Internat. J. Math. & Math. Sci.
Vol. 8 No. 4 (1985) 693-696

A VARIANT OF A FIXED POINT THEOREM OF BROWDER-FAN AND REICH

V. M. SEHGAL

Department of Mathematics
University of Wyoming
Laramie, Wyoming 82071 U.S.A.

(Received January 8, 1985)

ABSTRACT. Let S be a convex, weakly compact subset of a locally convex Hausdorff space (E, τ) and $f: S \to E$ be a continuous multifunction from its weak topology ω to τ. Let p be a continuous seminorm on (E, τ) and for subsets A, B, of E, let $p(A, B) = \inf\{p(x - y): x \in A, y \in B\}$. In this paper, sufficient conditions are developed for the existence of an $x \in S$ satisfying $p(x, fx) = p(fx, S)$. The result is then used to prove several fixed point theorems.

KEY WORDS AND PHRASES. Multifunctions, convex topology, fixed points.
1980 AMS SUBJECT CLASSIFICATION CODES. 47H10, 60H25.

1. INTRODUCTION

Let (E, τ) be a locally convex Hausdorff topological vector space with topology τ and $E^* = (E, \tau)^*$ be its topological dual. Let $\omega = \omega(E, E^*)$ be the weak topology of E. Let P and Q denote the family of continuous semi-norms generating the topologies τ and ω respectively. For sets A and B of E and a $p \in P$, let $p(A, B) = \inf\{p(x - y): x \in A, y \in B\}$. In this paper, we prove the following result.

THEOREM 1. Let S be a nonempty convex, ω-compact subset of E and $f: (S, \omega) \to (E, \tau)$ be a continuous multifunction such that $f(x)$ is convex and ω-compact for each $x \in S$. Then for each $p \in P$ there exists a $x \in S$ satisfying

$$p(x, fx) = p(fx, S). \tag{1.1}$$

Futher if $p(x, fx) > 0$ then $x \in \partial(S, \omega) \cap \partial(S, \tau)$ where ∂ denotes the boundary.

It may be remarked the since $\omega \subseteq \tau$, f in Theorem 1 is also a continuous multifunction from $(S, \omega) \to (E, \omega)$. Consequently it follows by Reich (Lemma 1.6 [1]) that each $q \in Q$ satisfies (1.1) for some $x \in S$. However, since $Q \subseteq P$, the lemma in [1] is not applicable for arbitrary $p \in P$. In fact, Theorem 1 contains the above lemma [1] (see Corollary 2) and it provides a generalization of a well-known result of Ky Fan [2] for single valued mappings.

2. PRELIMINARY RESULTS.

Recall that if X, Y are topological spaces then a multifunction $f: X \to Y$ ($fx \neq \emptyset$ for each x) is upper (lower) semicontinuous iff for each closed (open) subset A of Y, $f^{-1}(A) = \{x \in X: f(x) \cap A \neq \emptyset\}$ is a closed (open) subset of X. It follows by definition that f is l.s.c. iff $fx \cap U \neq \emptyset$ for some open set U of Y and x in X then $fz \cap U \neq \emptyset$ for each z in some neighborhood V of X. Further, it is well-known (i) that if f is u.s.c. and a net $x_\alpha \to x$ in X and $y_\alpha \to y$ in Y with $y_\alpha \in fx_\alpha$ then $y \in fx$; (ii) if X is compact and f is u.s.c. with compact values then fX is compact. A multifunction which is both u.s.c. and l.s.c. is called continuous.

We prove two lemmas that simplify the proof of Theorem 1. Throughout, let E be as stated in the beginning and S a nonempty subset of E.

LEMMA 1. Let A, B be ω-compact sets of E and $p \in P$. Then $p(A, B) = p(x, B) = p(x - y)$ for some $x \in A$, $y \in B$.

