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ON PAIRWISE S-CLOSED BITOPOLOGICAL SPACES

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ABSTRACT. The concept of pairwise S-closedness in bitopological spaces has been introduced and some properties of such spaces have been studied in this paper.

KEY WORDS AND PHRASES. Pairwise semi-open, Pairwise almost compact, Pairwise S-closed, Pairwise regularly open and regularly closed, Pairwise extremally disconnectedness, Pairwise semi-continuous and irresolute functions.
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1. INTRODUCTION.

Travis Thompson [1] in 1976 initiated the notion of S-closed topological spaces, which was followed by its further study by Thompson [2], T. Noiri [3,4] and others. It is now the purpose of this paper to introduce and investigate the corresponding concept, i.e., pairwise S-closedness in bitopological spaces. To make the exposition of this paper self-contained as far as possible, we shall quote some definitions and enunciate some theorems from [5,6,7].

DEFINITION 1.1. [7] Let (X, τ_1, τ_2) be a bitopological space.

(i) A subset A of X is called τ_i semi-open with respect to τ_j (abbreviated as τ_i s.o.w.r.t. τ_j) in X if there exists a τ_i open set B such that $B \subset A \subset \bar{B}^{\tau_j}$ (where $\bar{B}^{\tau_j}$ denotes the τ_j-closure of B in X), where $i, j = 1,2$ and $i \neq j$.

A is called pairwise semi-open (written as p.s.o) in X if A is τ_1 s.o.w.r.t. τ_1 as well as τ_2 s.o.w.r.t. τ_1 in X.

(ii) A subset A of X is called τ_1 semi-closed with respect to τ_2 (denoted as τ_1 s.cl.w.r.t. τ_2) if X - A is τ_1 s.o.w.r.t. τ_2. Definitions for τ_2 s.cl.w.r.t. τ_1 and p. s.cl. sets can be given similarly as in (i).
(iii) A subset N of X is called a τ_i semi-neighborhood of x w.r.t. τ_j, where $x \in X$, if there is a τ_i s.o. set w.r.t. τ_j containing x and contained in N. A point x of X is said to be a τ_i semi-accumulation point of a subset A of X w.r.t. τ_j, if every τ_i semi-neighborhood of x w.r.t. τ_j intersects A in at least one point other than x, where i,j = 1,2 and $i \neq j$.
(iv) The intersection of all τ_i s.cl. sets w.r.t. τ_j, each containing a subset A of X, is called the τ_i semi-closure of A w.r.t. τ_j and will be denoted by $\underline{A}_{\tau_i(\tau_j)}$, where i,j = 1,2 and $i \neq j$.

It has been proved in [7] that a subset A of a bitopological space (X, τ_1, τ_2) is τ_i s.cl. w.r.t. τ_j if and only if $A = \underline{A}_{\tau_i(\tau_j)}$ and moreover, $x \in \underline{A}_{\tau_i(\tau_j)}$ if and only if x is either a point of A or a τ_i semi-accumulation point of A w.r.t. τ_j, where $i \neq j$ and i, j = 1,2.

In [7], it was deduced that $A \subset (X, \tau_1, \tau_2)$ is τ_1 s.o.w.r.t τ_2 iff $\overline{A}^{\tau_2} = \overline{(A^{i_1})}^{\tau_2}$ where A^{i_1} denotes the τ_1-interior of A in X. Similarly we shall use A^{i_2} to mean the τ_2-interior of A in X.

It is very easy to see that every τ_i open set in (X, τ_1, τ_2) is τ_i s.o.w.r.t. τ_j and the union of any collection of sets that are τ_i s.o.w.r.t. τ_j, is also so, where i, j = 1,2; $i \neq j$. It was shown in [5] that the intersection of two τ_1 s.o. sets w.r.t. τ_2 is not necessarily τ_1 s.o.w.r.t. τ_2. But we have,

THEOREM 1.2. [5] If A is τ_i s.o.w.r.t. τ_j in (X, τ_1, τ_2) and $B \in \tau_1 \cap \tau_2$, then $A \cap B$ is τ_i s.o.w.r.t. τ_j, where i, j = 1,2 and $i \neq j$.

The first part of the following theorem was proved in [7] and the converse part in [5].

THEOREM 1.3. Let $A \subset Y \subset (X, \tau_1, \tau_2)$. If A is τ_i s.o.w.r.t. τ_j, then A is $(\tau_i)_Y$ s.o.w.r.t. $(\tau_j)_Y$. Conversely, if A is $(\tau_i)_Y$ s.o.w.r.t. $(\tau_j)_Y$ and $Y \in \tau_i$, then A is τ_i s.o.w.r.t. τ_j, where i, j = 1,2 and $i \neq j$.

DEFINITION 1.4. [6] (a) A bitopological space (X, τ_1, τ_2) is said to be τ_i almost compact w.r.t. τ_j (i, j = 1,2; $i \neq j$) if every τ_i open filterbase has a τ_j cluster point. (X, τ_1, τ_2) is called pairwise almost compact if it is τ_1

almost compact w.r.t. τ_2 and τ_2 almost compact w.r.t. τ_1.

(b) A bitopological space $(X^*, \tau_1^*, \tau_2^*)$ is called an extension of a bitopological space (X, τ_1, τ_2) if $X \subset X^*$, $\overline{X}^{\tau_i^*} = X^*$ and $(\tau_i^*)_X = \tau_i$, for i = 1,2.

A pairwise Hausdorff bitopological space (X, τ_1, τ_2) is called pairwise H-closed if the space cannot have any pairwise Hausdorff extension.

THEOREM 1.5. [6] (a) (X, τ_1, τ_2) is pairwise almost compact if and only if for each cover $\{G_\alpha: \alpha \in I\}$ of X by τ_i open sets, there exists a finite subcollection $\{G_{\alpha_1}, \ldots.., G_{\alpha_n}\}$ such that $X = \bigcup_{k=1}^{n} \overline{G_{\alpha_k}}^{\tau_j}$, where i, j = 1,2 and $i \neq j$.

(b) If (X, τ_1, τ_2) is τ_i regular w.r.t. τ_j and τ_i almost compact w.r.t. τ_j, then (X, τ_i) is compact, for i, j = 1,2 and $i \neq j$.

(c) A pairwise Hausdorff and pairwise almost compact bitopological space is pairwise H-closed.

In what follows, by (X, τ_1, τ_2) we shall always mean a bitopological space, i.e., a set X endowed with two topologies τ_1 and τ_2.

2. PAIRWISE S-CLOSED SPACES.

DEFINITION 2.1. Let $F = \{F_\alpha\}$ be a filterbase in (X, τ_1, τ_2) and $x \in X$. F is said to

(i) τ_i S-accumulate to x w.r.t. τ_j if for every τ_i s.o. set V w.r.t. τ_j containing x and each $F_\alpha \in F$, $F_\alpha \cap \overline{V}^{\tau_j} \neq \phi$.

(ii) τ_i S-converge w.r.t. τ_j to x, if corresponding to each τ_i s.o.set V w.r.t. τ_j containing x, there exists $F_\alpha \in F$ such that $F_\alpha \subset \overline{V}^{\tau_j}$.

