On Absolute Stability of Nonlinear Systems with Small Delays*

M.I. GIL'[a,†], A. AILON[b,‡] and B.H. AHN[b]

[a] Department of Mathematics, Ben Gurion University of the Negev, P.O. Box 653,
Beer-Sheva 84105, Israel; [b] Department of Mechatronics, Kwangju Institute of
Science and Technology, Kwangju 506-303, Korea

Nonlinear nonautonomous retarded systems with separated autonomous linear parts
and continuous nonlinear ones are considered. It is assumed that deviations of the
argument are sufficiently small. Absolute stability conditions are derived. They are
formulated in terms of eigenvalues of auxiliary matrices.

1 INTRODUCTION AND STATEMENT OF THE RESULT

Stability of nonlinear differential–difference systems has been dis-
cussed by many authors (see [1–3], and references given therein). The
basic method for the stability analysis is the Lyapunov functionals
one. By this method many very strong results are obtained. But finding
Lyapunov's functionals is usually difficult. In [4] (see also [2, p. 111])
an explicit absolute stability criterion for a class of retarded systems
was derived. It is formulated in the terms of the eigenvalues of the
characteristic matrix-valued functions. That result was further devel-
oped in [5].

The present paper is devoted to a class of nonlinear nonautonomous differential–difference systems having sufficiently small deviations of the argument. We will obtain absolute exponential stability conditions in terms of eigenvalues of auxiliary matrices. These conditions allow us to avoid the analysis of quasipolynomials. There have been many stability conditions for nonlinear systems with sufficiently small delays in recent years usually in the category of "robust stability" for time-delay systems (e.g. [6,7], and references therein). Besides, only systems with constant discrete delays are considered. Below we will consider systems with arbitrary (time-dependent and discrete, in general) delays.

Let $\mathbf{C}^n$ be a Euclidean space with the Euclidean norm $\|.\|_{\mathbf{C}^n}$. Moreover, $C([a, b], \mathbf{C}^n)$ is the space of continuous functions defined on the real segment $[a, b]$ with values in $\mathbf{C}^n$ and equipped with the norm

$$\|v\|_{C([a,b],\mathbf{C}^n)} := \sup_{a \le t \le b} \|v(t)\|_{\mathbf{C}^n} \quad (v \in C([a, b], \mathbf{C}^n)).$$

In addition, let $L^2([a, b], \mathbf{C}^n)$ be the space of n-vector-valued functions defined on $[a, b]$ with values in $\mathbf{C}^n$ and equipped with the norm

$$\|v\|_{L^2([a,b],\mathbf{C}^n)} := \left[\int_a^b \|v(t)\|_{\mathbf{C}^n}^2 \, dt \right]^{1/2} \quad (v \in L^2([a, b], \mathbf{C}^n)).$$

Let $\eta < \infty$ be a positive constant. As usually, for all $t \ge 0$ and $x \in C([0, \infty), \mathbf{C}^n)$, x_t is defined by the relation

$$x_t(\theta) \equiv x(\theta + t), \quad -\eta \le \theta \le 0,$$

cf. [1, p. 36], [2, p. 19]. That is, for any $x \in C([0, \infty), \mathbf{C}^n)$, x_t is some function from $C([-\eta, \infty), \mathbf{C}^n)$.

Consider in $\mathbf{C}^n$ the equation

$$\dot{x}(t) = \sum_{k=0}^m A_k x(t - h_k) + F(t, x_t) \quad (t \ge 0, \ \dot{x} = dx/dt), \qquad (1.1)$$

where $A_k \ (k = 0, \ldots, m)$ are constant matrices,

$$0 = h_0 < h_1 < \cdots < h_m = \eta$$

are numbers, and F continuously maps $[0, \infty) \times C([-\eta, 0], \mathbf{C}^n)$ into $\mathbf{C}^n$.

