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The Effect of Loss Functions on Empirical Bayes Reliability Analysis

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The aim of the present study is to investigate the sensitivity of empirical Bayes estimates of the reliability function with respect to changing of the loss function. In addition to applying some of the basic analytical results on empirical Bayes reliability obtained with the use of the "popular" squared error loss function, we shall derive some expressions corresponding to empirical Bayes reliability estimates obtained with the Higgins–Tsokos, the Harris and our proposed logarithmic loss functions. The concept of efficiency, along with the notion of integrated mean square error, will be used as a criterion to numerically compare our results.

It is shown that empirical Bayes reliability functions are in general sensitive to the choice of the loss function, and that the squared error loss does not always yield the best empirical Bayes reliability estimate.

Keywords: Reliability; Weibull and gamma underlying failure models; Empirical Bayes estimates; Integrated mean square error; Relative efficiency

1 INTRODUCTION

Empirical Bayes estimation was introduced by Robbins [8]. It parallels the Bayesian estimation philosophy except that the prior probability distribution is unknown and not assumed. Instead it assumes that realizations of the underlying failure model parameter have been estimated several times before. The obtained estimates that constitute

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some past information will help us construct the prior probability distribution empirically. Thus, the basic advantage in utilizing the empirical Bayes technique is the fact that we bypass having to assume the prior probability distribution function. The obtained decision function represents a good approximation to the Bayesian decision function [1,4,6,7].

In the early 1950s Epstein and Sobel [3] began to explore the field of parametric life testing. Under the assumptions of an exponential time-to-failure, they produced a series of papers [3] which were to influence future work in reliability and life parameter estimation.

Shortly thereafter, other failure distributions, more complex than the exponential, were used as failure models. For example, Kao [5] brought attention to the Weibull probability distribution, while Birnbaum and Saunders [2] suggested the gamma probability distribution.

In the present study, we shall analyze these classical and useful failure models in the Bayesian setting, that is, we shall consider the parameters inherent in these probability failure models to behave as random variables.

That is, we shall consider the three-parameter Weibull and the gamma underlying failure models that are respectively defined as follows:

$$f(x;a,b,c) = \frac{c}{b}(x-a)^{c-1}\exp\left(-\frac{(x-a)^c}{b}\right), \quad x \geq a;\ b,c > 0, \quad (1.1)$$

where a, b and c are respectively the location, the scale and the shape parameters, and

$$g(x;\alpha,\beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1}\exp\left(-\frac{x}{\beta}\right), \quad x > 0;\ \beta > 0, \qquad (1.2)$$

where α and β are respectively the shape and scale parameters.

For each of the above underlying probability failure models, we shall obtain empirical Bayes estimates of the reliability function by using the squared error, the Higgins–Tsokos, the Harris and a new proposed logarithmic loss functions that are respectively defined as follows.

Squared Error Loss
The squared error loss function is defined by:

$$L_{SE}\left(\hat{R}(t), R(t)\right) = \left(\hat{R}(t) - R(t)\right)^2,$$

where $R(t)$ and $\hat{R}(t)$ represent the true and estimated reliability functions,

$$R(t) = 1 - F(t) = 1 - \int_0^t f(x)\,\mathrm{d}x$$

with $f(x)$ being the failure probability distribution.

The aim of this loss function is that it places a small weight on estimates near the true value and proportionately more weight on extreme deviation from the true value of the $R(t)$. It is used extensively, and its popularity is due to its analytical tractability in Bayesian analysis.

Higgins–Tsokos Loss
The Higgins–Tsokos loss function is defined by

$$L_{HT}\left(\hat{R}, R\right) = \frac{f_1 \mathrm{e}^{-f_2(\hat{R}-R)} + f_2 \mathrm{e}^{-f_1(\hat{R}-R)}}{f_1 + f_2} - 1, \quad f_1, f_2 > 0.$$

This loss function has been shown to be very useful because it places a heavy penalty on extreme over- and underestimation. That is, it places an exponential weight on the extreme error.

Harris Loss
The Harris loss function is defined by

$$L_H\left(\hat{R}, R\right) = \left|\frac{1}{1-\hat{R}} - \frac{1}{1-R}\right|^k, \quad k > 0.$$

To our knowledge the properties of the Harris loss function have not been fully investigated. However, it is based on the premises that if the system is 0.99 reliable then on the average it should fail one time in

100, whereas if the reliability is 0.999, it should fail one time in 1000 and thus, it is ten times as good.

Logarithmic Loss

We propose the logarithmic loss function defined by

$$L_{\ln}(\hat{R}, R) = \left|\ln\left(\frac{\hat{R}}{R}\right)\right|^{l}, \quad l > 0.$$

The merit of this loss function is that it places a small weight on estimates whose ratios to the true value are close to one and proportionately more weight on estimates whose ratios to the true value are significantly different from one. In the present study we shall consider the cases where $l = k = 2$.

2 PRELIMINARY RESULTS

The reliability function corresponding to the three-parameter Weibull is defined as follows:

$$\begin{aligned} R(t) &= 1 - F(t) \\ &= 1 - \int_0^t \frac{c}{b}(x-a)^{a-1} \exp\left(-\frac{(t-a)^c}{b}\right) \mathrm{d}x, \quad b, c > 0, \ t > a. \end{aligned} \tag{2.1}$$

The reliability function corresponding to the gamma probability distribution is given by

$$R(t) = 1 - \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_0^t x^{\alpha-1} \exp\left(-\frac{c}{\beta}\right) \mathrm{d}x, \quad t > 0; \ \alpha, \beta > 0, \tag{2.2}$$

where $\hat{\gamma}(l_1, l_2)$ is the incomplete gamma function.

When α is an integer, Eq. (2.2) reduces to the following expression:

$$R(t) = \left(\sum_{i=0}^{\alpha-1} \frac{1}{i!}\left(\frac{t}{\beta}\right)^i\right) \mathrm{e}^{-t/\beta}.$$

In particular when $\alpha = 1$ we obtain

$$R(t) = e^{-t/\beta}, \quad t > 0. \tag{2.3}$$

Consider the situation where we have k independent random variables $X_1, X_2, \ldots, X_k$ with the same probability density function $dF(x \mid \theta)$, and each of them having n realizations:

$$\begin{aligned}
X_1&: x_{11}, x_{21}, \ldots, x_{n1},\\
X_2&: x_{12}, x_{22}, \ldots, x_{n2},\\
&\vdots\\
X_k&: x_{1k}, x_{2k}, \ldots, x_{nk}.
\end{aligned}$$

The Minimum Variance Unbiased Estimate (MVUE), $\hat{\theta}_j$, of the parameter θ_j is obtained with the n realizations $x_{1j}, x_{2j}, \ldots, x_{nj}$ where $j = 1, \ldots, k$.

