$$J(\epsilon, \theta, H, N, P) = \text{tr } P. \quad (3.2)$$

Note that this cost functional is used to choose a unique control law corresponding to the degree of robustness and performance specified by  $\gamma$ . More complex artificial cost functionals containing barrier functions to enforce  $P > 0$  and (2.11) or (2.16) are described in Collins *et al.* (1996, 1997), but our experience indicates that these cost functionals are not necessary.

To characterize the extremals define the Lagrangian

$$\mathcal{L}(\epsilon, \theta, H, N, P, Q) = J(\epsilon, \theta, H, N, P) + \text{tr } QW(\epsilon, \theta, H, N, P), \quad (3.3)$$

where  $W(\cdot)$  denotes the right-hand side of (2.12) or (2.17). The necessary conditions for a solution to (3.1) are given by

$$0 = \frac{\partial \mathcal{L}}{\partial \epsilon}, \quad 0 = \frac{\partial \mathcal{L}}{\partial \theta}, \quad 0 = \frac{\partial \mathcal{L}}{\partial H}, \quad 0 = \frac{\partial \mathcal{L}}{\partial N}, \quad (3.4)$$

$$0 = \frac{\partial \mathcal{L}}{\partial Q}, \quad 0 = \frac{\partial \mathcal{L}}{\partial P}. \quad (3.5)$$

The first equation in (3.5) recovers the Riccati equation (2.12) or (2.17) while the second equation in (3.5) results in a Lyapunov equation in  $Q$  whose coefficient and forcing matrices are functions of  $P$ .

If  $\theta$  is given by (2.5), the second equation in (3.4) is equivalent to

$$0 = \frac{\partial \mathcal{L}}{\partial A_c}, \quad 0 = \frac{\partial \mathcal{L}}{\partial B_c}, \quad 0 = \frac{\partial \mathcal{L}}{\partial C_c}. \quad (3.6)$$

The actual expressions for (3.6) corresponding to design for robust  $H_\infty$  performance are much more complex than the expressions corresponding to design for robust  $H_2$  performance. For example the expression for  $\partial \mathcal{L} / \partial B_c$  for robust  $H_2$  and robust  $H_\infty$  design are given respectively by

$$\frac{1}{2} \frac{\partial \mathcal{L}}{\partial B_c} = \frac{1}{\epsilon} P_{21} V_{12} + \frac{1}{\epsilon} [P_{22} B_c V_2] + [PQ]_{21} C_p^T \quad (3.7)$$

and

$$\begin{aligned} \frac{1}{2} \frac{\partial \mathcal{L}}{\partial B_c} = & [QP]_{12}^T + [M_1 \tilde{C} Q P]_{22}^T D_2^T + [\tilde{Y}^{-1} H^2 \tilde{C} Q P]_{22}^T D_2^T \\ & - [N \tilde{C}]_{22}^T \{ [\tilde{Y}^{-1} H^2 \tilde{C} Q P \tilde{B} \tilde{Y}^{-1}]_{22}^T + [\tilde{Y}^{-1} H^2 \tilde{C} Q P \tilde{B} \tilde{Y}^{-1}]_{22} \} D_2^T \\ & - [Q P \tilde{B} \tilde{Y}^{-1} N \tilde{C}]_{12}^T C_p^T + [\tilde{Y}^{-1} N \tilde{C} \tilde{A} Q P]_{22}^T D_2^T \\ & - [N \tilde{C}]_{22}^T \{ [\tilde{Y}^{-1} N \tilde{C} \tilde{A} Q P \tilde{B} \tilde{Y}^{-1}]_{22}^T + [\tilde{Y}^{-1} N \tilde{C} \tilde{A} Q P \tilde{B} \tilde{Y}^{-1}]_{22} \} D_2^T \\ & - [M_1 \tilde{C} Q P \tilde{B} \tilde{Y}^{-1} N \tilde{C}]_{22}^T - [\tilde{Y}^{-1} N \tilde{C} \tilde{B} M_1 \tilde{C} Q P]_{22}^T D_2^T \\ & + [N \tilde{C}]_{22}^T \{ [\tilde{Y}^{-1} N \tilde{C} \tilde{B} M_1 \tilde{C} Q P \tilde{B} \tilde{Y}^{-1}]_{22}^T \\ & + [\tilde{Y}^{-1} N \tilde{C} \tilde{B} M_1 \tilde{C} Q P \tilde{B} \tilde{Y}^{-1}]_{22} \} D_2^T \\ & + 14 \text{ additional terms.} \end{aligned} \quad (3.8)$$