It is assumed that for every $u \in L^2([-\eta, \infty), \mathbf{C}^n) \cap C([-\eta, \infty), \mathbf{C}^n)$, the inequality

$$\|F(\cdot, u_t)\|_{L^2([0,\infty),\mathbf{C}^n)} \equiv \left[\int_0^\infty \|F(t, u_t(t))\|_{\mathbf{C}^n}^2 \, dt \right]^{1/2} \leq q\|u\|_{L^2([-\eta,\infty),\mathbf{C}^n)}$$

(1.2)

is fulfilled. For instance, let there exist constants $b_j > 0$ and $q_j \geq 0$ ($j = 1, \ldots, m < \infty$), such that the relation

$$\|F(t, u_t)\|_{\mathbf{C}^n} \leq \sum_{k=1}^m q_j \|u(t - h_j(t))\|_{\mathbf{C}^n} \quad \text{for all } u \in L^2([-\eta, \infty), \mathbf{C}^n)$$

(1.3)

holds, where $h_j(t)$ ($j = 1, \ldots, m$) are differentiable scalar-valued functions with the properties

$$1 - \dot{h}_j(t) \geq b_j > 0 \quad \text{and} \quad 0 \leq h_j(t) \leq \eta \ (t \geq 0, j = 1, \ldots, m). \quad (1.4)$$

Then by Proposition 8.1.2 of [8], condition (1.2) holds with

$$q = \sum_{j=1}^m q_j b_j^{-1/2}.$$

(1.5)

Take the initial condition

$$x(t) = \Phi(t) \quad \text{for } -\eta \leq t \leq 0,$$

(1.6)

where $\Phi(t)$ is a given continuous vector-valued function defined on $[-\eta, 0]$. *A solution of* (1.1) is an absolutely continuous vector-valued function $x : [-\eta, \infty) \to \mathbf{C}^n$ which satisfies that equation over $R_+ := [0, \infty)$ almost everywhere with a given initial condition (1.6). The solution existence is assumed.

DEFINITION 1.1 We will say that the zero solution of (1.1) is absolutely exponentially stable in the class of nonlinearities (1.2) if there exist positive constants N and ϵ independent of the specific form of the function F (but dependent on q) such that for any solution $x(t)$

of (1.1) with an initial function Φ, the inequality

$$\|x(t)\|_{C^n} \leq N \exp[-\epsilon t] \|\Phi\|_{C([-\eta, 0], C^n)} \quad (t \geq 0)$$

is fulfilled.

Let A be an $n \times n$-matrix, and let $\lambda_k(A)$ $(k = 1, \ldots, n)$ denote the eigenvalues of A including with their multiplicities. The following quantity plays an essential role hereafter:

$$g(A) = \left(N^2(A) - \sum_{k=1}^{n} |\lambda_k(A)|^2\right)^{1/2},$$

where $N(A)$ is the Hilbert–Schmidt (Frobenius) norm of A, i.e. $N^2(A) = \mathrm{Trace}(AA^*)$. The following properties of $g(A)$ are proved in [8, Sections 1.1 and 1.3]: $g^2(A) \leq N^2(A) - |\mathrm{Trace}\, A^2|$,

$$g(A) \leq \sqrt{1/2} N(A^* - A) \quad \text{and} \quad g(Ae^{i\tau} + zI)$$
$$= g(A) \quad \text{for every } \tau \in \mathbf{R}, z \in \mathbf{C}. \tag{1.7}$$

Here and below, I is the unit matrix. In [9, p. 185], $g(A)$ is called the deviation from normality of A, since $g(A) = 0$ if A is a normal matrix, i.e., if $AA^* = A^*A$. Further, set

$$\tilde{A} = \sum_{k=0}^{m} A_k \quad \text{and} \quad V_2 = \sum_{k=1}^{m} h_k \|A_k\|_{C^n}.$$

Everywhere below it is assumed that $\tilde{A}$ is a Hurwitz matrix. Denote

$$\alpha(\tilde{A}) = \max_k \mathrm{Re}\lambda_k(\tilde{A}), \qquad \omega(\tilde{A}) = \max_k |\mathrm{Im}\lambda_k(\tilde{A})|,$$

and

$$\Gamma(\tilde{A}) = \sum_{k=1}^{n-1} \frac{g^k(\tilde{A})}{(k!)^{1/2} |\alpha(\tilde{A})|^{k+1}}.$$

The aim of this paper is to prove the following.

