A NOTE ON THE PROBLEM OF FINDING A (3,9)-CAGE

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ABSTRACT.   In this paper, we discuss the problem of finding a (3,9)-cage. A hamiltonian graph with girth 9 and 54 vertices is given. Except four vertices, each of the remaining vertices of this graph has valency three. This graph is obtained with the aid of a computer.

1. INTRODUCTION.

The problem of finding a smallest graph of girth 9 and valency 3 (a (3,9)-cage) is an old and difficult one. In 1952, R. M. Foster constructed the first trivalent graph with girth 9 and 60 vertices. Since then, at least 30 graphs of the same type have been obtained by various mathematicians. By using a computer, Biggs and Hoare [1] obtained a trivalent graph with girth 9 and 58 vertices. It is not known yet whether this graph is a (3,9)-cage.

Recently, with the aid of a computer, the author has completed the study of the (3,10)-cages [4], and he hopes to find a (3,9)-cage in this way. After considerable computation time was used in a computer, the author has obtained many partial solutions which are close to a (3,9)-cage. As an example, a hamiltonian graph with girth 9 and 54 vertices is given in this paper. Except four vertices, each of the remaining vertices of this graph has valency three. It is hoped that this graph may be useful in the search of a (3,9)-cage.

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2. A HAMILTONIAN GRAPH OF ORDER 54.