This extra complexity is due to the fact that  $\tilde{C}_0$  and  $\tilde{B}_0$  in (2.12) are *not* functions of  $\theta$  while  $\tilde{C}$  and  $\tilde{B}$  in (2.17) *are* functions of  $\theta$ . This extra complexity makes it much more difficult to practically derive and implement algorithms for the design of robust  $H_\infty$  controllers and increases the computational intensity of the resulting algorithms.

In Collins *et al.* (1996, 1997) probability-one homotopy algorithms are formulated based on the necessary conditions (3.4) and (3.5). In this paper probability-one homotopy algorithms based on the Popov multiplier have been developed and implemented for both robust  $H_2$  and robust  $H_\infty$  controllers.

#### 4 NUMERICAL EXAMPLE

To illustrate robust control synthesis with the probability-one homotopy algorithm, we consider the two-mass/spring benchmark system shown in Fig. 4 with uncertain stiffness  $k$ . A control force acts on the body 1 and the position of body 2 is measured, resulting in a noncollocated control problem. This benchmark problem is discussed in detail in Wei and Bernstein (1992).

The open-loop plant (for  $m_1 = m_2 = 1$ ) is given by

$$\dot{x} = A(k)x + Bu + D_1w, \quad (4.1)$$

$$y = Cx + D_2w, \quad (4.2)$$

$$z = E_1x, \quad (4.3)$$

where  $z = x_2$  is the output performance variable,  $y$  is a noise corrupted measurement of  $x_2$ ,  $w$  is the disturbance vector, and

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad A(k) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & 0 & 0 \\ k & -k & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \rho_2,$$

$$E_1 = \rho_1 C = \rho_1 [0 \ 1 \ 0 \ 0], \quad D_2 = \rho_2 [0 \ 1].$$

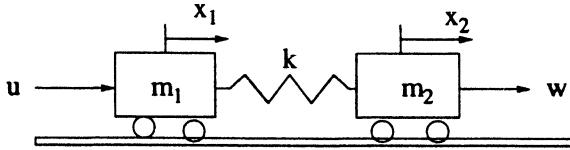


FIGURE 4 Two-mass/spring system.

Here  $E_2 = \rho_2$  and  $\rho_1$  are introduced artificially to act as control knobs. Decreasing  $\rho_2$  increases the controller authority and increasing  $\rho_1$  causes attenuation of the performance variable  $z$ . The parameters  $\rho_1$  and  $\rho_2$  are therefore made a function of the homotopy parameter  $\lambda$ . It is assumed that  $k = k_{\text{nom}} + \Delta k$ . The perturbation in  $A(k)$  due to a change  $\Delta k$  in the stiffness element  $k$  from the nominal value  $k_{\text{nom}}$  is given by

$$A(k) - A(k_{\text{nom}}) \stackrel{\Delta}{=} \Delta A = -B_0 \Delta k C_0, \quad (4.4)$$

where

$$B_0 = [0 \ 0 \ 1 \ -1]^T, \quad C_0 = [1 \ -1 \ 0 \ 0].$$

We desire to design a constant gain linear feedback compensator  $K(s)$  with realization (2.2) such that the closed-loop system is stable for  $0.5 < k < 2.0$  and for a unit impulse disturbance at  $t = 0$ , the performance variable  $z$  has a settling time of about 15 s for the nominal system (with  $k = k_{\text{nom}} = 1$ ). The uncertain plant and the controller can now be put into the form shown in Fig. 1.