THEOREM 1.2 *Let the condition*

$$[(\alpha^2(\tilde{A}) + \omega^2(\tilde{A}))^{1/2} V_2 + q] \Gamma(\tilde{A}) < 1 \qquad (1.8)$$

hold. Then the zero solution of Eq. (1.1) is absolutely exponentially stable in the class of nonlinearities (1.2).

Remark 1.3 Since $V_2 = h_1 \|A_1\|_{\mathbf{C}^n} + \cdots + h_m \|A_m\|_{\mathbf{C}^n}$, if $q\Gamma(\tilde{A}) < 1$, then sufficently small delays provide the stability.

If $\tilde{A}$ is a normal (in particular, Hermitian) matrix. Then $g(\tilde{A}) = 0$, $\Gamma(\tilde{A}) = |\alpha(\tilde{A})|^{-1}$, and condition (1.8) takes the form

$$[(\alpha^2(\tilde{A}) + \omega^2(\tilde{A}))^{1/2} V_2 + q] |\alpha(\tilde{A})|^{-1} < 1 \qquad (1.9)$$

Remark 1.4 Theorem 1.2 is exact, and condition (1.8) cannot be improved without additional restrictions.

In fact, let us consider the system

$$\dot{x}(t) = Ax(t) + F(t, x(t)), \qquad (1.10)$$

where A is a constant Hurwitz matrix, and F continuously maps $[0, \infty) \times \mathbf{C}^n$ into $\mathbf{C}^n$ with the property

$$\|F(t, h)\|_{\mathbf{C}^n} \le q\|h\|_{\mathbf{C}^n} \quad (h \in \mathbf{C}^n; \, t \ge 0). \qquad (1.11)$$

Under consideration $V_2 = 0$, and (1.8) takes the form $q\Gamma(A) < 1$. If A is normal, then the inequality

$$q|\alpha(A)|^{-1} < 1 \qquad (1.12)$$

gives the absolute stability conditions of the zero solution of Eq. (1.10) in the class of nonlinearities (1.11). Take $F(t, h) = qh$ $(h \in \mathbf{C})$. Then for the asymptotic stability of Eq. (1.10) it is necessary and sufficient that $\alpha(A) + q < 0$. But this condition coincides with (1.12).

2 PROOF OF THEOREM 1.2

For brevity put $\|.\|_{L^2([0, \infty), \mathbf{C}^n)} = \|.\|_L$, and

$$\|y(i\omega)\|_{\tilde{L}} \equiv \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \|y(i\omega)\|_{\mathbf{C}^n}^2 \, d\omega \right]^{1/2} \quad (y \in L^2(\mathbf{R}, \mathbf{C}^n)).$$

Denote

$$c_0(\tilde{A}) = \sup_{\omega \in \mathbf{R}} \|(\mathrm{i}\omega I - \tilde{A})^{-1}\|_{\mathbf{C}^n} \quad \text{and} \quad c_1(\tilde{A}) = \sup_{\omega \in \mathbf{R}} \|\omega(\mathrm{i}\omega I - \tilde{A})^{-1}\|_{\mathbf{C}^n}.$$

Let us consider the equation

$$\dot{x}(t) = \sum_{k=0}^{m} A_k x(t - h_k) + f(t) \quad (t \geq 0). \tag{2.1}$$

LEMMA 2.1   *Under the condition* $c_1(\tilde{A})V_2 < 1$, *let* $f \in L^2([0, \infty), \mathbf{C}^n)$. *Then the bound*

$$\|x\|_L \leq [1 - c_1(\tilde{A})V_2]^{-1} c_0(\tilde{A}) \|f\|_L + m_1 \|\Phi\|_{C([-\eta,0],\mathbf{C}^n)} \tag{2.2}$$

is valid with a constant m_1 *independent of* Φ *for any solution* $x(t)$ *of* (2.1) *with a continuous initial function* Φ.