In (i) and (ii) above, $i \neq j$ and i,j = 1,2. F is said to pairwise S-converge to x if F is τ_1 S-convergent to x w.r.t. τ_2 as well as τ_2 S-convergent to x w.r.t. τ_1. The definition of pairwise S-accumulation point of F is similar.

DEFINITION 2.2. (X, τ_1, τ_2) is called τ_1 S-closed w.r.t. τ_2 if for each cover $\{V_\alpha: \alpha \in I\}$ of X with τ_1 s.o. sets w.r.t. τ_2, there is a finite subfamily $\{V_{\alpha_i}: i = 1,2, \ldots.., n\}$ such that $\bigcup_{i=1}^{n} \overline{V}_{\alpha_i}^{\tau_2} = X$ (where I is some index set). X is called pairwise S-closed if it is τ_1 S-closed w.r.t. τ_2 and τ_2 S-closed w.r.t. τ_1.

THEOREM 2.3. Let F be an ultrafilter in X. Then F τ_1 S-accumulates to a point

$x_0 \in X$ w.r.t. τ_2 if and only if F is τ_1 S-convergent to x_0 w.r.t. τ_2.

PROOF: Let F be τ_1 S-convergent w.r.t. τ_2 to x_0 and let it not τ_1 S-accumulate w.r.t. τ_2 to x_0. Then there exist a τ_1 s.o. set V w.r.t. τ_2 (containing x_0) and some $F_\alpha \in F$ such that $F_\alpha \cap \overline{V}^{\tau_2} = \phi$. Then $F_\alpha \subset X - \overline{V}^{\tau_2}$ and hence

$$X - \overline{V}^{\tau_2} \in F \ldots\ldots (2.1) .$$

Since F is τ_1 S-convergent w.r.t. τ_2 to x_0, corresponding to V there exists $F_\beta \in F$ such that $F_\beta \subset \overline{V}^{\tau_2}$. Then $\overline{V}^{\tau_2} \in F$ ...... (2.2). Clearly (2.1) and (2.2) are incompatible. Note that for this part we do not need maximality of F.

Conversely, if F does not τ_1 S-converge w.r.t. τ_2 to x_0, there exists a τ_1 s.o. set V w.r.t. τ_2 containing x_0, such that $F_\alpha \not\subset \overline{V}^{\tau_2}$, for each $F_\alpha \in F$. But F has x_0 as a τ_1 S-accumulation point w.r.t. τ_2. Hence $F_\alpha \cap \overline{V}^{\tau_2} \neq \emptyset$, for each $F_\alpha \in F$. Thus $F_\alpha \cap \overline{V}^{\tau_2} \neq \emptyset$ and $F_\alpha \cap (X - \overline{V}^{\tau_2}) \neq \emptyset$, for each $F_\alpha \in F$. Since F is maximal, this shows that $\overline{V}^{\tau_2}$ and $X - \overline{V}^{\tau_2}$ both belong to F, which is a contradiction.

NOTE 2.4. In the above theorem, the indices 1 and 2 could be interchanged.

THEOREM 2.5. In a bitopological space (X, τ_1, τ_2) the following are equivalent:

(a) X is τ_1 S-closed w.r.t. τ_2.

(b) Every ultrafilterbase F is τ_1 S-convergent w.r.t. τ_2.

(c) Every filterbase τ_1 S-accumulates w.r.t. τ_2 to some point of X.

(d) For every family $\{F_\alpha\}$ of τ_1 s.cl. sets w.r.t. τ_2, with $\bigcap F_\alpha = \emptyset$, there exists a finite subcollection $\{F_{\alpha_i}\}_{i=1}^{n}$ of $\{F_\alpha\}$ such that $\bigcap_{i=1}^{n} (F_{\alpha_i})^{i_2} = \emptyset$.

PROOF: (a) => (b) Let $F = \{F_\alpha\}$ be an ultrafilterbase in X, which does not τ_1 S-converge w.r.t. τ_2 to any point of X. Then by Theorem 2.3, F has no τ_1 S-accumulation point w.r.t. τ_2. Thus for every $x \in X$, there is a τ_1 s.o. set $V(x)$ w.r.t. τ_2 containing x and an $F_{\alpha(x)} \in F$ such that $F_{\alpha(x)} \cap \overline{V(x)}^{\tau_2} = \emptyset$. Evidently, $\{V(x): x \in X\}$ is a cover of X with sets that are τ_1 s.o.w.r.t. τ_2 and by (a), there exists a finite subcollection $\{V(x_i): i = 1,2,\ldots,n\}$ of $\{V(x): x \in X\}$ such that $\bigcup_{i=1}^{n} \overline{V(x_i)}^{\tau_2} = X$.

Now, F being a filterbase, there exists $F_0 \in F$ such that

$$F_0 \subset \bigcap_{i=1}^{n} F_{\alpha(x_i)} .$$

Then $F_0 \cap \overline{V(x_i)}^{\tau_2} = \emptyset$ for i = 1,2 ....., n.

$\Rightarrow F_0 \cap (\bigcup_{i=1}^{n} \overline{V(x_i)}^{\tau_2}) = F_0 \cap X = \emptyset \Rightarrow F_0 = \emptyset$ which is a contradiction.

(b) => (c) Every filterbase F is contained in an ultrafilter base F* and F* is τ_1 S-convergent w.r.t. τ_2 to some point x_0 by (b), and hence x_0 is a τ_1 S-accumulation point of F* w.r.t. τ_2. Since $F \subset F^*$, x_0 is also a τ_1 S-accumulation point of F w.r.t. τ_2.

(c) => (d) Let $F = \{F_\alpha\}$ be a family of τ_1 s.cl. sets w.r.t. τ_2 with $\cap F_\alpha = \emptyset$ and be such that for every finite subfamily $\{F_{\alpha_i}\}_{i=1}^{n}$ (say), $\bigcap_{i=1}^{n} (F_{\alpha_i})^{i_2} \neq \emptyset$. Thus $\mathcal{F} = \{\bigcap_{i=1}^{n} (F_{\alpha_i})^{i_2}$: n = positive integer, $F_{\alpha_i} \in F\}$ forms a filterbase in X and hence by hypothesis has a τ_1 S-accumulation point x_0 w.r.t. τ_2. Then for any τ_1 s.o. set $V(x_0)$ w.r.t. τ_2 containing x_0, $(F_\alpha)^{i_2} \cap \overline{V(x_0)}^{\tau_2} \neq \emptyset$, for each $F_\alpha \in F$. Since $\cap F_\alpha = \emptyset$, there is some $F_{\alpha_0} \in F$ such that $x_0 \notin F_{\alpha_0}$. Hence $x_0 \in X - F_{\alpha_0}$ which is τ_1 s.o.w.r.t. τ_2. Hence $(F_{\alpha_0})^{i_2} \cap \overline{(X - F_{\alpha_0})}^{\tau_2} \neq \emptyset$ or, $(F_{\alpha_0})^{i_2} \cap (X - (F_{\alpha_0})^{i_2}) \neq \emptyset$ which is impossible.