Repeating independently the same estimation k times, we obtain the following MVUEs for the parameters θ_js: $\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k$.

Using the θ_js and the probability density function of the $\hat{\theta}_j$s, we obtain a smooth empirical Bayes estimate for the parameter θ. A continuously smooth empirical Bayes estimate is computed with the use of the θ_js, the probability density function of the $\hat{\theta}_j$s, and a weighting function. In this study the exponential weighting function will be used. It is defined as follows:

$$g(\theta) = \exp\left(-\frac{|\theta - \hat{\theta}|}{k^{-1/5}}\right), \quad \theta > 0. \tag{2.4}$$

3 MAIN RESULTS

3.1 TheThree-Parameter Weibull Failure Model

For the three-parameter Weibull underlying failure model, we shall consider the location and shape parameters, a and c, fixed. The scale parameter b will be assumed to have a random variable and will correspond to the behavior of the parameter θ as we discussed above.

The likelihood function corresponding to n independent random variables following our three-parameter Weibull probability failure model can be written in the following form:

$$\ell(x, a, c \mid b) = \exp\left(-\frac{S_n}{b} - n\ln(b) + (c-1)\sum_{i=1}^{n} \ln(x_1 - a) + n\ln(c) \right) \tag{3.1}$$

where

$$S_n = \sum_{i=1}^{n} (x_i - a)^c.$$

Therefore, if we consider the scale parameter b, Eq. (3.1) satisfies the factorization theorem and S_n is a sufficient statistic for b with its MVUE given by

$$\hat{b} = \frac{S_n}{n}. \tag{3.2}$$

Furthermore, the expected value and conditional expectation of $\hat{b}$ are

$$E(\hat{b}) \quad \text{and} \quad E(\hat{b} \mid S_n) = \hat{b}. \tag{3.3}$$

The probability density function of $Y = (X - a)^c$, where X follows the three-parameter Weibull probability density function, is given by

$$\begin{aligned} f(y \mid b) &= \frac{c}{b}(y^{1/c})^{c-1} \mathrm{e}^{-(1/b)y} \left(\frac{1}{c} y^{1/c-1} \right), \quad y > 0 \\ &= \frac{1}{b} \mathrm{e}^{-(1/b)y}, \quad y > 0; \ b > 0. \end{aligned} \tag{3.4}$$

The moment generating function of Y is therefore given by

$$\begin{aligned} E(\mathrm{e}^{\mu y}) &= \frac{1}{b} \int_0^{\infty} \mathrm{e}^{-y(1/b - \mu)} \, \mathrm{d}y \\ &= (1 - \mu b)^{-1}. \end{aligned} \tag{3.5}$$

Using Eq. (3.5) and the fact that the X_is are independent, the moment generating function of the MVUE $\hat{b}$ is given by

$$E(e^{\mu\hat{b}}) = \prod_{i=1}^{n} E(e^{(\mu/n)(x_i-a)^c}) = \left(1-\mu\frac{b}{n}\right)^{-n}. \tag{3.6}$$

Equation (3.6) corresponds to the moment generating function of the gamma probability distribution $G(n, b/n)$. Therefore, the probability density function of the MVUE $\hat{b}$ conditional on b is given by

$$g(\hat{b}, a, c\,|\,b) = \frac{n^n}{\Gamma(n)b^n}(\hat{b})^{n-1}e^{-(n/b)\hat{b}}, \quad \hat{b} > 0;\ b > 0. \tag{3.7}$$

To determine a smooth empirical Bayes estimator of b corresponding to the squared error loss for the realization b_k, we use the probability density function of $\hat{b}$ given by Eq. (3.7) and obtain

$$\begin{aligned}\bar{b}_{1kk} &= \frac{\sum_{j=1}^{k} b_j([n^n/\Gamma(n)b_j^n](\hat{b}_k)^{n-1}e^{-(n/b_j)\hat{b}_k})}{\sum_{j=1}^{k}[n^n/\Gamma(n)b_j^n](\hat{b}_k)^{n-1}e^{-(n/b_j)\hat{b}_k}} \\ &= \frac{\sum_{j=1}^{k}(\hat{b}_k/b_j)^{n-1}e^{-(n/b_j)\hat{b}_k}}{\sum_{j=1}^{k}[(\hat{b}_k)/b_j^n]e^{-(n/b_j)\hat{b}_k}}.\end{aligned} \tag{3.8}$$

Replacing b_j by its MVUE $\hat{b}_j$ in Eq. (3.8), we obtain the following smooth empirical Bayes estimate of b corresponding to the realization b_k. That is,

$$\bar{b}_{1k} = \frac{\sum_{j=1}^{k}[1/(\hat{b}_j)^{n-1}]e^{-(n/b_j)\hat{b}_k}}{\sum_{j=1}^{k}[1/(\hat{b}_j)^{n}]e^{-(n/b_j)\hat{b}_k}}. \tag{3.9}$$

By the same process we obtain smooth empirical Bayes estimators for b corresponding to the Higgins–Tsokos, the Harris, and the new logarithmic loss functions. They are given by

$$\bar{b}_{2k} = \frac{1}{f_1+f_2}\ln\left(\frac{\sum_{j=1}^{k}[1/(\hat{b}_j)^n]e^{-(n/b_j)\hat{b}_k+f_1\hat{b}_j}}{\sum_{j=1}^{k}[1/(\hat{b}_j)^n]e^{-n/\hat{b}_k-f_2\hat{b}_j}}\right), \tag{3.10}$$

$$\bar{b}_{3k} = \frac{\sum_{j=1}^{k}[1/(\hat{b}_j)^{n-1}(1-\hat{b}_j)]\mathrm{e}^{-(n/\hat{b}_j)\hat{b}_k}}{\sum_{j=1}^{k}[1/(\hat{b}_j)^{n}(1-\hat{b}_j)]\mathrm{e}^{-(n/\hat{b}_j)\hat{b}_k}}, \quad \hat{b}_j \neq 1, \tag{3.11}$$

and

$$\bar{b}_{4k} = \exp\left(\frac{\sum_{j=1}^{k}[\ln(\hat{b}_j)/\hat{b}_j^{n}]\mathrm{e}^{-(n/\hat{b}_j)\hat{b}_k}}{\sum_{j=1}^{k}[1/\hat{b}_j^{n}]\mathrm{e}^{-(n/\hat{b}_j)\hat{b}_k}}\right), \tag{3.12}$$

respectively.