Proof   Rewrite (2.1) in the form

$$\dot{x}(t) = \tilde{A}x(t) + \sum_{k=1}^{m} A_k(x(t - h_k) - x(t)) + f(t) \quad (t \geq 0). \tag{2.3}$$

The Laplace transform of $x(t - h_k)$ is $\tilde{x}(\lambda)\mathrm{e}^{-h_k\lambda} + v_k(\lambda)$ where $\tilde{x}(\lambda)$ is the Laplace transform of $x(t)$, and

$$v_k(\lambda) = \int_0^{h_k} \mathrm{e}^{-t\lambda} x(t - h_k)\,\mathrm{d}t = \int_0^{h_k} \mathrm{e}^{-t\lambda} \Phi(t - h_k)\,\mathrm{d}t.$$

Thus the application of the Laplace transformation to Eq. (2.3) gives

$$\lambda\tilde{x}(\lambda) - x(0) = \tilde{A}\tilde{x}(\lambda) + \sum_{k=1}^{m} A_k[\tilde{x}(\lambda)(\mathrm{e}^{-h_k\lambda} - 1)$$
$$+ v_k(\lambda)] + \tilde{F}(\lambda) \quad (\lambda \in \mathbf{C}),$$

where $\tilde{F}(\lambda)$ is the Laplace transform of the function $f(t)$. Hence,

$$\tilde{x}(\lambda) = (\lambda I - \tilde{A})^{-1}\left[\sum_{k=1}^{m} A_k(\mathrm{e}^{-h_k\lambda} - 1)x(\lambda) + \tilde{F}(\lambda)\right] + e(\lambda) \quad (\lambda \in \mathbf{C}),$$

$$\tag{2.4}$$

where

$$e(\lambda) = (\lambda I - \tilde{A})^{-1} \left(x(0) + \sum_{k=1}^{m} A_k v_k(\lambda) \right).$$

Since $|1 - e^{-h i \omega}| \leq h|\omega|$ $(\omega \in \mathbf{R}, h > 0)$, we have

$$\left\| (\tilde{A} - i\omega I)^{-1} \sum_{k=1}^{m} A_k (1 - e^{-h_k i\omega}) \tilde{x}(i\omega) \right\|_{\tilde{L}}$$

$$\leq \| \omega(\tilde{A} - i\omega I)^{-1} \|_{C^n} \sum_{k=1}^{m} h_k \|A_k\|_{C^n} \|\tilde{x}(i\omega)\|_{\tilde{L}}$$

$$\leq c_1(\tilde{A}) V_2 \|\tilde{x}(i\omega)\|_{\tilde{L}} \quad (\omega \in \mathbf{R}). \tag{2.5}$$

Due to (2.4) and (2.5), it can be written

$$\|\tilde{x}(i\omega)\|_{\tilde{L}} \leq V_2 c_1(\tilde{A}) \|\tilde{x}(i\omega)\|_{\tilde{L}} + c_0(\tilde{A}) \|\tilde{F}\|_{\tilde{L}} + \|e(i\omega)\|_{\tilde{L}}.$$

Hence, taking into account that $c_1(\tilde{A}) V_2 < 1$, we get

$$\|\tilde{x}(i\omega)\|_{\tilde{L}} \leq (1 - V_2 c_1(\tilde{A}))^{-1} (c_0(\tilde{A}) \|\tilde{F}\|_{\tilde{L}} + \|e(i\omega)\|_{\tilde{L}}). \tag{2.6}$$