## ***H*<sub>2</sub> Performance**

The closed-loop system can now be put into the form shown in Fig. 2, with a realization given by (2.6). The matrices  $\tilde{A}$ ,  $\tilde{B}_0$ ,  $\tilde{C}_0$  in (2.12) of Theorem 1 are given by (2.7)–(2.9). The upper bound on the  $H_2$  cost functional is given by (2.13) in Theorem 1. It can be seen that the diagonal  $H$  and  $N$  of the Popov multiplier reduce to scalars for this particular example. The parameter vector  $x$  with respect to which the

Lagrangian (3.3) is to be minimized is given by

$$x = [H \ N \ \epsilon \ \text{vec}^T(A_c) \ \text{vec}^T(B_c) \ \text{vec}^T(C_c)]^T. \quad (4.5)$$

The initial point  $x_0$  is chosen in the following manner.  $H_0$ ,  $N_0$ , and  $\epsilon_0$  are chosen arbitrarily as 10, 10, and 1, respectively. The initial controller  $(A_{c,0}, B_{c,0}, C_{c,0})$  is an LQG controller for the plant corresponding to  $k = 1.25$ ,  $\rho_1 = 1$  and  $\rho_2 = \sqrt{0.1}$ . No robustness is expected of this controller and hence the initial  $M_1$  and  $M_2$  in (2.3) are chosen close to zero (i.e.,  $M_{1,0} = -0.01$  and  $M_{2,0} = 0.01$ ). It was found that the desired robustness and performance level could not be achieved without increasing the authority of the controller. Hence  $M_1$  and  $M_2$  are increased to  $+0.75$  and  $-0.75$  respectively and  $\rho_2$  is decreased to  $\sqrt{0.001}$ , during the course of the homotopy algorithm.

### $H_\infty$ Performance

As shown in Section 2, the problem may be formulated to minimize an  $H_\infty$  instead of an  $H_2$  performance index. The closed-loop system may now be formulated as shown in Fig. 3 and is given by the realization

$$\tilde{G} \sim \left[ \begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & 0 \end{array} \right] \quad (4.6)$$

where  $\tilde{A}$ ,  $\tilde{B}$ , and  $\tilde{C}$  are given by (2.7) and (2.15). The formulation is similar to that in Braatz and Morari (1992). However, unlike in Braatz and Morari (1992), we let the performance block  $\Delta_p$  be full, i.e.,  $\Delta_p \in \mathcal{C}^{2 \times 2}$ .  $\bar{M}_1$  and  $\bar{M}_2$  are diagonal matrices of the form given in (2.14). Notice that in this formulation  $\Delta$  is *mixed* since  $\Delta_k$  is real whereas  $\Delta_p$  is allowed to be complex. Hence the multiplier elements  $H$  and  $N$  have the structure

$$H = \text{diag}\{h_{11}, h_{22}, h_{22}\}, \quad N = \text{diag}\{n_{11}, 0, 0\} \quad (4.7)$$

as discussed in Theorem 2. The parameter vector  $x$  with respect to which the Lagrangian (3.3) is to be minimized is given by

$$x = [h_{11} \ h_{22} \ n_{11} \ \epsilon \ \text{vec}(A_c)^T \ \text{vec}(B_c)^T \ \text{vec}(C_c)^T]^T. \quad (4.8)$$

The initial point  $x_0$  is chosen in the following manner.  $H_0$ ,  $N_0(1, 1)$ , and  $\epsilon_0$  are chosen arbitrarily as  $10I_3$ , 10 and 1, respectively. The initial controller  $(A_{c,0}, B_{c,0}, C_{c,0})$  is an  $H_\infty$  controller corresponding to  $k = 1.25$ ,  $\rho_1 = 1$ ,  $\rho_2 = 0.01$ ,  $\bar{M}_1 = \text{block-diag}\{-0.001, -0.1I_2\}$  and  $\bar{M}_2 = \text{block-diag}\{0.001, 0.1I_2\}$ . The initial controller provides a fair amount of performance but no robustness. It was found that the required robustness level could not be achieved without increasing the controller authority. Hence,  $\bar{M}_1(1, 1)$  and  $\bar{M}_2(1, 1)$  are changed to  $-0.75$  and  $+0.75$  respectively, and  $\rho_2$  is decreased to 0.0001, during the course of the homotopy algorithm. The parameter  $\rho_1$  is increased to 2.5 to improve the performance.

### Complex $\mu$ Synthesis

The formulation is identical to the robust  $H_\infty$  performance case except that the  $\Delta$  is complex structured (*not mixed*). The weights are adjusted till  $\mu$  for *robust stability* is 1 and the settling time constraints are met. The  $D$ -scales are constrained to be of zeroth order to keep the order of the controller fixed at 4, so as to be able to make a fair comparison between different synthesis techniques.