The empirical Bayes estimates of the reliability function corresponding to the above smooth empirical Bayes estimates of b are expressed under the following form:

$$R(t, a, c \mid \bar{b}_k) = \exp\left(-\frac{(t-a)^c}{\bar{b}_k}\right), \quad t > a, \tag{3.13}$$

where $\bar{b}_k$ stands respectively for $\bar{b}_{1k}$, $\bar{b}_{2k}$, $\bar{b}_{3k}$ and $\bar{b}_{4k}$.

It can be shown that the use of the above exponential weighting function yields the following continuously smooth empirical Bayes estimator for b corresponding to the realization b_k when the squared error loss function is used

$$\bar{\bar{b}}_{1k} = \frac{\sum_{j=1}^{k}\int_{\min(\hat{b}_j)}^{\max(\hat{b}_j)}[1/b^{n-1}]\mathrm{e}^{-(n/b)\hat{b}_k}\mathrm{e}^{-[|b-\hat{b}_j|/k^{-1/5}]}\,\mathrm{d}b}{\sum_{j=1}^{n}\int_{\min(\hat{b}_j)}^{\max(\hat{b}_j)}[1/b_n]\mathrm{e}^{-(n/b)\hat{b}_k}\mathrm{e}^{-[|b-\hat{b}_j|/k^{-1/5}]}\,\mathrm{d}b}. \tag{3.14}$$

By the same process we obtain continuously smooth empirical Bayes estimates of b using the Higgins–Tsokos, the Harris and the new logarithmic loss functions. They are given by

$$\bar{\bar{b}}_{2k} = \ln\left(\frac{\sum_{j=1}^{n}\int_{\min(\hat{b}_j)}^{\max(\hat{b}_j)}(1/b^n)\mathrm{e}^{f_1 b-(n/b)\hat{b}_k}\mathrm{e}^{-[|b-\hat{b}_j|/k^{-1/5}]}\,\mathrm{d}b}{\sum_{j=1}^{n}\int_{\min(\hat{b}_j)}^{\max(\hat{b}_j)}(1/b^n)\mathrm{e}^{-f_2 b-(n/b)\hat{b}_k}\mathrm{e}^{-[|b-\hat{b}_j|/k^{-1/5}]}\,\mathrm{d}b}\right), \tag{3.15}$$

$$\bar{\bar{b}}_{3k} = \frac{\sum_{j=1}^{k}\int_{\min(\hat{b}_j)}^{\max(\hat{b}_j)}[1/b^{n-1}(1-b)]\mathrm{e}^{-(n/b)\hat{b}_k}\mathrm{e}^{-[|b-\hat{b}_j|/k^{-1/5}]}\,\mathrm{d}b}{\sum_{j=1}^{k}\int_{\min(\hat{b}_j)}^{\max(\hat{b}_j)}[1/b^{n}(1-b)]\mathrm{e}^{-(n/b)\hat{b}_k}\mathrm{e}^{-[|b-\hat{b}_j|/k^{-1/5}]}\,\mathrm{d}b}, \tag{3.16}$$

$$\max(b_j) \neq 1, \quad \min(b_j) \neq 1,$$

and

$$\tilde{\tilde{b}}_{4k} = \exp\left(\frac{\sum_{j=1}^{k} \int_{\min(\hat{b}_j)}^{\max(\hat{b}_j)} (\ln(b)/b^n) \mathrm{e}^{-(n/b)\hat{b}_k - [|b-\hat{b}_j|/k^{-1/5}]} \, \mathrm{d}b}{\sum_{j=1}^{k} \int_{\min(\hat{b}_j)}^{\max(\hat{b}_j)} (1/b^n) \mathrm{e}^{-(n/b)\hat{b}_k - [|b-\hat{b}_j|/k^{-1/5}]} \, \mathrm{d}b} \right), \tag{3.17}$$

respectively, where $j = 1, 2, \ldots, k$.

The empirical Bayes reliability estimates corresponding to the above continuously smooth empirical Bayes estimates of b are expressed under the following form:

$$R(t, a, c \mid \tilde{\tilde{b}}_k) = \exp\left(-\frac{(t-a)^c}{\tilde{\tilde{b}}_k} \right), \quad t > a, \tag{3.18}$$

where $\tilde{\tilde{b}}_k$ stands for $\tilde{\tilde{b}}_{1k}$, $\tilde{\tilde{b}}_{2k}$, $\tilde{\tilde{b}}_{3k}$ and $\tilde{\tilde{b}}_{4k}$, respectively.

3.2 The Gamma Failure Model

The likelihood function corresponding to n independent random variables following our two-parameter gamma underlying failure model can be written as follows:

$$\ell(x, \alpha \mid \beta) = \mathrm{e}^{-(1/\beta)S'_n - n\alpha \ln(\beta)} \mathrm{e}^{(\alpha-1)\sum_{i=1}^{n} \ln(x_i) - n \ln(\Gamma(\alpha))}, \tag{3.19}$$

where

$$S'_n = \sum_{i=1}^{n} x_i. \tag{3.20}$$

Thus, Eq. (3.19) satisfies the factorization theorem and S'_n is a sufficient statistic for the scale parameter β and its MVUE is given by

$$\hat{\beta} = \frac{\sum_{i=1}^{n} x_i}{n\alpha}. \tag{3.21}$$

Furthermore, we have

$$E(\hat{\beta}) = \beta \quad \text{and} \quad E(\hat{\beta} \mid S'_n) = \hat{\beta}. \tag{3.22}$$

The moment generating function of $\hat{\beta}$ is given by

$$E(e^{\mu\hat{\beta}}) = \prod_{i=1}^{n} E(e^{\mu x_i/n\alpha}) = \left(1 - \mu\frac{\beta}{n\alpha}\right)^{-n\alpha}. \quad (3.23)$$