(d) => (a) Let $\{V_\alpha\}$ be a covering of X with sets that are τ_1 s.o.w.r.t. τ_2. Then $\cap (X - V_\alpha) = X - \cup V_\alpha = \emptyset$. By (d), there exists finite number of indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\bigcap_{k=1}^{n} (X - V_{\alpha_k})^{i_2} = \emptyset$, i.e., $\bigcap_{k=1}^{n} (X - \overline{V_{\alpha_k}}^{\tau_2}) = \emptyset$, or $X - \bigcup_{k=1}^{n} \overline{V_{\alpha_k}}^{\tau_2} = \emptyset$, or $\bigcup_{k=1}^{n} \overline{V_{\alpha_k}}^{\tau_2} = X$ and hence X is τ_1 S-closed w.r.t. τ_2.

NOTE 2.6. Obviously, in the above theorem, the indices 1 and 2 could have been interchanged and hence the statement (a) can be replaced by "X is pairwise S-closed" with corresponding alterations in (b), (c) and (d).

DEFINITION 2.7. A subset Y of (X, τ_1, τ_2) will be called τ_i S-closed w.r.t. τ_j in X if and only if for every cover $\{V_\alpha : \alpha \in I\}$ of Y by τ_i s.o. sets w.r.t. τ_j of X, there exists a finite set of indices $\alpha_1, \alpha_2, \ldots, \alpha_n \in I$ such that

$$Y \subset \bigcup_{k=1}^{n} \{\overline{V_{\alpha k}}^{\tau_j}\}, \text{ where } i, j = 1,2 \text{ and } i \neq j.$$

THEOREM 2.8. A subset Y of (X, τ_1, τ_2) will be $(\tau_i)_Y$ S-closed w.r.t. $(\tau_j)_Y$ if Y is τ_i S-closed w.r.t. τ_j in X and $Y \in \tau_i$, where $i, j = 1,2$ and $i \neq j$.

PROOF: We prove the theorem by taking $i = 1$ and $j = 2$. Similar will be the proof when $i = 2$ and $j = 1$. By virtue of Theorem 1.3, every cover $\{V_\alpha : \alpha \in I\}$ of Y by sets that are $(\tau_1)_Y$ s.o.w.r.t. $(\tau_2)_Y$ can be regarded as a cover of Y by sets that are τ_1 s.o.w.r.t. τ_2. Then by hypothesis, there is a finite number of indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$Y \subset \bigcup_{k=1}^{n} \bar{V}_{\alpha_k}^{\tau_2} \Rightarrow Y = \bigcup_{k=1}^{n} \overline{V_{\alpha_k}}^{(\tau_2)_Y}$$ and the theorem follows.

THEOREM 2.9. If $Y\ (\subset (X, \tau_1, \tau_2))$ is $(\tau_i)_Y$ S-closed w.r.t. $(\tau_j)_Y$ and $Y \in \tau_1 \cap \tau_2$, then Y is τ_i S-closed w.r.t. τ_j in X, for $i, j = 1,2$ and $i \neq j$.

PROOF: We prove only the case when $i = 1$ and $j = 2$. Let $\{G_\alpha\}$ be a cover of Y, where each G_α is τ_1 s.o.w.r.t. τ_2. Then by Theorem 1.2, $G_\alpha \cap Y$ is τ_1 s.o.w.r.t. τ_2 for each α and hence by Theorem 1.3, $G_\alpha \cap Y$ is $(\tau_1)_Y$ s.o.w.r.t. $(\tau_2)_Y$ for each α. By hypothesis, there exists a finite number of indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$Y = \bigcup_{k=1}^{n} \overline{(G_{\alpha_k} \cap Y)}^{(\tau_2)_Y} \Rightarrow Y \subset \bigcup_{k=1}^{n} \overline{G_{\alpha_k}}^{\tau_2} \Rightarrow Y$$ is τ_1 S-closed w.r.t. τ_2 in X.

DEFINITION 2.10. [7] A subset A in (X, τ_1, τ_2) is called τ_1 regularly open (closed) w.r.t. τ_2 if and only if $A = (\bar{A}^{\tau_2})^{i_1}$ (respectively if and only if $A = \overline{(A^{i_2})}^{\tau_1}$). Similarly we define sets that are τ_2 regularly open (closed) w.r.t. τ_1.

It has been shown in [7] that a subset B of (X, τ_1, τ_2) is τ_i regularly closed w.r.t. τ_j iff $(X - B)$ is τ_i regularly open w.r.t. τ_j, for $i, j = 1,2$ and $i \neq j$.

LEMMA 2.11. If a subset A of a bitopological space (X, τ_1, τ_2) is τ_j regularly closed w.r.t. τ_i, then A is τ_i s.o.w.r.t. τ_j, where $i, j = 1,2$ and $i \neq j$.

PROOF: Proof is done only in the case when $i = 1$ and $j = 2$.

A is τ_2 regularly closed w.r.t. $\tau_1 \Rightarrow (X - A)$ is τ_2 regularly open w.r.t. τ_1

$$\Rightarrow X - A = \left[\overline{(X - A)}^{\tau_1}\right]^{i_2} \tag{2.3}$$

Let $0 = X - \overline{(X - A)}^{\tau_1}$. Then 0 is τ_1 open and

$$\overline{0}^{\tau_2} = \overline{\left[X - \overline{(X - A)}^{\tau_1}\right]}^{\tau_2} = X - \left[X - \overline{(X - A)}^{\tau_1}\right]^{i_2} = A \text{ (by (2.3))}.$$

Thus $0 \subset A \subset \overline{0}^{\tau_2}$ and $0 \in \tau_1$. Hence A is τ_1 s.o.w.r.t. τ_2.

LEMMA 2.12. If a subset A of (X, τ_1, τ_2) is τ_i s.o.w.r.t. τ_j then $\overline{A}^{\tau_j}$ is τ_j regularly closed w.r.t. τ_i, where $i \neq j$ and i, j = 1,2.

PROOF: As before we consider the case i = 1 and j = 2. Since A is τ_1 s.o.w.r.t. τ_2, we have $A^{i_1} \subset A \subset \overline{A^{i_1}}^{\tau_2}$. Then $\overline{A}^{\tau_2} = \overline{(A_2^{i_1})}^{\tau_2}$ ..... . (2.4)

It has been shown in [7] that a set A in (X, τ_1, τ_2) is τ_i regularly closed w.r.t τ_j (i, j = 1,2; $i \neq j$) if it is τ_i closure of some τ_j open set. Since A^{i_1} is τ_1 open, by virtue of (2.4) the result follows.

THEOREM 2.13. A bitopological space (X, τ_1, τ_2) is τ_i S-closed w.r.t. τ_j if and only if every proper τ_j regularly open set w.r.t. τ_i of X is τ_i S-closed w.r.t. τ_j, for i, j = 1,2 and $i \neq j$.

PROOF: We only take up the case i = 1 and j = 2.

Let X be τ_1 S-closed w.r.t. τ_2 and F be a proper τ_2 regularly open set of X w.r.t τ_1. Let $\{V_\alpha : \alpha \in I\}$ be a cover of F by sets that are τ_1 s.o.w.r.t. τ_2. Since X - F is τ_2 regularly closed w.r.t. τ_1, by Lemma 2.11, (X - F) is τ_1 s.o.w.r.t. τ_2 and hence $(X - F) \cup \{V_\alpha : \alpha \in I\}$ is a cover of X by τ_1 s.o. sets w.r.t. τ_2. Since X is τ_1 S-closed w.r.t. τ_2, there exists a finite-number of indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $X = \overline{(X - F)}^{\tau_2} \cup [\bigcup_{k=1}^{n} (\overline{V}_{\alpha_k}^{\tau_2})]$.