The simple calculations show that

$$\|v_k(i\omega)\|_{C^n} \leq h_k \|\Phi\|_{C([-\eta,0],\mathbf{C}^n)} \quad (\omega \in \mathbf{R})$$

and

$$\|e(i\omega)\|_{\tilde{L}} \leq \|(i\omega I - \tilde{A})^{-1}\|_{\tilde{L}} \left(\|x(0)\|_{C^n} + \sum_{k=1}^{m} h_k \|A_k\|_{C^n} \sup_{\omega \in \mathbf{R}} \|v_k(i\omega)\|_{C^n} \right)$$

$$= \|(i\omega I - \tilde{A})^{-1}\|_{\tilde{L}} \left(\|\Phi(0)\|_{C^n} + V_2 \|\Phi\|_{C([-\eta,0],\mathbf{C}^n)} \right).$$

Hence,

$$\|e(i\omega)\|_{\tilde{L}} \leq \|(i\omega I - \tilde{A})^{-1}\|_{\tilde{L}} (1 + V_2) \|\Phi\|_{C([-\eta,0],\mathbf{C}^n)}.$$

Relation (2.6) yields

$$\|\tilde{x}(i\omega)\|_{\tilde{L}} \leq (1 - V_2 c_1(\tilde{A}))^{-1} c_0(\tilde{A}) \|\tilde{F}\|_{\tilde{L}} + m_1 \|\Phi\|_{C([-\eta,0],\mathbf{C}^n)}.$$

Now due to the Parseval equality, we arrive at the required result.

LEMMA 2.2   *Under condition (1.2), let the inequality*

$$c_1(\tilde{A})V_2 + c_0(\tilde{A})q < 1 \qquad (2.7)$$

be fulfilled. Then for any solution $x(t)$ of (1.1) with a continuous initial function $\Phi(t)$, the bound

$$\|x\|_L \leq m_0[1 - (c_1(\tilde{A})V_2 + qc_0(\tilde{A}))]^{-1}\|\Phi\|_{C([-\eta,0],\mathbf{C}^n)}$$

is valid with a constant m_0.

Proof   Rewrite (1.1) in the form (2.1) with $f(t) = F(t, x_t(t))$, where x is the solution of (1.1). Combining the previous lemma with condition (1.2), we easily get

$$\begin{aligned}\|x\|_L &\leq [1 - c_1(\tilde{A})V_2)]^{-1}c_0(\tilde{A})\|f\|_L + m_1\|\Phi\|_{C([-\eta,0],\mathbf{C}^n)}\\ &\leq [1 - c_1(\tilde{A})V_2)]^{-1}c_0(\tilde{A})q(\|x\|_L + \|\Phi\|_{L^2([-\eta,0],\mathbf{C}^n)})\\ &\quad + m_1\|\Phi\|_{C([-\eta,0],\mathbf{C}^n)}.\end{aligned}$$

Consequently,

$$\|x\|_L \leq [1 - c_1(\tilde{A})V_2)]^{-1}c_0(\tilde{A})q\|x\|_L + m_2\|\Phi\|_{C([-\eta,0];\mathbf{C}^n)}$$

with a constant m_2. Condition (2.7) implies

$$[1 - c_1(\tilde{A})V_2)]^{-1}c_0(\tilde{A})q < 1.$$

Thus,

$$\begin{aligned}\|x\|_L &\leq (1 - c_0(\tilde{A})q[1 - c_1(\tilde{A})V_2)]^{-1})m_2\|\Phi\|_{C([-\eta,0],\mathbf{C}^n)}\\ &= (1 - c_0(\tilde{A})q - c_1(\tilde{A})V_2)^{-1}m_0\|\Phi\|_{C([-\eta,0],\mathbf{C}^n)}.\end{aligned}$$

As claimed.

LEMMA 2.3   *The relations $c_0(\tilde{A}) \leq \Gamma(\tilde{A})$ and*

$$c_1(\tilde{A}) \leq \Gamma(\tilde{A})[\alpha^2(\tilde{A}) + \omega^2(\tilde{A})]^{1/2}$$

are true.