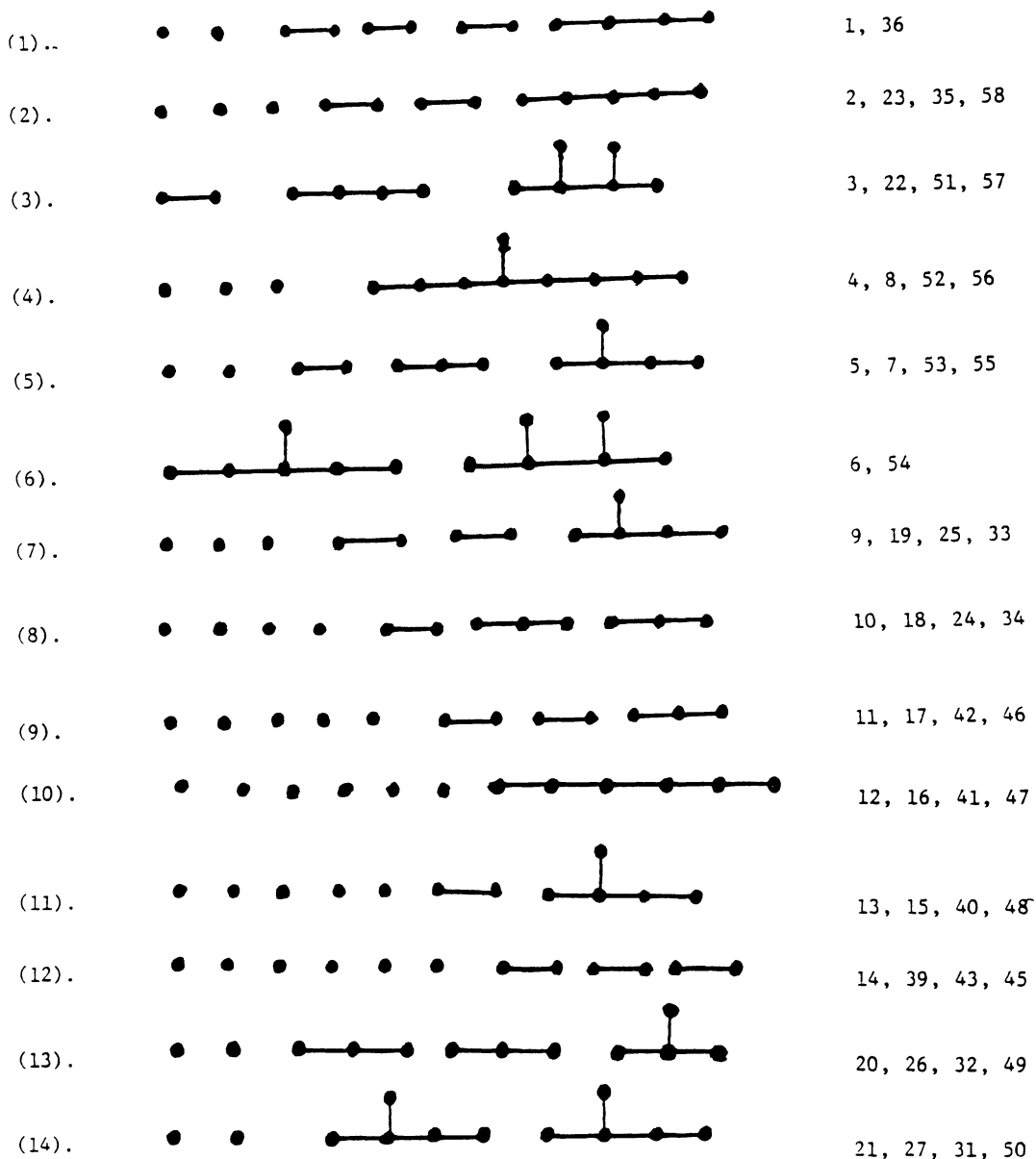Let f(3,9) be the number of vertices of a (3,9)-cage. It is known that 54 ≤ f(3,9) ≤ 58 (see [1]). The determination of the exact value of f(3,9) is of interest. By using a method similar to that in [4], the author hopes to construct a (3,9)-cage. He came close to obtaining one on many occasions. As an example, a hamiltonian graph with girth 9 and 54 veritces is given here.
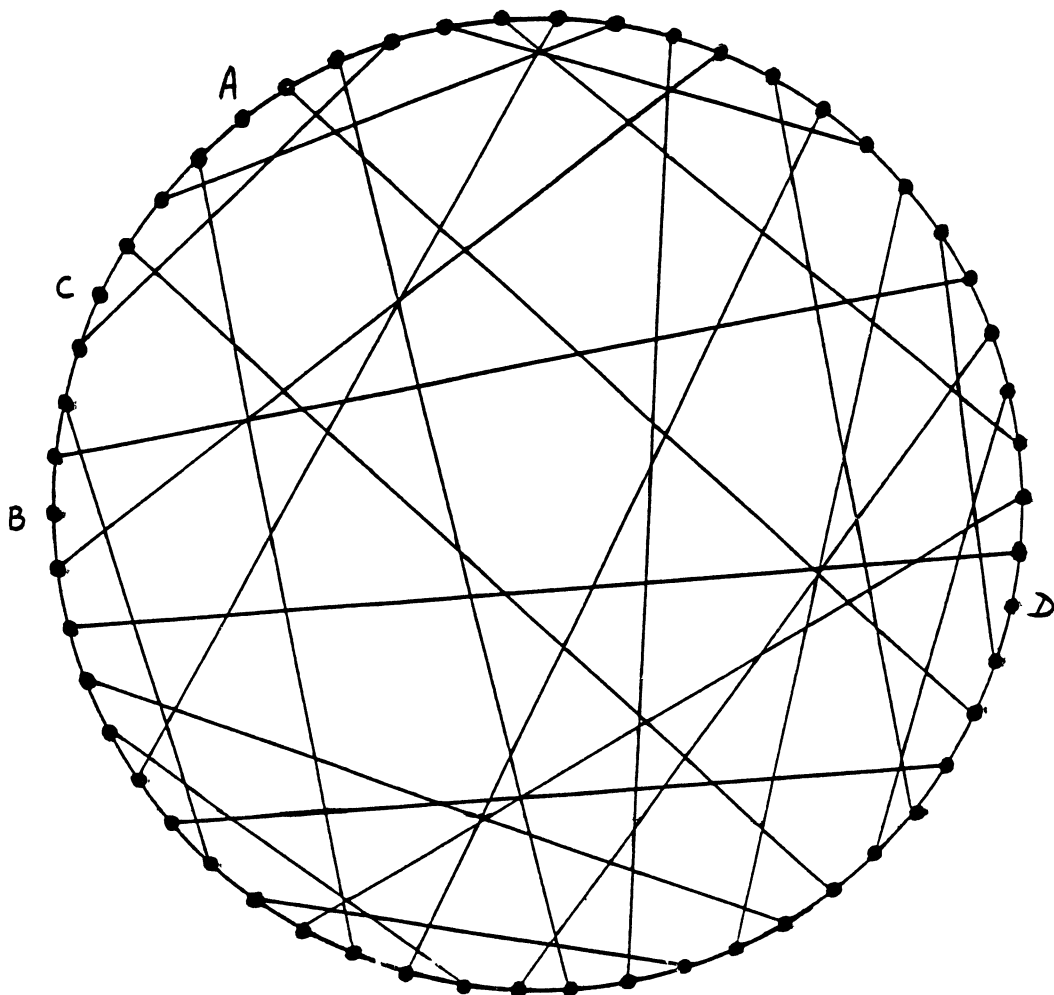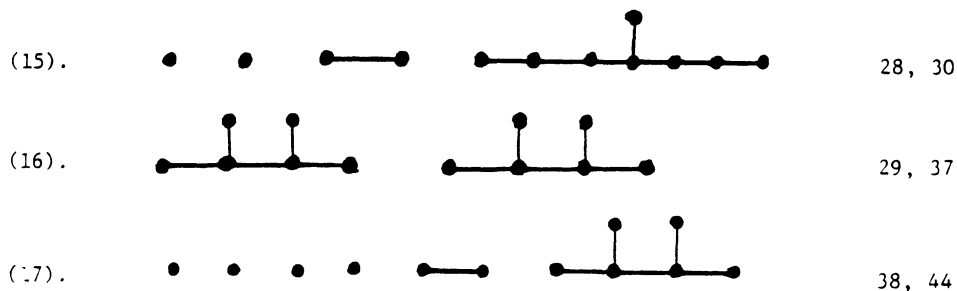