### Observations

All three controllers are guaranteed by the theory to be robust for the range  $0.5 < k < 2.0$  and this was also verified by a direct search. The actual cost and the upper bound on the worst case cost (for  $0.5 < k < 2.0$ ), as guaranteed by the respective algorithms, have been plotted for all three controllers in Figs. 7, 10 and 13. It can be seen that the upper bound on the worst case cost for both the robust  $H_2$  and robust  $H_\infty$  controllers are fairly ‘tight’, whereas that for complex  $\mu$  synthesis is clearly very conservative. The robust  $H_2$  controller is stable for  $0.35 < k < 2.39$ ; the robust  $H_\infty$  controller is stable for  $0.4 < k < 2.45$  and the controller obtained by complex  $\mu$  synthesis is stable for  $0.32 < k < 6.7$ , as can also be seen from the familiar ‘cost buckets’ in Figs. 7, 10 and 13. Clearly the controllers obtained using the Popov multiplier approach are less conservative than that obtained by complex  $\mu$  synthesis.

The settling time for the system was chosen to be the time required for the displacement of mass 2 to reach and stay within the interval  $[-0.1 \text{ m}, 0.1 \text{ m}]$ . All three controllers are seen to satisfy the settling time objectives when connected to the nominal model corresponding to  $k = 1 \text{ N/m}$ , as can be seen from the impulse response of the closed-loop system in Figs. 5, 8 and 11. It can also be seen that the settling time objective is satisfied for the entire family of plants ( $0.5 < k < 2.0$ ), which, though not a design requirement, is a very desirable characteristic of the controllers. It is seen that the robust  $H_2$  and robust  $H_\infty$  controllers obtained using the Popov multiplier approach yield similar time responses. The control effort required is shown in Figs. 6, 9 and 12. It is seen that nearly similar control effort is required by both the robust  $H_2$  and the robust  $H_\infty$  controllers and it is significantly less than that required by the complex  $\mu$  controller. It can also be seen from Fig. 14 that both the robust  $H_2$  and the robust  $H_\infty$  controllers have bandwidths which are significantly smaller than the bandwidth of the complex  $\mu$  controller.

It is observed that the algorithm for robust  $H_\infty$  performance is much more computationally intensive than that for robust  $H_2$  performance.