Proof   By Corollary 1.2.4 of [8], for any $n \times n$-matrix A, we have the inequality

$$\|(A - I\lambda)^{-1}\|_{C^n} \leq \sum_{k=0}^{n-1} \frac{g^k(A)}{\sqrt{k!}\rho^{k+1}(A, \lambda)} \quad \text{for all regular } \lambda, \qquad (2.8)$$

where $\rho(A, \lambda)$ is the distance between the spectrum of A and a complex point λ. Since $\tilde{A}$ is a Hurwitz matrix,

$$\rho(\tilde{A}, i\omega) \geq |\alpha(\tilde{A})| \quad (\omega \in \mathbf{R}), \qquad (2.9)$$

and therefore $c_0(\tilde{A}) \leq \Gamma(\tilde{A})$.
  Further, we have

$$|\lambda_k(\tilde{A}) - i\omega|^2 \geq \alpha^2(\tilde{A}) + (\omega_k - \omega)^2$$

for any eigenvalue $\lambda_k(\tilde{A})$ of $\tilde{A}$ with $\mathrm{Im}\lambda_k(\tilde{A}) = \omega_k$. Hence simple calculations show that

$$\max_{\omega \in \mathbf{R}} \omega^2 |\lambda_k(\tilde{A}) - i\omega|^{-2} \leq \frac{\alpha^2(\tilde{A}) + \omega_k^2}{\alpha^2(\tilde{A})}.$$

Thus

$$\omega^2 \rho^{-2}(\tilde{A}, \omega) \leq \frac{\alpha^2(\tilde{A}) + \omega^2(\tilde{A})}{\alpha^2(\tilde{A})} \quad (\omega \in \mathbf{R}).$$

Therefore according to (2.8) and (2.9),

$$\|\omega(\tilde{A} - i\omega I)^{-1}\| \leq \sum_{k=0}^{n-1} \frac{|\omega| g^k(\tilde{A})}{\sqrt{k!}\rho^{k+1}(\tilde{A}, \omega)}$$

$$\leq (\alpha^2(\tilde{A}) + \omega^2(\tilde{A}))^{1/2} \sum_{k=0}^{n-1} \frac{g^k(\tilde{A})}{\sqrt{k!}|\alpha(\tilde{A})|^{k+1}}$$

$$= (\alpha^2(\tilde{A}) + \omega^2(\tilde{A}))^{1/2} \Gamma(\tilde{A}) \quad (\omega \in \mathbf{R}).$$

As claimed.

Lemmas 2.2 and 2.3 yield

COROLLARY 2.4   *Let conditions (1.2) and (1.8) hold. Then for any solution $x(t)$ of (1.1) with an initial function $\Phi(t)$, the bound*

$$\|x\|_L \leq m_0[1 - \Gamma(\tilde{A})(V_2(\alpha^2(\tilde{A}) + \omega^2(\tilde{A}))^{1/2} + q)]^{-1}\|\Phi\|_{C([-\eta,0],\,\mathbf{C}^n)}$$
$$(m_0 = \text{const.})$$

is valid.

Proof of Theorem 1.2   From (1.1) it follows the inequality

$$\|\dot{x}\|_L \leq \sum_{k=0}^{m} \|A_k\|_{C^n}\|x(t - h_k)\|_L + \|F\|_L$$
$$\leq V_1(\|x\|_L + \|x\|_{L^2([-\eta,0],\mathbf{C}^n)}) + \|F\|_L,$$

where

$$V_1 = \sum_{k=0}^{m} \|A_k\|_{C^n}.$$

Due to (1.2) and (1.6) this gives

$$\|\dot{x}\|_L \leq (q + V_1)(\|x\|_L + \|\Phi\|_{L^2([-\eta,0],\mathbf{C}^n)}).$$