Since F is τ_2 open, $F \cap \overline{X - F}^{\tau_2} = \emptyset$ and hence $F \subset \bigcup_{k=1}^{n}(\overline{V}_{\alpha_k}^{\tau_2})$, proving that F is τ_1 S-closed w.r.t. τ_2. Conversely, let $\{V_\alpha : \alpha \in I\}$ be a cover of X by sets that are τ_1 s.o.w.r.t. τ_2. If $X = \overline{V}_\alpha^{\tau_2}$, for each $\alpha \in I$, then the theorem is proved. So, suppose $X \neq \overline{V}_\beta^{\tau_2}$, for some $\beta \in I$ and $V_\beta \neq \emptyset$. Then $\overline{V}_\beta^{\tau_2}$ is a proper subset of X. Since V_β is τ_1 s.o.w.r.t. τ_2, by Lemma 2.12, $\overline{V}_\beta^{\tau_2}$ is τ_2 regularly closed w.r.t. τ_1, so that $X - \overline{V}_\beta^{\tau_2}$ is proper τ_2 regularly open w.r.t. τ_1 and by hypothesis, it is τ_1 S-closed w.r.t. τ_2. Then there exists a finite set of indices $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that $X - \bar{V}_\beta^{\tau_2} \subset \bigcup_{k=1}^{m} \bar{V}_{\alpha_k}^{\tau_2}$. Hence $X = \bar{V}_\beta^{\tau_2} \cup (\bigcup_{k=1}^{m} \bar{V}_{\alpha_k}^{\tau_2})$ and X is τ_1 S-closed w.r.t. τ_2.

THEOREM 2.14. A subset A in (X, τ_1, τ_2) is τ_i S-closed w.r.t. τ_j in X if and only if every cover of A by sets that are τ_j regularly closed w.r.t. τ_i in X, has a finite subcover, where $i, j = 1,2$ and $i \neq j$.

PROOF: We consider only the case $i = 1$ and $j = 2$. Let A be τ_1 S-closed w.r.t. τ_2 in X and $\{V_\alpha\}$ be a collection of τ_2 regularly closed sets in X w.r.t. τ_1, which is a cover of A. Then each V_α is τ_1 s.o.w.r.t. τ_2, by Lemma 2.11 and hence there exists a finite set of indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $A \subset \bar{V}_{\alpha_1}^{\tau_2} \cup \ldots\ldots\ldots \cup \bar{V}_{\alpha_n}^{\tau_2} = V_{\alpha_1} \cup \ldots\ldots \cup V_{\alpha_n}$ (since each V_{α_i} is τ_2 closed). Conversely, let the given condition hold and $\{V_\alpha\}$ be a τ_1 s.o. cover of A w.r.t. τ_2. Then $\bar{V}_\alpha^{\tau_2}$ is τ_2 regularly closed w.r.t. τ_1 for each α, by Lemma 2.12, and $\{\bar{V}_\alpha^{\tau_2}\}$ is a cover of A. Then by hypothesis, there exist a finite number of indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $A \subset \bigcup_{k=1}^{n} \bar{V}_{\alpha_k}^{\tau_2}$, showing that A is τ_1 S-closed w.r.t. τ_2.

THEOREM 2.15. If A and B are τ_i S-closed w.r.t. τ_j in (X, τ_1, τ_2), then $A \cup B$ is also so, where $i, j = 1,2$ and $i \neq j$.

PROOF: Let $\{V_\alpha\}$ be a cover of $A \cup B$ by sets that are τ_i s.o.w.r.t. τ_j in X. Then it is a cover of A as well as of B. By hypothesis, there will exist a finite number of indices $\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1k}$ and $\alpha_{21}, \alpha_{22}, \ldots, \alpha_{2r}$ such that $A \subset \bigcup_{k=1}^{k} \bar{V}_{\alpha_{1k}}^{\tau_j}$ and $B \subset \bigcup_{k=1}^{r} \bar{V}_{\alpha_{2k}}^{\tau_j}$. Then $A \cup B \subset (\bigcup_{k=1}^{k} \bar{V}_{\alpha_{1k}}^{\tau_j}) \cup (\bigcup_{k=1}^{r} \bar{V}_{\alpha_{2k}}^{\tau_j})$ and hence $A \cup B$ is τ_i S-closed w.r.t. τ_j.

THEOREM 2.16. If A is τ_1 S-closed w.r.t. τ_2 in (X, τ_1, τ_2) then $\bar{A}^{\tau_2}$ is also so.

PROOF: Let $\{V_\alpha\}$ be a cover of $\bar{A}^{\tau_2}$ by sets that are τ_1 s.o.w.r.t. τ_2, then it is also a cover of A. Thus there exists a finite number of indices $\alpha_1, \ldots, \alpha_n$ such that $A \subset \bigcup_{i=1}^{n} \bar{V}_{\alpha_i}^{\tau_2} \Rightarrow \bar{A}^{\tau_2} \subset \bigcup_{i=1}^{n} \bar{V}_{\alpha_i}^{\tau_2}$ and the result follows. From

Theorem 2.9 and Theorem 2.16 we get:

COROLLARY 2.17. If A (X, τ_1, τ_2) is pairwise open and $(A, (\tau_1)_A, (\tau_2)_A)$ is pairwise S-closed, then $\bar{A}^{\tau_i}$ is pairwise S-closed in X, for i = 1,2.

COROLLARY 2.18. A space (X, τ_1, τ_2) is τ_i S-closed w.r.t. τ_j if there exists a τ_i S-closed subset A w.r.t. τ_j in X, which is τ_j dense in X, where i, j = 1,2 and i ≠ j.

THEOREM 2.19. Let $A \subset (X, \tau_1, \tau_2)$ be τ_1 S-closed w.r.t. τ_2 and B is τ_2 regularly open w.r.t. τ_1 in X. Then $A \cap B$ is τ_1 S-closed w.r.t. τ_2.

PROOF: Let $\{V_\alpha: \alpha \in I\}$ be a τ_1 s.o. cover of $A \cap B$ w.r.t. τ_2, where I is some index set. Since X-B is τ_2 regularly closed w.r.t. τ_1, by Lemma 2.11, (X-B) is τ_1 s.o.w.r.t. τ_2. Thus $A \subset \bigcup_{\alpha \in I} \{V_\alpha\} \cup (X-B)$ and A is τ_1 S-closed w.r.t. τ_2.

Then there exist indices $\alpha_1, \alpha_2, \ldots, \alpha_n$,, finite in number, such that

$$A \subset \bigcup_{i=1}^{n} \bar{V}_{\alpha_i}^{\tau_2} \cup \overline{(X - B)}^{\tau_2} = \bigcup_{i=1}^{n} \bar{V}_{\alpha_i}^{\tau_2} \cup (X - B).$$

Thus $A \cap B \subset \bigcup_{i=1}^{n} \bar{V}_{\alpha_i}^{\tau_2}$ and $A \cap B$ is τ_1 S-closed w.r.t. τ_2.