Equation (3.23) gives the moment generating function of the gamma probability distribution $G(n\alpha, \beta/n\alpha)$. Therefore, the probability density function of the MVUE, $\hat{\beta}$, conditional on β, is given by

$$h(\hat{\beta}, \alpha \mid \beta) = \frac{(n\alpha)^{n\alpha}}{\Gamma(n\alpha)\beta^{n\alpha}}(\hat{\beta})^{n\alpha-1}e^{-(n\alpha/\beta)\hat{\beta}}, \quad \hat{\beta} > 0. \quad (3.24)$$

When we use the squared error loss, an empirical Bayes estimator for the scale parameter β corresponding to the realization β_k is obtained using the following expression:

$$\bar{\beta}_{1kk} = \frac{\sum_{i=1}^{k} \beta_j[(n\alpha)^{n\alpha}/\Gamma(n\alpha)\beta_j^{n\alpha}](\hat{\beta}_k)^{n\alpha-1}e^{-(n\alpha/\beta_j)\hat{\beta}_k}}{\sum_{i=1}^{k}[(n\alpha)^{n\alpha}/\Gamma(n\alpha)\beta_j^{n\alpha}](\hat{\beta}_k)^{n\alpha-1}e^{-(n\alpha/\beta_j)\hat{\beta}_k}}. \quad (3.25)$$

Replacing β_j by the corresponding MVUE $\hat{\beta}_j$ in Eq. (3.25), a smooth empirical Bayes estimator for the scale parameter β corresponding to the realization β_k is given by

$$\bar{\beta}_{1k} = \frac{\sum_{j=1}^{k}[1/(\hat{\beta}_j)^{n\alpha-1}]e^{-(n\alpha/\beta_j)\hat{\beta}_k}}{\sum_{j=1}^{k}[1/(\hat{\beta}_j)^{n\alpha}]e^{-(n\alpha/\beta_j)\hat{\beta}_k}}. \quad (3.26)$$

By the same process we derive smooth empirical Bayes estimates for β corresponding to the Higgins–Tsokos, the Harris and the new logarithmic loss functions. They are defined by

$$\bar{\beta}_{2k} = \frac{1}{f_1 + f_2}\ln\left(\frac{\sum_{j=1}^{k}[1/(\hat{\beta}_j)^{n\alpha}]e^{-(n\alpha/\beta_j)\hat{\beta}_k + f_1\hat{\beta}_j}}{\sum_{j=1}^{k}[1/(\hat{\beta}_j)^{n\alpha}]e^{-(n\alpha/\beta_j)\hat{\beta}_k - f_2\hat{\beta}_j}}\right), \quad (3.27)$$

$$\bar{\beta}_{3k} = \frac{\sum_{j=1}^{k}[1/(\hat{\beta}_j)^{n\alpha-1}(1-\hat{\beta}_j)]e^{(n\alpha/\beta_j)\hat{\beta}_k}}{\sum_{j=1}^{k}[1/(\hat{\beta}_j)^{n\alpha}(1-\hat{\beta}_j)]e^{(n\alpha/\beta_j)\hat{\beta}_k}}, \quad \hat{\beta}_j \neq 1 \quad (3.28)$$

and

$$\bar{\beta}_{4k} = \exp\left(\frac{\sum_{j=1}^{k}[\ln(\bar{\beta}_j)/\hat{\beta}_j^{n\alpha}]\mathrm{e}^{-(n\alpha/\hat{\beta}_j)\hat{\beta}_k}}{\sum_{j=1}^{k}(1/\hat{\beta}_j^{n\alpha})\mathrm{e}^{-(n\alpha/\hat{\beta}_j)\hat{\beta}_k}}\right), \quad \hat{\beta}_j > 0, \tag{3.29}$$

respectively.

Empirical Bayes estimates of the reliability function are therefore obtained by replacing β by its above smooth empirical Bayesian estimates. Thus we can write:

$$R(t,\alpha,\beta \mid \bar{\beta}_k) = 1 - \frac{\gamma(\alpha, t/\bar{\beta}_k)}{\Gamma(\alpha)}, \quad t > 0;\ \ \alpha > 0, \tag{3.30}$$

where $\bar{\beta}_k$ stands for $\bar{\beta}_{1k}, \bar{\beta}_{2k}, \bar{\beta}_{3k}$ and $\bar{\beta}_{4k}$, respectively.

Considering the squared error loss, and the exponential weighting function, we obtain the following continuously smooth empirical Bayes estimator for the scale parameter β corresponding to the realization β_k.

$$\begin{aligned}\tilde{\tilde{\beta}}_{1kk} &= \frac{\sum_{j=1}^{k}\int_{\min(\hat{\beta}_j)}^{\max(\hat{\beta}_j)} \beta[(n\alpha)^{n\alpha}/\Gamma(n\alpha)\beta^{n\alpha}](\hat{\beta}_k)^{n\alpha-1}\mathrm{e}^{-(n\alpha/\beta)\hat{\beta}_k}\mathrm{e}^{[|\beta-\hat{\beta}_j|/k^{-1/5}]}\,\mathrm{d}\beta}{\sum_{j=1}^{k}\int_{\min(\hat{\beta}_j)}^{\max(\hat{\beta}_j)} [(n\alpha)^{n\alpha}/\Gamma(n\alpha)\beta^{n\alpha}](\hat{\beta}_k)^{n\alpha-1}\mathrm{e}^{-(n\alpha/\beta)\hat{\beta}_k}\mathrm{e}^{-[|\beta-\hat{\beta}_j|/k^{-1/5}]}\,\mathrm{d}\beta} \\ &= \frac{\sum_{j=1}^{k}\int_{\min(\hat{\beta}_j)}^{\max(\hat{\beta}_j)}(1/\beta^{n\alpha-1})\mathrm{e}^{-(n\alpha/\beta)\hat{\beta}_k-[|\beta-\beta_j|/k^{-1/5}]}\,\mathrm{d}\beta}{\sum_{j=1}^{k}\int_{\min(\hat{\beta}_j)}^{\max(\hat{\beta}_j)}(1/\beta^{n\alpha})\mathrm{e}^{-(n\alpha/\beta)\hat{\beta}_k-[|\beta-\beta_j|/k^{-1/5}]}\,\mathrm{d}\beta}.\end{aligned} \tag{3.31}$$