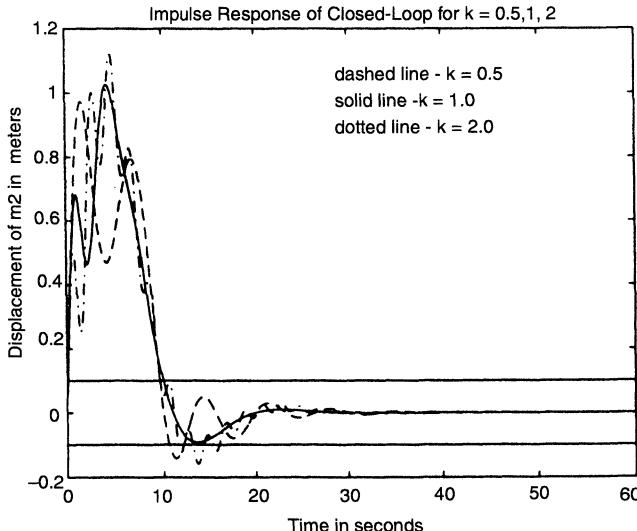
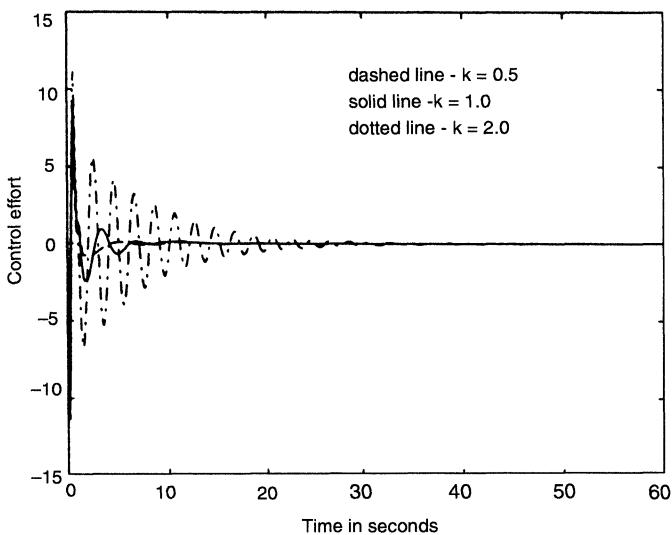
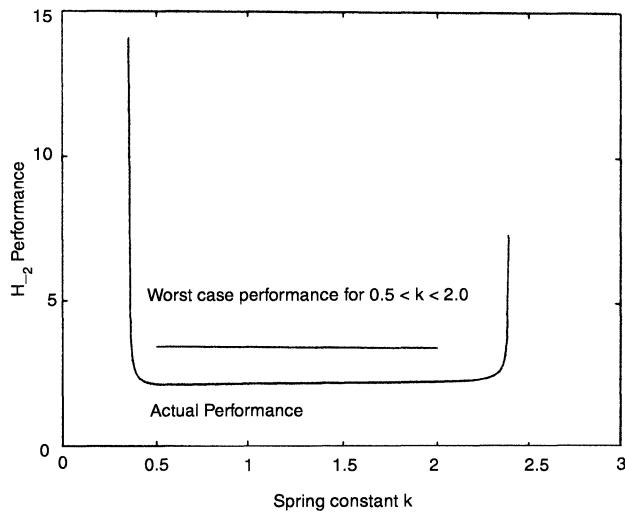
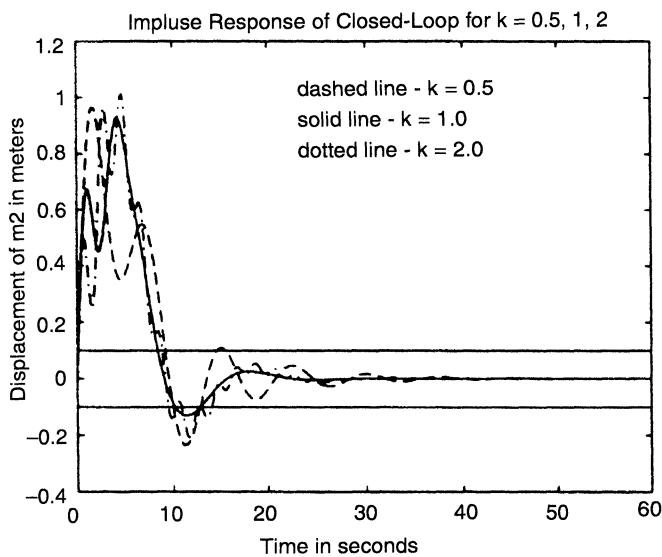
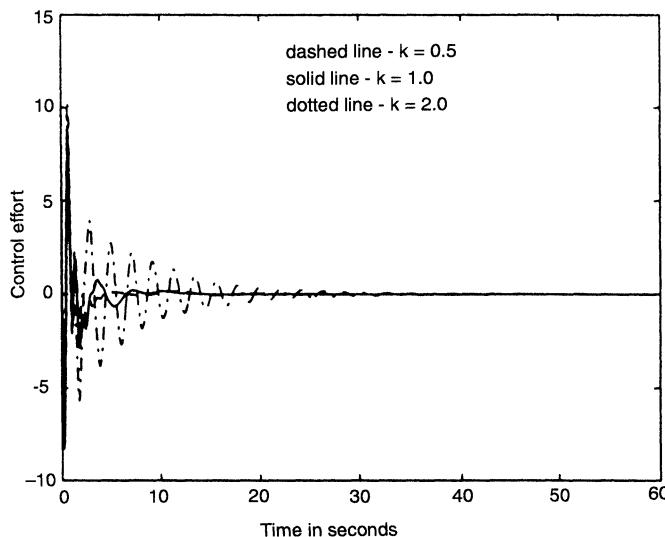


FIGURE 5 Impulse response – robust  $H_2$  controller.