COROLLARY 2.20. Let $A \subset (X, \tau_1, \tau_2)$ be τ_1 S-closed w.r.t. τ_2 and B is τ_2 regularly open w.r.t. τ_1, then

(a) B is τ_1 S-closed w.r.t. τ_2 if $B \subset A$.

(b) A^{i_2} is τ_1 S-closed w.r.t. τ_2 if A is τ_1 closed in X.

PROOF: (a) Follows immediately from Theorem 2.19.

(b) Since $(\bar{A}^{\tau_1})^{i_2}$ is τ_2 regularly open w.r.t. τ_1 and $(\bar{A}^{\tau_1})^{i_2} \cap A = A^{i_2} \cap A$ $= A^{i_2}$, the result follows by virtue of Theorem 2.19.

THEOREM 2.21. If (X, τ_1, τ_2) is τ_i regular w.r.t. τ_j and τ_i S-closed w.r.t. τ_j, then (X, τ_i) is compact, where i, j = 1,2; i ≠ j.

<u>Proof</u> By virtue of Theorem 1.5(a), we see that every τ_i S-closed space w.r.t. τ_j is τ_i almost compact w.r.t. τ_j. Hence by Theorem 1.5(b) the result follows.

In Theorem 3.7 we shall prove a partial converse of the above theorem.

3. PAIRWISE EXTREMALLY DISCONNECTEDNESS AND S-CLOSED SPACE.

DEFINITION 3.1. A bitopological space (X, τ_1, τ_2) is said to be τ_i extremally disconnected w.r.t. τ_j if and only if for every τ_i open set A of X,

$\overline{A}^{\tau_j}$ is τ_i open, where i, j = 1,2 and $i \neq j$. X is called pairwise extremally disconnected if and only if it is τ_1 extremally disconnected w.r.t. τ_2 and τ_2 extremally disconnected w.r.t. τ_1.

Datta in [8] has defined pairwise extremally disconnected bitopological space identically as above, we shall show (see Corollary 3.4) that the concept can be defined by a weaker condition.

The conclusion of the following theorem was also derived in [8] under the hypothesis that the space is pairwise Hausdorff and pairwise extremally disconnected. We prove a much stronger result here.

THEOREM 3.2. Let (X, τ_1, τ_2) be τ_1 extremally disconnected w.r.t. τ_2 or τ_2 extremally disconnected w.r.t. τ_1. Then for every pair of disjoint sets A, B in X, where $A \in \tau_1$ and $B \in \tau_2$, one has $\overline{A}^{\tau_2} \cap \overline{B}^{\tau_1} = \emptyset$.

PROOF: Suppose (X, τ_1, τ_2) is τ_1 extremally disconnected w.r.t. τ_2 and $A \in$ τ_1,

$B \in \tau_2$ with $A \cap B = \emptyset$. Then $\overline{A}^{\tau_2} \cap B = \emptyset$ ... (1). Now, if $\overline{A}^{\tau_2} \cap \overline{B}^{\tau_1} \neq \emptyset$, then there exists $x \in \overline{B}^{\tau_1}$ and $x \in \overline{A}^{\tau_2} \in \tau_1$. Hence $\overline{A}^{\tau_2} \cap B \neq \emptyset$ contradicting (1). Similarly the other case can be handled.

We prove a stronger converse of the above theorem.

THEOREM 3.3. (X, τ_1, τ_2) is pairwise extremally disconnected if for every pair of disjoint sets A and B, where $A \in \tau_1$ and $B \in \tau_2$, $\overline{A}^{\tau_2} \cap \overline{B}^{\tau_1} = \emptyset$ holds.

PROOF: Suppose (X, τ_1, τ_2) is not τ_1 extremally disconnected w.r.t. τ_2. Then there is a τ_1 open set A such that $\overline{A}^{\tau_2}$ τ_1. Then $X - \overline{A}^{\tau_2} \in \tau_2$ and $A \in \tau_1$ such that $A \cap (X - \overline{A}^{\tau_2}) = \emptyset$. Hence by hypothesis, $\overline{A}^{\tau_2} \cap \overline{(X - \overline{A}^{\tau_2})}^{\tau_1} = \emptyset$. Then $\overline{(X - \overline{A}^{\tau_2})}^{\tau_1} = X - \overline{A}^{\tau_2}$ and $X - \overline{A}^{\tau_2}$ is τ_1 closed. Thus $\overline{A}^{\tau_2}$ is τ_1 -open. A contradiction.

Similarly, (X, τ_1, τ_2) is τ_2 extremally disconnected w.r.t. τ_1.

From Theorems 3.2 and 3.3 we have,

COROLLARY 3.4. (X, τ_1, τ_2) is pairwise extremally disconnected if and only if it is either τ_1 extremally disconnected w.r.t. τ_2 or τ_2 extremally disconnected w.r.t. τ_1.

LEMMA 3.5. If (X, τ_1, τ_2) is pairwise extremally disconnected, then for every τ_1

s.o. set V w.r.t. τ_2, $\underline{V}_{\tau_2(\tau_1)} = \overline{V}^{\tau_2}$ and for every τ_2 s.o. set U w.r.t τ_1, $\underline{U}_{\tau_1(\tau_2)} = \overline{U}^{\tau_1}$.

PROOF: Obviously, $\underline{V}_{\tau_2(\tau_1)} \subset \overline{V}^{\tau_2}$.

Now, if $x \notin \underline{V}_{\tau_2}(\tau_1)$, then there exists a τ_2 s.o. set W w.r.t τ_1, containing x such that $V \cap W = \emptyset$. Then V^{i_1} and W^{i_2} are nonempty disjoint sets, respectively τ_1 open and τ_2 open. Since (X, τ_1, τ_2) is pairwise extremally disconnected, we have

$\overline{V^{i_1}}^{\tau_2} \cap \overline{W^{i_2}}^{\tau_1} = \emptyset$, i.e., $\overline{V}^{\tau_2} \cap \overline{W}^{\tau_1} = \emptyset$. Thus $x \notin \overline{V}^{\tau_2}$. Hence $\underline{V}_{\tau_2(\tau_1)} = \overline{V}^{\tau_2}$.
Similarly the other part can be proved.

LEMMA 3.6. In a pairwise extremally disconnected space (X, τ_1, τ_2), every τ_i regularly open set w.r.t. τ_j is τ_i open and τ_j closed, where $i, j = 1,2$ and $i \neq j$.

PROOF: Let A be a τ_1 regularly open set in X w.r.t. τ_2, so that $(\overline{A}^{\tau_2})^{i_1} = A$. Now, $(X - \overline{A}^{\tau_2})$ and A are disjoint sets, respectively τ_2 open and τ_1 open. Since (X, τ_1, τ_2) is pairwise extremally disconnected, we have

$\overline{(X - \overline{A}^{\tau_2})}^{\tau_1} \cap \overline{A}^{\tau_2} = \emptyset$, by Theorem 3.2. Then $\overline{(X - \overline{A}^{\tau_2})}^{\tau_1} = X - \overline{A}^{\tau_2}$ and $X - \overline{A}^{\tau_2}$ is τ_1 -closed. Hence $\overline{A}^{\tau_2}$ is τ_1-open, so that $\overline{A}^{\tau_2} = (\overline{A}^{\tau_2})^{i_1} = A$ is τ_1 open and τ_2 -closed.