By the same process we obtain empirical Bayes estimators for the parameter β corresponding respectively to the Higgins–Tsokos, the Harris and the new logarithmic loss function. They are defined as follows:

$$\tilde{\tilde{\beta}}_{2k} = \frac{1}{f_1+f_2}\ln\left(\frac{\sum_{j=1}^{k}\int_{\min(\hat{\beta}_j)}^{\max(\hat{\beta}_j)}(1/\beta^{n\alpha})\mathrm{e}^{f_1\beta-(n\alpha/\beta)\hat{\beta}_k-[|\beta-\hat{\beta}_j|/k^{-1/5}]}\,\mathrm{d}\beta}{\sum_{j=1}^{k}\int_{\min(\hat{\beta}_j)}^{\max(\hat{\beta}_j)}(1/\beta^{n\alpha})\mathrm{e}^{-f_2\beta-(n\alpha/\beta)\hat{\beta}_k-[|\beta-\hat{\beta}_j|/k^{-1/5}]}\,\mathrm{d}\beta}\right), \tag{3.32}$$

$$\tilde{\tilde{\beta}}_{3k} = \frac{\sum_{j=1}^{k} \int_{\min(\hat{\beta}_j)}^{\max(\hat{\beta}_j)} [1/\beta^{n\alpha-1}(1-\beta)] e^{(-n\alpha/\beta)\hat{\beta}_k - [|\beta-\hat{\beta}_j|/k^{-1/5}]} \, d\beta}{\sum_{j=1}^{k} \int_{\min(\hat{\beta}_j)}^{\max(\hat{\beta}_j)} [1/\beta^{n\alpha-1}(1-\beta)] e^{(-n\alpha/\beta)\hat{\beta}_k - [|\beta-\hat{\beta}_j|/k^{-1/5}]} \, d\beta};$$
$$\max(\hat{\beta}_j) \neq 1, \ \min(\hat{\beta}_j) \neq 1, \qquad (3.33)$$

and

$$\tilde{\tilde{\beta}}_{4k} = \exp\left(\frac{\sum_{j=1}^{k} \int_{\min(\hat{\beta}_j)}^{\max(\hat{\beta}_j)} (\ln(\beta)/\beta^{n\alpha}) e^{-(n\alpha/\beta)\hat{\beta}_k - [|\beta-\hat{\beta}_j|/k^{-1/5}]} \, d\beta}{\sum_{j=1}^{k} \int_{\min(\hat{\beta}_j)}^{\max(\hat{\beta}_j)} (1/\beta^{n\alpha}) e^{-(n\alpha/\beta)\hat{\beta}_k - [|\beta-\hat{\beta}_j|/k^{-1/5}]} \, d\beta} \right), \qquad (3.34)$$

respectively.

Replacing β by the above continuously smooth empirical Bayes estimates for the realization β_k in Eqs. (2.4), we obtain empirical Bayes reliability estimates given by:

$$R(t, \alpha, \mid \tilde{\tilde{\beta}}_k) = 1 - \frac{\gamma(\alpha, t/\tilde{\tilde{\beta}}_k)}{\Gamma(\alpha)}, \quad t > 0; \ \alpha > 0 \qquad (3.35)$$

where $\tilde{\tilde{\beta}}_k$ stands for $\tilde{\tilde{\beta}}_{1k}, \tilde{\tilde{\beta}}_{2k}, \tilde{\tilde{\beta}}_{3k}$ and $\tilde{\tilde{\beta}}_{4k}$, respectively.

3.3 Relative Efficiency

In Bayesian estimation the popular loss function that is commonly used is the square error loss function. The reason that scientists use this loss function is because of its analytical tractability. In the present study, we shall investigate the robustness of the square error loss function in empirical Bayes reliability estimates when it is challenged by the Higgins–Tsokos, Harris and the Logarithmic loss functions. To accomplish this objective, we shall use the concept of *relative efficiency*. Relative efficiency is defined as the ratio of the *integrated mean square error*, IMSE, of the empirical Bayes reliability estimate using one of the challenging loss functions to that of the mean square error loss function, where

$$\text{IMSE} = \int_0^\infty \left(\hat{R}_E(t) - R(t)\right)^2 dt.$$

Thus, the relative efficiencies corresponding respectively to the Higgins–Tsokos, Harris and our proposed logarithmic loss function are defined as follows:

$$\begin{aligned}\mathrm{EFF(HT)} &= \frac{\mathrm{IMSE}\big(\hat{R}_{\mathrm{E(HT)}}(t)\big)}{\mathrm{IMSE}\big(\hat{R}_{\mathrm{E(SE)}}(t)\big)} \\ &= \frac{\int_0^\infty \big(\hat{R}_{\mathrm{E(HT)}}(t) - R(t)\big)^2 \,\mathrm{d}t}{\int_0^\infty \big(\hat{R}_{\mathrm{E(SE)}}(t) - R(t)\big)^2 \,\mathrm{d}t},\end{aligned} \tag{3.36}$$

$$\begin{aligned}\mathrm{EFF(H)} &= \frac{\mathrm{IMSE}\big(\hat{R}_{\mathrm{E(H)}}(t)\big)}{\mathrm{IMSE}\big(\hat{R}_{\mathrm{E(SE)}}(t)\big)} \\ &= \frac{\int_0^\infty \big(\hat{R}_{\mathrm{E(H)}}(t) - R(t)\big)^2 \,\mathrm{d}t}{\int_0^\infty \big(\hat{R}_{\mathrm{E(SE)}}(t) - R(t)\big)^2 \,\mathrm{d}t},\end{aligned} \tag{3.37}$$

and

$$\begin{aligned}\mathrm{EFF(ln)} &= \frac{\mathrm{IMSE}\big(\hat{R}_{\mathrm{E(ln)}}(t)\big)}{\mathrm{IMSE}\big(\hat{R}_{\mathrm{E(SE)}}(t)\big)} \\ &= \frac{\int_0^\infty \big(\hat{R}_{\mathrm{E(ln)}}(t) - R(t)\big)^2 \,\mathrm{d}t}{\int_0^\infty \big(\hat{R}_{\mathrm{E(SE)}}(t) - R(t)\big)^2 \,\mathrm{d}t}.\end{aligned} \tag{3.38}$$

4 NUMERICAL SIMULATIONS

Monte Carlo simulation has been used to generate information from the three-parameter Weibull $W(a = 1, b = 1, c = 2)$ and the two-parameter gamma $G(\beta = 1, \beta = 1)$ probability distributions.