FIGURE 6 Control effort – robust  $H_2$  controller.FIGURE 7  $H_2$  performance for range of values of spring constant  $k$  – robust  $H_2$  controller.

FIGURE 8 Impulse response – robust  $H_\infty$  controller.FIGURE 9 Control effort – robust  $H_\infty$  controller.

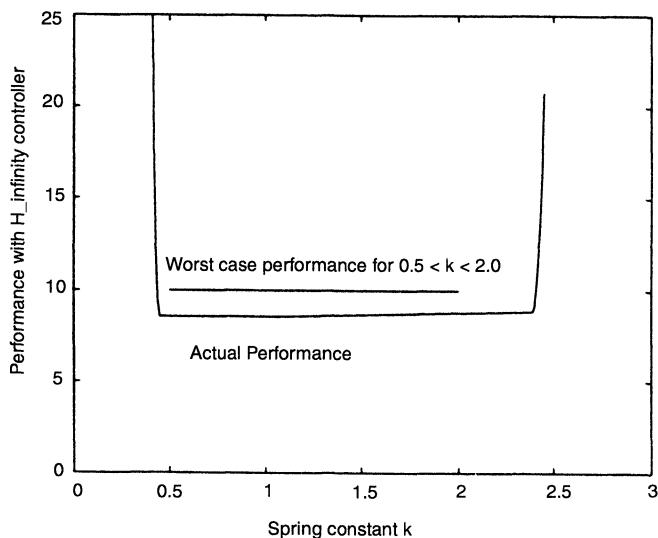


FIGURE 10  $H_{\infty}$  performance vs. spring constant  $k$  – robust  $H_{\infty}$  controller.

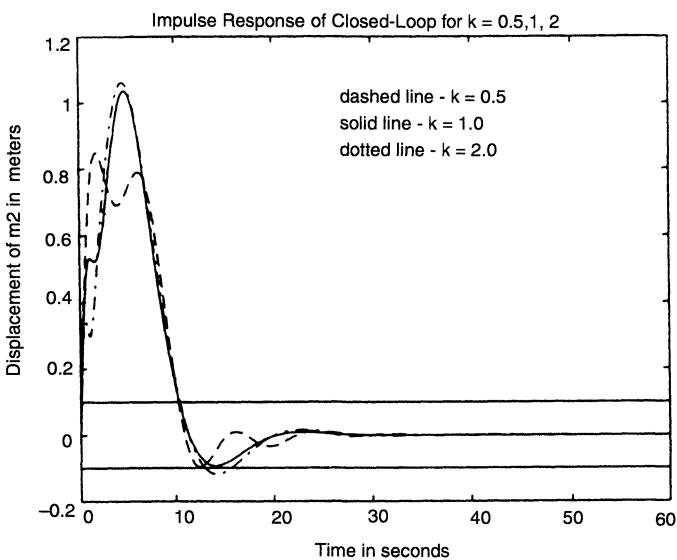


FIGURE 11 Impulse response – complex  $\mu$  controller.

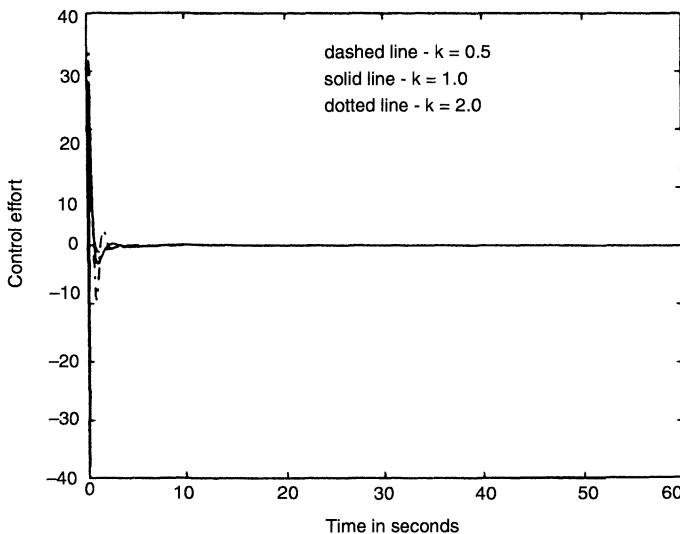


FIGURE 12 Control effort – complex  $\mu$  controller.