Similarly, we can show that every τ_2 regularly open set in X w.r.t. τ_1 is τ_2 -open and τ_1 -closed.

THEOREM 3.7. If (X, τ_1, τ_2) is pairwise extremally disconnected and (X, τ_1) is compact, then (X, τ_1, τ_2) is τ_1 S-closed w.r.t. τ_2.

PROOF: Let $\{V_\alpha : \alpha \in I\}$ be a cover of X by sets that are τ_1 s.o.w.r.t. τ_2. For each $x \in X$, there is a V_{α_x} containing x, for some $\alpha_x \in I$. Then there exists a τ_1 open set 0_{α_x} such that $0_{\alpha_x} \subseteq V_{\alpha_x} \subset \overline{0}_{\alpha_x}^{\tau_2}$. Since X is pairwise extremally disconnected, $\overline{0}_{\alpha_x}^{\tau_2}$ is τ_1 open for each $x \in X$. By compactness of (X, τ_1) there exists a finite set of points $x_1, x_2, \ldots, x_n$ of X such that $X = \bigcup_{k=1}^{n} \{\overline{0}_{\alpha_{x_k}}^{\tau_2}\}$. But $0_{\alpha_x} \subset V_{\alpha_x}$, for each x. Hence $\overline{0}_{\alpha_x}^{\tau_2} \subset \overline{V}_{\alpha_x}^{\tau_2}$.

Hence $X = \bigcup_{k=1}^{n} \{\overline{V}_{\alpha_{x_k}}^{\tau_2}\}$ and X is τ_1 S-closed w.r.t. τ_2.

We have earlier observed that every τ_i S-closed space (X, τ_1, τ_2) w.r.t. τ_j is always τ_i almost compact w.r.t. τ_j for i, j = 1,2 and $i \neq j$. Now we have:

THEOREM 3.8. If (X, τ_1, τ_2) is τ_1 almost compact w.r.t. τ_2 and pairwise extremally disconnected, then (X, τ_1, τ_2) is τ_1 S-closed w.r.t. τ_2.

PROOF: Let us consider a cover $\{V_\alpha: \alpha \in I\}$ of X with sets that are τ_1 s.o.w.r.t. τ_2. For each $\alpha \in I$, we consider the set $U_\alpha = (\overline{V}_\alpha^{\tau_2})^{i_1}$ which is τ_1 regularly open w.r.t τ_2. Then $U_\alpha \subset U_\alpha \cup V_\alpha \subset \overline{V}_\alpha^{\tau_2} = \overline{[(\overline{V}_\alpha^{\tau_2})^{i_1}]}^{\tau_2} = \overline{U}_\alpha^{\tau_2}$. Since U_α is τ_1 regularly open w.r.t. τ_2, by Lemma 3.6, U_α is τ_2 -closed and hence, $U_\alpha \subset U_\alpha \cup V_\alpha \subset \overline{U}_\alpha^{\tau_2} = U_\alpha$. Thus $U_\alpha = U_\alpha \cup V_\alpha$. Again, U_α being τ_1 -open, for each $\alpha \in I$, it follows that $\{U_\alpha \cup V_\alpha: \alpha \in I\}$ is a τ_1 -open cover of (X, τ_1, τ_2). (X, τ_1, τ_2) being τ_1 almost compact w.r.t. τ_2, there exists a finite subfamily

I_0 of I such that $X = \bigcup_{\alpha \in I_0} \{\overline{U_\alpha \cup V_\alpha}^{\tau_2}\}$.. Now, since $U_\alpha \cup V_\alpha \subset \overline{V}_\alpha^{\tau_2}$, for each $\alpha \in I$, we have $\overline{U_\alpha \cup V_\alpha}^{\tau_2} \subset \overline{V}_\alpha^{\tau_2}$ for each α and hence $X = \bigcup_{\alpha \in I_0} \{\overline{V}_\alpha^{\tau_2}\}$. Hence (X, τ_1, τ_2) is τ_1 S-closed w.r.t. τ_2.

4. SEMI CONTINUITY, IRRESOLUTE FUNCTIONS AND S-CLOSEDNESS.

DEFINITION 4.1. [7] A function f from a bitopological space (X, τ_1, τ_2) into a bitopological space (Y, σ_1, σ_2) is called $\tau_1 \sigma_1$ semi-continuous w.r.t. τ_2 if for each $A \in \sigma_1$, $f^{-1}(A)$is τ_1 s.o.w.r.t. τ_2. Similar goes the definition of $\tau_2 \sigma_2$ semi-continuity of f w.r.t. τ_1. f is called pairwise semi-continuous if f is $\tau_1 \sigma_1$ semi-continuous w.r.t. τ_2 and $\tau_2 \sigma_2$ semi-continuous w.r.t. τ_1.

LEMMA 4.2. If a function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $\tau_1 \sigma_1$ semi-continuous w.r.t. τ_2, then for any subset A of X, $f(\underline{A}_{\tau_1(\tau_2)}) \subset \overline{f(A)}^{\sigma_1}$.

PROOF: Let $y \in f(\underline{A}_{\tau_1(\tau_2)})$ and $y \in V \in \sigma_1$. Then there exists $x \in \underline{A}_{\tau_1(\tau_2)}$ such that $f(x) = y$ and $x \in f^{-1}(V)$ and $f^{-1}(V)$ is τ_1 s.o.w.r.t. τ_2. Hence $f^{-1}(V) \cap A \neq \emptyset \Rightarrow f(f^{-1}(V) \cap A) \neq \emptyset \Rightarrow V \cap f(A) \neq \emptyset \Rightarrow y \in \overline{f(A)}^{\sigma_1}$.

THEOREM 4.3. Pairwise semi-continuous surjection of a pairwise S-closed space onto a pairwise Hausdorff space is pairwise H-closed.

PROOF: Let $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be a pairwise semi-continuous surjection, where X is pairwise S-closed. We first show that (Y, σ_1, σ_2) is σ_1 almost compact w.r.t. σ_2. Let $\{V_\alpha: \alpha \in I\}$ be a σ_1 open cover of Y. Then