Now Corollary 2.4 implies the relation

$$\|\dot{x}\|_L \leq N_1\|\Phi\|_{C([-\eta,0],\mathbf{C}^n)} \quad (N_1 = \text{const.}).$$

By the trivial Lemma 8.4.5 of [8], we get

$$\|x\|_{C([0,\infty],\mathbf{C}^n)} \leq (2\|x\|_L\|\dot{x}\|_L)^{1/2} \leq N_2\|\Phi\|_{C([-\eta,0],\mathbf{C}^n)} \quad (N_2 = \text{const.}).$$
$$\tag{2.10}$$

Further, substitute in (1.1) the relation

$$x(t) = x_\epsilon(t)e^{-\epsilon t} \tag{2.11}$$

with a small enough $\epsilon > 0$. Applying our reasoning above, we can assert according to (2.10) that

$$\|x_\epsilon\|_{C([-\eta,\infty], \mathbf{C}^n)} \leq N_\epsilon \|\Phi\|_{C([-\eta,0],\mathbf{C}^n)} \quad (N_\epsilon = \text{const.}).$$

Hence, (2.11) yields the absolute exponential stability.

3 EXAMPLE

Let us consider in $\mathbf{C}^n$ the equation

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \eta) + f\left(t, x(t), \int_0^\eta T(t,s)x(t - s)\,ds\right), \quad (3.1)$$

where the function $f: [0, \infty) \times \mathbf{C}^{2n} \to \mathbf{C}^n$ has the property

$$\|f(t, h, w)\|_{\mathbf{C}^n} \leq q_0 \|h\|_{\mathbf{C}^n} + q_1 \|w\|_{\mathbf{C}^n} \quad (h, w \in \mathbf{C}^n; t \geq 0), \quad (3.2)$$

and the matrix-valued function $T(t, s)$ defined on $[0, \infty) \times [0, \eta]$ satisfies the inequality

$$m_T \equiv \sup_{t \geq 0} \int_0^\eta \|T(t,s)\|_{\mathbf{C}^n}\,ds < \infty. \quad (3.3)$$

In the considered case

$$F(t, x_t) = f\left(t, x(t), \int_0^\eta T(t,s)x(t - s)\,ds\right).$$

Taking into account, that

$$\int_0^\infty \left\| \int_0^\eta T(t,s)x(t - s)\,ds \right\|_{\mathbf{C}^n}^2 dt$$

$$\leq \int_0^\infty \left(\int_0^\eta \|T(t,s)\|_{\mathbf{C}^n} \|x(t - s)\|_{\mathbf{C}^n}\,ds \right)^2 dt$$

$$\leq \sup_{t \geq 0} \left(\int_0^\eta \|T(t,s)\|_{\mathbf{C}^n}\,ds \right)^2 \int_{-\eta}^\infty \|x(t)\|_{\mathbf{C}^n}^2\,dt = m_T^2 \|x\|_{L^2([-\eta,0], \mathbf{C}^n)}^2,$$

we easily get the inequality

$$\left[\int_0^\infty \left\| f(t, u(t), \int_0^\eta T(t, s)u(t - s)\,\mathrm{d}s) \right\|_{\mathbf{C}^n}^2 \mathrm{d}t \right]^{1/2}$$
$$\leq q_0 \|u\|_{L^2([0,\infty];\,\mathbf{C}^n)} + q_1 m_T \|u\|_{L^2([-\eta,\infty],\mathbf{C}^n)}$$

for any $u \in L^2([-\eta, \infty), \mathbf{C}^n) \cap C([-\eta, \infty), \mathbf{C}^n)$. Hence it follows that (1.2) holds with

$$F(t, u_t) = f\left(t, u(t), \int_0^\eta T(t, s)u(t - s)\,\mathrm{d}s \right) \text{ and } q = q_0 + q_1 m_T. \quad (3.4)$$

Clearly, under consideration

$$\tilde{A} = A_0 + A_1, \quad V_2 = \eta \|A_1\|_{\mathbf{C}^n}. \quad (3.5)$$

Now Theorem 1.2 yields the following result. Let $\tilde{A}$, q and V_2 be defined by (3.4) and (3.5). Let condition (1.8) hold. Then the zero solution of Eq. (3.1) is absolutely exponentially stable in the class of nonlinearities satisfying conditions (3.2) and (3.3).