From each of the above underlying failure models, three sets of thirty failure times each have been randomly generated to obtain three minimum variance unbiased estimates for the scale parameters b and β. Empirical Bayes estimates for the reliability functions have been obtained and compared using the concept of relative efficiency.

4.1 The Three-Parameter Weibull $W(a=1,\ b=1,\ c=2)$

For the three random samples of size 30 that have been generated, the corresponding MVUE of b are:

$$\hat{b}_1 = 1.1408084120, \quad \hat{b}_2 = 1.0091278197, \quad \hat{b}_3 = 0.9991267092.$$

In Table I we present the various empirical Bayes reliability estimates under study, along with the relative efficiencies as defined above.

Note that $\hat{R}_{\mathrm{Eb1(SE)}}(t)$, $\hat{R}_{\mathrm{Eb2(SE)}}(t)$, $\hat{R}_{\mathrm{Eb1(HT)}}(t)$, $\hat{R}_{\mathrm{Eb2(HT)}}(t)$, $\hat{R}_{\mathrm{Eb1(H)}}(t)$, $\hat{R}_{\mathrm{Eb2(H)}}(t)$, $\hat{R}_{\mathrm{Eb1(ln)}}(t)$ and $\hat{R}_{\mathrm{Eb2(ln)}}(t)$ correspond to the empirical Bayes reliability estimates obtained with smooth and continuously smooth empirical Bayes estimates of b when the squared error, the Higgins–Tsokos with $f_1 = f_2 = 1$ and $f_1 = 2$, $f_2 = 1$, the Harris and the new logarithmic loss functions are used, respectively.

In Table II we present for certain units of time, t, the actual empirical Bayes estimates of the reliability under different situations.

Note that the above empirical Bayes reliability estimates are good estimates of the true reliability function. These results are illustrated by the graphs in Fig. 1.

4.2 The Gamma Failure Model $G(1, 1)$

Similarly, three random samples of size 30 were generated for the gamma probability distribution with $\alpha = 1$ and $\beta = 1$ and the MVUE of β were calculated. That is,

$$\hat{\beta}_1 = 1.009127916, \quad \hat{\beta}_2 = 1.140808468, \quad \hat{\beta}_3 = 0.9991268436.$$

In Table III we present the different empirical Bayes estimates of the reliability functions under study, along with the relative efficiencies with respect to the best empirical Bayes reliability estimate.

Note that $\hat{R}_{\mathrm{E}\beta 1(\mathrm{SE})}(t)$, $\hat{R}_{\mathrm{E}\beta 2(\mathrm{SE})}(t)$, $\hat{R}_{\mathrm{E}\beta 1(\mathrm{HT})}(t)$, $\hat{R}_{\mathrm{E}\beta 2(\mathrm{HT})}(t)$, $\hat{R}_{\mathrm{E}\beta 1(\mathrm{H})}(t)$, $\hat{R}_{\mathrm{E}\beta 2(\mathrm{H})}(t)$, $\hat{R}_{\mathrm{E}\beta 1(\ln)}(t)$ and $\hat{R}_{\mathrm{E}\beta 2(\ln)}(t)$ correspond to the empirical Bayes reliability estimates obtained with smooth and continuously smooth empirical Bayes estimates of β when the squared error, the Higgins–Tsokos, the Harris and the new logarithmic loss functions are used, respectively.

In Table IV we present for various units of time, t, the numerical estimates of the empirical Bayes estimates of the reliability function.

TABLE I Empirical Bayes reliability estimates corresponding to the smooth and continuously smooth empirical bayes estimates of b

	$\hat{R}_{\mathrm{Eb1(SE)}}(t)$	$\hat{R}_{\mathrm{Eb2(SE)}}(t)$	$\hat{R}_{\mathrm{Eb1(HT)}}(t)$	$\hat{R}_{\mathrm{Eb2(HT)}}(t)$	$\hat{R}_{\mathrm{Eb1(H)}}(t)$	$\hat{R}_{\mathrm{Eb2(H)}}(t)$	$\hat{R}_{\mathrm{Eb1(ln)}}(t)$	$\hat{R}_{\mathrm{Eb2(ln)}}(t)$
Approximation	$e^{-(1/1.0424)(t-1)^2}$	$e^{-(1/1.0670)(t-1)^2}$	$e^{-(1/1.0424)(t-1)^2}$	$e^{-(1/1.0670)(t-1)^2}$	$e^{-(1/0.9971)(t-1)^2}$	$e^{-(1/1.0910)(t-1)^2}$	$e^{-(1/1.0406)(t-1)^2}$	$e^{-(1/1.0660)(t-1)^2}$
IMSE	2.945172×10^{-4}	7.191048×10^{-4}	2.945172×10^{-4}	7.191048×10^{-4}	1.437346×10^{-6}	1.298339×10^{-3}	2.704883×10^{-4}	6.984286×10^{-4}
Efficiency with respect to $\hat{R}_{\mathrm{Eb1(SE)}}(t)$	1.0000000	2.4416394	1.0000000	2.4416394	0.004880346	4.4083639	0.9184125	2.3714357

TABLE II Some values taken by the different empirical bayes reliability estimates corresponding to the Weibull distribution

Time t	$\hat{R}(t)$	$\hat{R}_{\mathrm{Eb1(SE)}}(t)$	$\hat{R}_{\mathrm{Eb2(SE)}}(t)$	$\hat{R}_{\mathrm{Eb1(ln)}}(t)$	$\hat{R}_{\mathrm{Eb2(ln)}}(t)$	$\hat{R}_{\mathrm{Eb1(HT)}}(t)$	$\hat{R}_{\mathrm{Eb2(HT)}}(t)$	$\hat{R}_{\mathrm{Eb1(H)}}(t)$	$\hat{R}_{\mathrm{Eb2(H)}}(t)$
1.00001	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.25	0.9394	0.9418	0.9431	0.9417	0.9431	0.9418	0.9431	0.9392	0.9443
1.50	0.7788	0.7868	0.7911	0.7864	0.7909	0.7868	0.7911	0.7782	0.7952
1.75	0.5698	0.5830	0.5903	0.5824	0.5900	0.5830	0.5903	0.5689	0.5972
2.00	0.3679	0.3832	0.3917	0.3825	0.3914	0.3832	0.3917	0.3668	0.3999
2.25	0.2096	0.2234	0.2312	0.2228	0.2310	0.2234	0.2312	0.2087	0.2388
2.50	0.1054	0.1155	0.1214	0.1151	0.1212	0.1155	0.1214	0.1047	0.1272
2.75	0.0468	0.0530	0.0567	0.0527	0.0565	0.0530	0.0567	0.0464	0.0604
3.00	0.0183	0.0216	0.0235	0.0214	0.0235	0.0216	0.0235	0.0181	0.0256
3.25	0.0063	0.0078	0.0087	0.0077	0.0087	0.0078	0.0087	0.0062	0.0097
3.50	0.0019	0.0025	0.0029	0.0025	0.0028	0.0025	0.0029	0.0019	0.0033
3.75	0.0005	0.0007	0.0008	0.0007	0.0008	0.0007	0.0008	0.0005	0.0010
4.00	0.0001	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0001	0.0003