$\{f^{-1}(V_\alpha): \alpha \in I\}$ is a cover of X by sets that are τ_1 s.o.w.r.t. τ_2. Since X is τ_1 S-closed w.r.t. τ_2, there exists a finite subfamily I_0 of I, such that $X = \bigcup_{\alpha \in I_0} \overline{f^{-1}(V_\alpha)}^{\tau_2}$. We show that $\underline{\bigcup_{\alpha \in I_0} f^{-1}(V_\alpha)}_{\tau_2(\tau_1)} = X$. In fact, let $x \in X$ and W be any τ_2 s.o. set w.r.t. τ_2, containing x. Then there exists $U \in \tau_2$ such that $U \subset W \subset \bar{U}^{\tau_1}$ and $U \neq \emptyset$. Since $\bigcup_{\alpha \in I_0} f^{-1}(V_\alpha)$ is τ_2 dense in X, every nonempty τ_2 open set must intersect $\bigcup_{\alpha \in I_0} f^{-1}(V_\alpha)$ and hence $U \cap [\bigcup_{\alpha \in I_0} f^{-1}(V_\alpha)] \neq \emptyset$. Then $W \cap (\bigcup_{\alpha \in I_0} f^{-1}(V_\alpha)) \neq \emptyset$ and hence

$$x \in \underline{\bigcup_{\alpha \in I_0} f^{-1}(V_\alpha)}_{\tau_2(\tau_1)}. \text{ Now,}$$

$$Y = f(X) = f\ [\underline{\bigcup_{\alpha \in I_0} f^{-1}(V_\alpha)}_{\tau_2(\tau_1)}]$$

$$\subset \overline{f(\bigcup_{\alpha \in I_0} f^{-1}(V_\alpha))}^{\sigma_2}$$

$$= \bigcup_{\alpha \in I_0} \bar{V}_\alpha^{\sigma_2}.$$

(using Lemma 4.2 and the fact that f is $\tau_2\,\sigma_2$ semi-continuous w.r.t τ_1). Thus by Theorem 1.5(a), Y is σ_1 almost compact w.r.t. σ_2. Similarly, Y is σ_2 almost compact w.r.t. σ_1. Since Y is pairwise Hausdorff, it finally follows by virtue of Theorem 1.5(c) that (Y, σ_1, σ_2) is pairwise H-closed.

DEFINITION 4.4. A function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called $\tau_1\sigma_1$ -irresolute w.r.t. τ_2 if for every σ_1 s.o. set V w.r.t. σ_2, $f^{-1}(V)$ is τ_1 s.o.w.r.t. τ_2. Functions that are $\tau_2\,\sigma_2$ irresolute w.r.t. τ_1 and pairwise irresolute can be defined in the usual manner.

Clearly, every $\tau_i\,\sigma_i$ irresolute function w.r.t. τ_j is $\tau_i\,\sigma_i$ semi-continuous w.r.t. τ_j, where $i, j = 1,2$ but $i \neq j$, but it can be shown that the converse is not true, in general. This converse is true if the function f is, in addition, pairwise open [7].

LEMMA 4.5. A function f from a bitopological space (X, τ_1, τ_2) to a bitopological space (Y, σ_1, σ_2) is $\tau_1\,\sigma_1$ irresolute w.r.t τ_2 if and only if for every subset A of X, $f(\underline{A}_{\tau_1(\tau_2)}) \subset \underline{f(A)}_{\sigma_1(\sigma_2)}$.

PROOF: Let $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be $\tau_1\,\sigma_1$ -irresolute w.r.t. τ_2 and $A \subset X$. Then $f^{-1}(\underline{f(A)}_{\sigma_1(\sigma_2)})$ is τ_1 s.cl.w.r.t. τ_2. Since $A \subset f^{-1}(f(A)) \subset f^{-1}(\underline{f(A)}_{\sigma_1(\sigma_2)})$, we have $\underline{A}_{\tau_1(\tau_2)} \subset f^{-1}(\underline{f(A)}_{\sigma_1(\sigma_2)})$ and hence

$f(\underline{A}_{\tau_1(\tau_2)})\quad f\,f^{-1}\,(\underline{f(A)}_{\sigma_1(\sigma_2)})$, i.e. $f(\underline{A}_{\tau_1(\tau_2)}) \subset \underline{f(A)}_{\sigma_1(\sigma_2)}$.

Conversely, let B be σ_1 s.cl.w.r.t. σ_2 in Y. By hypothesis, $f(\underline{f^{-1}(B)}_{\tau_1(\tau_2)}) \subset \underline{f\,f^{-1}(B)}_{\sigma_1}\,(\sigma_2) \subset \underline{B}_{\sigma_1}\,(\sigma_2) = B$.

Then $\underline{f^{-1}(B)}_{\tau_1(\tau_2)} \subset f^{-1}(B)$ and hence $f^{-1}(B) = \underline{f^{-1}(B)}_{\tau_1(\tau_2)}$. This shows that $f^{-1}\,(B)$ is τ_1 s.cl.w.r.t. τ_2 and then f is $\tau_1\,\sigma_1$ irresolute w.r.t. τ_2.

COROLLARY 4.6. If a function $f\colon (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is $\tau_i\,\sigma_i$ irresolute w.r.t. τ_j, then for any subset A of X, $f(\underline{A}_{\tau_i(\tau_j)}) \subset \overline{f(A)}^{\sigma_i}$, where $i, j = 1,2$ and $i \neq j$.

PROOF: For every subset B of a bitopological space (X, τ_1, τ_2) we always have $\underline{B}_{\tau_i(\tau_j)} \subset \overline{B}^{\tau_i}$, for $i, j = 1, 2$ and $i \neq j$. Hence by Lemma 4.5, the corollary follows.

NOTE 4.7. Following a similar line of proof as in Lemma 4.2, we could also prove the above corollary 4.6.

THEOREM 4.8. Let (X, τ_1, τ_2) be pairwise extremally disconnected and $f\colon (x, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be pairwise irresolute, where (Y, σ_1, σ_2) is a bitopological space. If a subset G of X is pairwise S-closed in X, then $f(G)$ is pairwise S-closed in Y.

PROOF: Let $\{A_\alpha\colon \alpha \in I\}$ be a cover of $f(G)$ by sets that are σ_1 s.o.w.r.t. σ_2 in Y. Then $f^{-1}(A_\alpha)$ is τ_1 s.o.w.r.t. τ_2 in X, for each $\alpha \in I$ and $\{f^{-1}(A_\alpha)\colon \alpha \in I\}$ is a cover of G. Since G is pairwise S-closed in X, there exist a finite number of indices $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $G \subset \bigcup_{k=1}^{n} (\overline{f^{-1}(A_{\alpha_k})}^{\tau_2})$.

By Lemma 3.5, we have $\overline{f^{-1}(A_{\alpha_k})}^{\tau_2} = \underline{f^{-1}(A_{\alpha_k})}_{\tau_2(\tau_1)}$ for $k = 1,2, \ldots, n$. Since f is $\tau_2\,\sigma_2$ irresolute w.r.t. τ_1, we have by Lemma 4.5 $f(\underline{f^{-1}(A_{\alpha_k})}_{\tau_2(\tau_1)}) \subset \left(\underline{f\,(f^{-1}(A_{\alpha_k})}_{\sigma_2(\sigma_1)}\right) \subset \underline{A_{\alpha_k}}_{\sigma_2(\sigma_1)} \subset \overline{A_{\alpha_k}}^{\sigma_2}$, for $k = 1,2 \ldots, n$.

Hence $f(G) \subset f\left[\bigcup_{k=1}^{n} \overline{f^{-1}(A_{\alpha_k})}^{\tau_2}\right] \subset \bigcup_{k=1}^{n} \overline{A_{\alpha_k}}^{\sigma_2}$ and then $f(G)$ is σ_1 S-closed w.r.t. σ_2 in Y. Similarly, $f(G)$ is σ_2 S-closed w.r.t. σ_1 in Y. Hence $f(G)$ is pairwise S-closed in Y. This completes the proof.