4   CONCLUDING REMARKS

We have derived the sufficient absolute stability conditions for Eq. (1.1) in the class of nonlinearities (1.2) in terms of the eigenvalues of the matrix $\tilde{A}$. These conditions are exact according to Remark 1.4. As it was mentioned, the stability conditions suggested in [6,7] can be applied to systems with constant discrete delays only, while Theorem 1.2 allows us to investigate systems containing time-variant and distributed delays (see relations (1.3) and (3.4) above). Consider now the equation

$$\dot{x} = A_0 x(t) + F(x(t - h)), \quad (4.1)$$

where A_0 is a Hurwitz matrix, $F: \mathbf{C}^n \to \mathbf{C}^n$ satisfies the inequality

$$\|F(z)\|_{\mathbf{C}^n} \leq L \|z\|_{\mathbf{C}^n} \quad (L = \text{const.}, z \in \mathbf{C}^n).$$

In the paper [10] among other very interesting results the following stability condition for Eq. (4.1) is established:

$$\Lambda(A_0) + L < 0, \tag{4.2}$$

where

$$\Lambda(A_0) = \lim_{h \to 0+} (\|I + hA_0\| - 1)/h$$

with some norm $\| \cdot \|$ in $\mathbf{C}^n$. But if A_0 is not dissipative with respect to that norm, condition (4.2) cannot be applied. At that same time, as the example shows, Theorem 1.2 gives the stability conditions for (4.1) if L is sufficiently small.

References

[1] J.K. Hale and S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*. Springer-Verlag, Berlin, 1993.

[2] V. Kolmanovskii and A. Myshkis, *Applied Theory of Functional Differential Equations*, Kluwer Academic Publishers, London, 1992.

[3] V. Kolmanovskii and V. Nosov, *Stability of Functional Differential Equations*, Academic Press, New York, 1986.

[4] M.I. Gil', On absolute stability of differential-delay systems, *IEEE Trans. Automatic Control* **39**(12) (1994), 2481–2484.

[5] M.I. Gil', On absolute stability of neutral type functional differential systems, *Non-linear Studies* **3**(1) (1996), 59–72.

[6] K.K. Shyu and J.J. Yan, Robust stability of uncertain time-delay systems, and its stabilization by variable structure control. *Int. Journal of Control* **57**(1) (1993), 237–246.

[7] S.S. Wang, B.S. Chen and T.P. Lin, Robust stability of uncertain time-delay systems. *Int. Journal of Control* **46**(3) (1987), 963–976.

[8] M.I. Gil', *Norm Estimations for Operator-valued Functions and Applications*. Marcel Dekker Inc., New York, 1995.

[9] L. Collatz, *Functional Analysis and Numerical Mathematics*. Academic Press, New York, Lodon, 1966.

[10] V. Kolmanovskii, Applications of differential inequalities for stability of some functional differential equations. *Nonlinear Analysis, TMA* **25**(9–10) (1995), 1017–1028.

Special Issue on
Time-Dependent Billiards

Call for Papers

This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

Manuscript Due	December 1, 2008
First Round of Reviews	March 1, 2009
Publication Date	June 1, 2009

Guest Editors

Edson Denis Leonel, Departamento de Estatística, Matemática Aplicada e Computação, Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista, Avenida 24A, 1515 Bela Vista, 13506-700 Rio Claro, SP, Brazil ; edleonel@rc.unesp.br

Alexander Loskutov, Physics Faculty, Moscow State University, Vorob'evy Gory, Moscow 119992, Russia; loskutov@chaos.phys.msu.ru