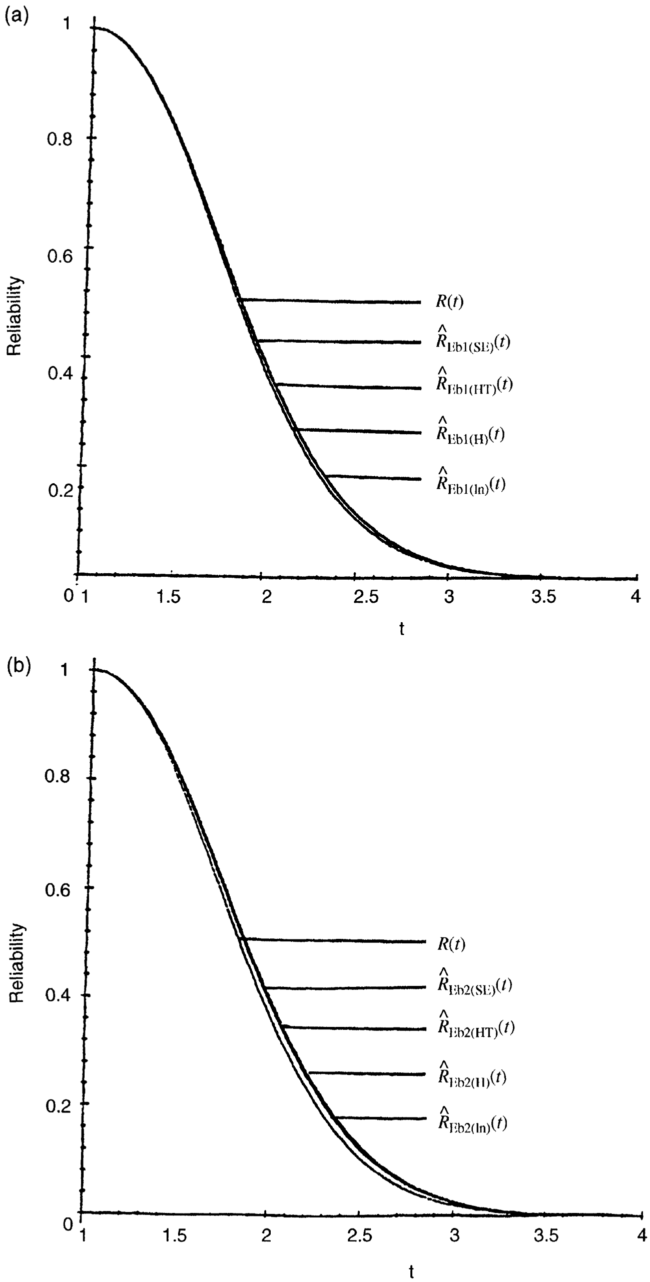

FIGURE 1 True reliability and empirical Bayes reliability estimates for the Weibull distribution, with (a) smooth and (b) continuously smooth empirical Bayes estimates of b.

TABLE III Empirical Bayes reliability estimates corresponding to the smooth and continuously smooth empirical Bayes estimates of β

	$\hat{R}_{E\beta 1(SE)}(t)$	$\hat{R}_{E\beta 2(SE)}(t)$	$\hat{R}_{E\beta 1(HT)}(t)$	$\hat{R}_{E\beta 2(HT)}(t)$	$\hat{R}_{E\beta 1(H)}(t)$	$\hat{R}_{E\beta 2(H)}(t)$	$\hat{R}_{E\beta 1(\ln)}(t)$	$\hat{R}_{E\beta 2(\ln)}(t)$
Approximation	$e^{-(t/1.0424)}$	$e^{-(t/1.0670)}$	$e^{-(t/1.0424)}$	$e^{-(t/1.0670)}$	$e^{-(t/0.9971)}$	$e^{-(t/1.0910)}$	$e^{-(t/1.0406)}$	$e^{-(t/1.0660)}$
IMSE	4.401098×10^{-4}	1.085873×10^{-3}	4.401098×10^{-4}	1.085873×10^{-3}	2.105549×10^{-6}	1.980152×10^{-3}	4.038897×10^{-4}	1.054210×10^{-3}
Efficiency with respect to $\hat{R}_{E\beta 1(SE)}(t)$	1.0000000	2.4672775	1.0000000	2.4672775	0.0047844	4.4992227	0.9177021	2.3953341

TABLE IV Some values taken by the maximum likelihood and the different empirical Bayes reliability estimates corresponding to the gamma distribution

Time t	$\hat{R}(t)$	$\hat{R}_{E\beta 1(SE)}(t)$	$\hat{R}_{E\beta 2(SE)}(t)$	$\hat{R}_{E\beta 1(\ln)}(t)$	$\hat{R}_{E\beta 2(\ln)}(t)$	$\hat{R}_{E\beta 1(HT)}(t)$	$\hat{R}_{E\beta 2(HT)}(t)$	$\hat{R}_{E\beta 1(H)}(t)$	$\hat{R}_{E\beta 2(H)}(t)$
10^{-100}	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.00	0.3679	0.3832	0.3917	0.3825	0.3914	0.3832	0.3917	0.3668	0.3999
2.00	0.1353	0.1468	0.1534	0.1463	0.1532	0.1468	0.1534	0.1346	0.1599
3.00	0.9498	0.0562	0.0601	0.0557	0.0599	0.0562	0.0601	0.0494	0.0639
4.00	0.0183	0.0216	0.0235	0.0214	0.0235	0.0216	0.0235	0.0181	0.0256
5.00	0.0067	0.0083	0.0092	0.0082	0.0092	0.0083	0.0092	0.0066	0.0102
6.00	0.0025	0.0032	0.0036	0.0031	0.0036	0.0032	0.0036	0.0024	0.0041
7.00	0.0009	0.0012	0.0014	0.0012	0.0014	0.0012	0.0014	0.0009	0.0016
8.00	0.0003	0.0005	0.0006	0.0005	0.0006	0.0005	0.0006	0.0003	0.0007
9.00	0.0001	0.0002	0.0002	0.0002	0.0002	0.0002	0.0002	0.0001	0.0003