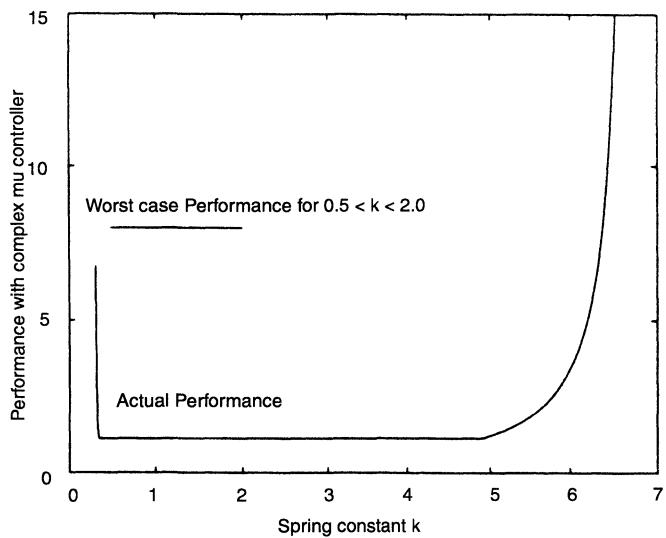


FIGURE 13  $H_\infty$  performance vs. spring constant  $k$  – complex  $\mu$  controller.

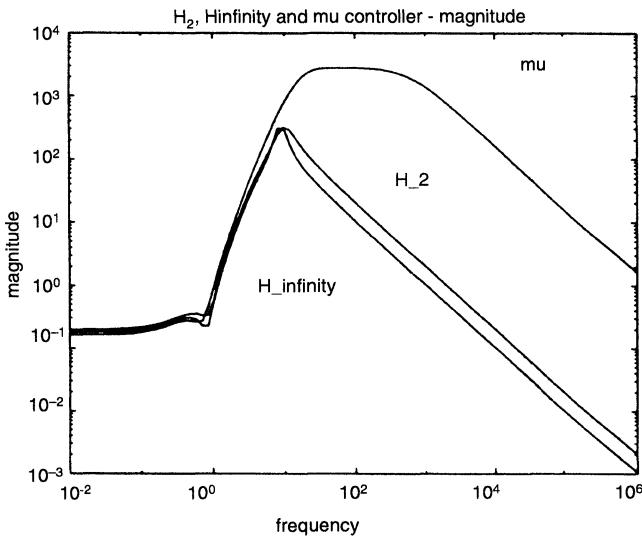


FIGURE 14 Controller bandwidths.

This is because the expressions for the gradient and Hessian for  $H_\infty$  design are far more complex than those for  $H_2$  design.

## 5 CONCLUSIONS

In this paper the Popov multiplier has been used to develop probability-one homotopy algorithms for the design of robust controllers with guaranteed  $H_2$  or  $H_\infty$  performance. The formulation closely follows that presented in Collins *et al.* (1996, 1997) and extends it to the case of robust controllers with  $H_\infty$  performance. Though the formulation for both the robust  $H_2$  and the robust  $H_\infty$  problems are very similar, the gradient and the Hessian expressions for the  $H_\infty$  formulation are more complex. A numerical benchmark example is presented for both the robust  $H_2$  and  $H_\infty$  controllers. Both controllers are found to have smaller bandwidth, smaller control authority and to be significantly less conservative than controllers obtained by complex  $\mu$  synthesis. It is seen that the algorithms for the robust  $H_\infty$  controllers are

more computationally intensive than algorithms for robust  $H_2$  controllers, as is expected.

Certainly if the uncertainty is mixed, and the performance requirements are in terms of  $H_\infty$  cost, it is preferable to use the multiplier based algorithms with guaranteed  $H_\infty$  performance (as described in this paper) than complex  $\mu$  synthesis. The fact that the robust  $H_2$  and the robust  $H_\infty$  algorithms produce controllers with similar characteristics, suggests that when the performance specifications are not directly in terms of either  $H_2$  or  $H_\infty$  cost, one may use either of the two algorithms. In this case, due to the significant difference in computational complexity, it is advantageous to use the algorithm for  $H_2$  performance.

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