NOTE 4.9. If the set G of Theorem 4.8 is the whole space X, then we do not require the condition that (X, τ_1, τ_2) is pairwise extremally disconnected. In fact, proceeding in a similar fashion as in Theorem 4.3 and using Corollary 4.6, we can have :

THEOREM 4.10. If $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is pairwise irresolute and surjective, where (X, τ_1, τ_2) is pairwise S-closed, then (Y, σ_1, σ_2) is also pairwise S-closed.

THEOREM 4.11. Let $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ be $\tau_1\ \sigma_1$ semi-continuous w.r.t. σ_2, $f: (X, \tau_2) \to (Y, \sigma_2)$ is continuous and open. If $G \subset X$ is τ_1 S-closed w.r.t. τ_2 in X, then $f(G)$ is σ_1 S-closed w.r.t. σ_2 in Y.

PROOF: Let $\{U_\alpha: \alpha \in I\}$ be a cover of $f(G)$ by sets that are σ_1 s.o.w.r.t. σ_2. For each α, there is $V_\alpha \in \sigma_1$ such that $V_\alpha \subset U_\alpha \subset \overline{V_\alpha}^{\sigma_2}$. Since $f: (X, \tau_2) \to (Y, \sigma_2)$ is open, we have $f^{-1}(\overline{V_\alpha}^{\sigma_2}) \subset \overline{f^{-1}(V_\alpha)}^{\tau_2}$. Since f is $\tau_1\ \sigma_1$ semi-continuous w.r.t. τ_2, $f^{-1}(V_\alpha)$ is τ_1 s.o.w.r.t. τ_2 and hence there exists $O \in \tau_1$, such that

$O \subset f^{-1}(V_\alpha) \subset \overline{O}^{\tau_2} \Rightarrow O \subset \overline{f^{-1}(V_\alpha)}^{\tau_2} \subset \overline{O}^{\tau_2}$. Thus $O \subset f^{-1}(V_\alpha) \subset f^{-1}(U_\alpha) \subset f^{-1}(\overline{V_\alpha}^{\sigma_2})$ $\subset \overline{f^{-1}(V_\alpha)}^{\tau_2} \subset \overline{O}^{\tau_2}$. That is, $O \subset f^{-1}(U_\alpha) \subset \overline{O}^{\tau_2}$ and $O \in \tau_1$. Therefore, $f^{-1}(U_\alpha)$ is τ_1 s.o.w.r.t. τ_2, for each $\alpha \in I$, and $\{f^{-1}(U_\alpha): \alpha \in I\}$ is a cover of G. Then there exists a finite number of indices $\alpha_1, \ldots, \alpha_n$ such that

$G \subset \bigcup_{i=1}^{n} \overline{f^{-1}(U_{\alpha_i})}^{\tau_2}$. Since $f: (X, \tau_2) \to (Y, \sigma_2)$ is continuous,

$f\left[\overline{f^{-1}(U_{\alpha_i})}^{\tau_2}\right] \subset \overline{U_{\alpha_i}}^{\sigma_2}$, for $i = 1,2 \ldots, n$. Therefore, $f(G) \subset \bigcup_{i=1}^{n} \overline{U_{\alpha_i}}^{\sigma_2}$ and then $f(G)$ is σ_1 S-closed w.r.t. σ_2 in Y.

COROLLARY 4.12. Pairwise S-closedness is a bitopological invariant.

PROOF: Since every pairwise continuous function is pairwise semi-continuous, the corollary follows by virtue of Theorem 4.11.

COROLLARY 4.13. Let $\{(X_\alpha, \tau_\alpha^1, \tau_\alpha^2): \alpha \in I\}$ be a family of bitopological spaces and (X, τ^1, τ^2) be their product space. If (X, τ^1, τ^2) is pairwise S-closed, then each $(X_\alpha, \tau_\alpha^1, \tau_\alpha^2)$ is also pairwise S-closed.

PROOF: Since $P_\alpha:(X, \tau^i) \to (X_\alpha, \tau_\alpha^i)$ is an open, continuous surjection, for $i = 1,2$ and for each $\alpha \in I$,, the corollary becomes evident because of Theorem 4.11.

THEOREM 4.14. The pairwise irresolute image of a pairwise S-closed and pairwise extremally disconnected bitopological space in any pairwise Hausdorff bitopological space is pairwise closed.

PROOF: Let f be a pairwise irresolute function from a pairwise S-closed and pairwise extremally disconnected space (X, τ_1, τ_2) into a pairwise Hausdorff space (Y, σ_1, σ_2). Let $y \in \overline{f(X)}^{\sigma_2}$ and $N_1(y)$ denote the σ_1 -open neighborhood system at y in (Y, σ_1, σ_2). Then $F = \{f^{-1}(V): V \in N_1(y)\}$ is a filter-base in X. Since X is τ_2 S-closed w.r.t. τ_1, F has a τ_2 S-accumulation point x w.r.t. τ_1.

We show that $f(F)$ has $f(x)$ as a σ_2 accumulation point. In fact, let $f(x) \in V \in \sigma_2$. Then $f^{-1}(V)$ is τ_2 s.o.w.r.t. τ_1 and contains x. Now, for each $W \in N_1(y)$, $f^{-1}(W) \in F$ and hence $f^{-1}(W) \cap \overline{f^{-1}(V)}^{\tau_1} \neq \emptyset$. Since (X, τ_1, τ_2) is pairwise extremally disconnected, we then must have $[f^{-1}(W)]^{i_1} \cap [f^{-1}(V)]^{i_2} \neq \emptyset$. Indeed, if $[f^{-1}(W)]^{i_1} \cap [f^{-1}(V)]^{i_2} = \emptyset$, then $\overline{[f^{-1}(W)]^{i_1}}^{\tau_2} \cap \overline{[f^{-1}(V)]^{i_2}}^{\tau_1} = \emptyset$, i.e., $\overline{f^{-1}(W)}^{\tau_2} \cap \overline{f^{-1}(V)}^{\tau_1} = \emptyset$ which is not the case.
Now, $\emptyset \neq f[(f^{-1}(W))^{i_1} \cap f^{-1}(V))^{i_2}] \subset f[f^{-1}(W) \cap f^{-1}(V)] \subset W \cap V$. Hence $W \cap V \neq \emptyset$. This shows that $f(x)$ is a σ_2 accumulation point of $f(F)$ in Y. But $f(F)$ being finer than $N_1(y)$, $N_1(y)$ also σ_2 accumulates to $f(x)$. Now, if $y \neq f(x)$, by pairwise Hausdorff property of (y, σ_1, σ_2), there exist σ_1 open set A and σ_2 open set B such that $y \in A$, $f(x) \in B$ and $A \cap B = \emptyset$. Since $A \in N_1(y)$, $f(f^{-1}(A)) \in f(F)$ and hence $B \cap f(f^{-1}(A) \neq \emptyset$, because $f(x)$ is a σ_2 accumulation point of $f(F)$. In other words $B \cap A \neq \emptyset$ which is a contradiction. Hence $y = f(x)$ and then $y \in f(X)$. Consequently $f(X)$ is σ_2 closed in Y. Similarly $f(X)$ is σ_1 closed in Y. This completes the proof.

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