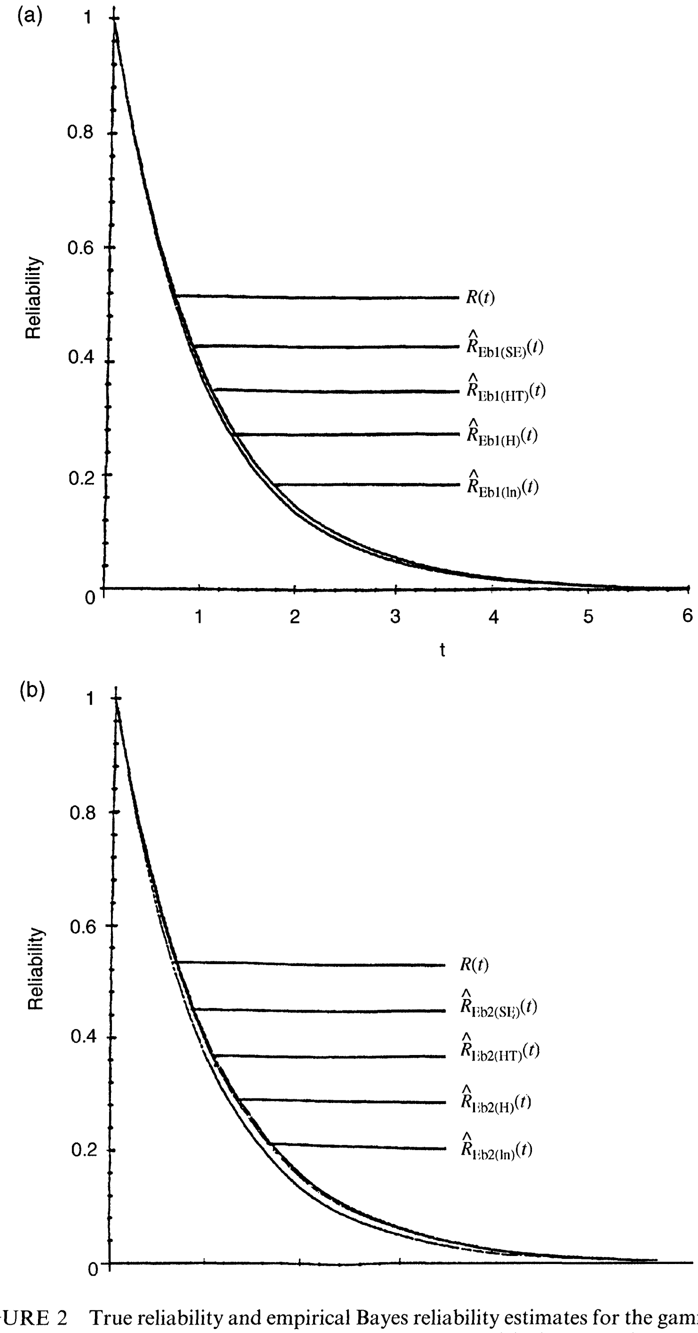

FIGURE 2 True reliability and empirical Bayes reliability estimates for the gamma distribution, with (a) smooth and (b) continuously smooth empirical Bayes estimates of β.

Note that these empirical Bayes estimates corresponding to the two-parameter gamma underlying failure model are good estimates of the true reliability function. This is well-illustrated by the graphs presented in Fig. 2.

5 CONCLUSION

In the present study we have obtained smooth empirical Bayes estimates and continuous smooth empirical Bayes estimates of the reliability function of the three-parameter Weibull failure model and the gamma probability distribution under four different loss functions. The four loss functions that were used are: the mean square error, the Higgins–Tsokos, the Harris, and a new logarithmic loss function that we propose. The analytical expressions of these estimates are quite complicated; however, one can easily computerize them for implementation. We employed the concept of relative efficiency to compare our empirical Bayes estimates using the popular mean square error loss function with the other three challenging loss functions.

Based on our numerical simulation, we can conclude the following:

(i) The obtained empirical Bayes reliability estimates are good estimates of the true reliability function.
(ii) Empirical Bayes reliability estimates are in general sensitive to the choice of the loss function, for both the three-parameter Weibull and the gamma underlying failure models.
(iii) Empirical Bayes reliability estimates are also sensitive to the method used to approximate the prior empirically.
(iv) Empirical Bayes reliability estimates corresponding to the squared error loss function do not always yield the best approximations to the true reliability function. In fact, the empirical Bayes estimates obtained with the Higgins–Tsokos, the Harris and our proposed logarithmic loss functions are sometimes equally efficient if not better.

References

[1] G.K. Bennett, "Smooth Empirical Bayes Estimation with Application to the Weibull Distribution," NASA Technical Memorandum X-58048 (1970).

[2] Z.W. Birnbaum and S.C. Saunders, A statistical model for life-length of material, *Jour. Amer. Statist. Assoc.* **53** (1958), 151–160.

[3] B. Epstein and M. Sobel, Life testing, *Jour. Amer. Statist. Assoc.* **48** (1953), 486–502.

[4] R.V. Hogg and A.T. Craig, *Introduction to Mathematical Statistics*, Macmillan Co., New York, 1965.

[5] J.H.K. Kao, A new-life-quality measurer for electron tubes, *TRE Transactions on Reliability and Quality Control*, 1956.

[6] G.H. Lemon and R.G. Krutchkoff, An empirical Bayes smoothing technique, *Biometrica* **54** (1969), 361–365.

[7] J.S. Maritz, Smooth empirical Bayes estimation for one-parameter discrete distributions, *Biometrica* **54** (1967), 417–429.

[8] H. Robbins, An empirical Bayes estimation problem, *Proc. National Academy of Science, USA* **77** (1980), no. 12, 6988–6989.

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