In the above graph, except vertices A, B, C and D, each of the remaining vertices of the graph has valency three. If A is adjacent to B and C is adjacent to D ,then this graph becomes a trivalent graph of girth 8.

In [1], Biggs and Hoare obtained a trivalent graph with girth 9 and 58

vertices. The author has tried to obtain a symmetrical design of this graph by
using the mathod suggested in [3]. For each vertex v of this graph, we draw a
tree T_v of order 46 in which all internal vertices have valency 3, containing
the vertex v in the middle. There are 12 vertices of the graph, which do not belong
to T_v . These 12 vertices are called the opposite vertices (with respect to
vertex v) (see [2] and [3]). These 12 opposite vertices form 17 different cases of
arrangement, which are given in the figure below. (A simple computer program can
be used to obtain this information.)

(1).⸳⸳   1, 36

(2).   2, 23, 35, 58

(3).   3, 22, 51, 57

(4).   4, 8, 52, 56

(5).   5, 7, 53, 55

(6).   6, 54

(7).   9, 19, 25, 33

(8).   10, 18, 24, 34

(9).   11, 17, 42, 46

(10).   12, 16, 41, 47

(11).   13, 15, 40, 48

(12).   14, 39, 43, 45

(13).   20, 26, 32, 49

(14).   21, 27, 31, 50

(15).   28, 30

(16).   29, 37

(17).   38, 44

According to the above figure, the vertices of the graph of Biggs and
Hoare (Figure 1 in [1]) can be roughly divided into 17 different sets (e.g., set
(1) consists of vertices 1 and 36, etc.). Therefore, this graph has at least 17
different types of vertices and so it is much less symmetrical than all known
cages. The author could not get a reasonably good design of this graph. This,
together with the graph given above seems to suggest that f(3,9) ≠ 58.

For the other known cages (except those obtained by using the properties
of finite projective geometry), a partial solution which contains about 80% of the
total number of edges of a cage can always be completed to yield a cage of that
type. For example, in order to shorten the computer program (used in [2]) to
obtain a (7,6)-cage, about 75% of the total number of edges of this cage was
completed by a computer. Only a few partial solutions were obtained, each of which
was checked by hand to determine whether it is a solution. The author assumed that
this was true for all other cages. However, the graph given here disproves this
conjecture. This also seems to confirm that the construction of a (3,9)-cage is
indeed a very difficult problem.

<div align="center">REFERENCES</div>

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