PROOF. Choose sequences $\{x_n\} \subseteq A$, $\{y_n\} \subseteq B$ such that $p(x_n - y_n) \downarrow p(A, B)$. We may assume that $x_n \to x$ weakly for some $x \in A$ and $y_n \to y$ weakly for some $y \in B$. By Hahn Banach Theorem (see [3], Cor. 2, p. 29) there exists a $x^* \in E^*$ with $x^*(x - y) = p(x - y)$ and $|x^*(u)| \le p(u)$ for each $u \in E$. Consequently, since $x_n - y_n \to x - y$ weakly,

$$p(x, B) \le p(x - y) = x^*(x - y) = \lim|x^*(x_n - y_n)| \le \underline{\lim}\, p(x_n - y_n) = p(A, B) \le p(x, B)$$

LEMMA 2. Let S be ω-compact subset of E and $f: (S, \omega) \to (E, \tau)$ be a l.s.c. multifunction with weakly compact values. If a net $x_\alpha \to x$ weakly in S, then for each $p \in P$ and $\varepsilon > 0$, $p(fx_\alpha, S) \le p(fx, S) + \varepsilon$ eventually.

PROOF. It follows by Lemma 1 that there is a $y \in fx$ with $p(fx, S) = p(y, S)$. Let $U = \{x \in E: p(x - y) < \varepsilon\}$. Then U is τ-open and $y \in fx \cap U$. Hence by l.s.c., $fx_\alpha \cap U \neq \emptyset$ eventually. For such α, let $y_\alpha \in fx_\alpha \cap U$. Then eventually,

$$p(fx_\alpha, S) \le p(y_\alpha, S) \le p(y_\alpha - y) + p(y, S) \le p(fx, S) + \varepsilon.$$

3. MAIN RESULTS.

PROOF OF THEOREM 1. Let $p \in P$. Define a multifunction $g: (S, \omega) \to (S, \omega)$ by

$$g(x) = \{y \in S: p(y, fx) = p(fx, S)\}.$$

Then by Lemma 1, $g(x) \neq \emptyset$ and is clearly convex. Further, since S is τ-closed and for any $y, z \in g(x)$, the triangular inequality implies

$$|p(y, fx) - p(z, fx)| \le p(y - z).$$

It follows $g(x)$ is τ-closed convex and hence a ω-compact subset of S. We show that g is u.s.c. Let C be a weakly closed (hence weakly compact) aubset of S. We show that $x \in g^{-1}(C)$, that is $g(x) \cap C \neq \emptyset$. Choose for each α, $y_\alpha \in gx_\alpha \cap C$. We may assume that $y_\alpha \to y$ weakly for some $y \in C$. Also since $p(y_\alpha, fx_\alpha) = p(fx_\alpha, S)$, there exists $z_\alpha \in fx_\alpha$ with $p(y_\alpha - z_\alpha) = p(fx_\alpha, S)$. Further $f: (S, \omega) \to (E, \omega)$ being u.s.c., it follows that fS is weakly compact and hence we may assume that $z_\alpha \to z$ weakly for some $z \in fx$. Thus $y_\alpha - z_\alpha \to y - z$ weakly. Choose as before a $x^* \in E^*$ such that $x^*(y - z) = p(y - z)$ and $|x^*(u)| \le p(u)$ for each $u \in E$. Let $\varepsilon > 0$. Choose $\alpha_0 \in \Delta$ such that $p(fx_\alpha, S) \le p(fx, S) + \varepsilon$ for $\alpha \ge \alpha_0$. Consequently, for $\alpha \ge \alpha_0$ $|x^*(y_\alpha - z_\alpha)| \le p(fx_\alpha, S) \le p(fx, S) + \varepsilon$ and hence

$$p(y, fx) \le p(y - z) = \lim|x^*(y_\alpha - z_\alpha)| \le p(fx, S) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and $p(fx, S) \leq p(y, fx)$, we have $p(y, fx) = p(fx, S)$ that is $y \in g(x) \cap C$. Thus g is u.s.c. Hence by Glicksberg [4] there exists a $x \in S$ with $x \in g(x)$. This implies $p(x, fx) = p(fx, S)$.

Now, suppose $p(x, fx) > 0$. Then $fx \cap S = \emptyset$. Choose by Lemma 1, a $y \in fx$ satisfying $p(x - y) = p(x, fx)$. Now, if $x \in int(S, \omega)$ or $int(S, \tau)$, then since S being weakly closed and convex, there is a $z \in (x, y) \cap S$ with $0 < p(fx, S) \leq p(y - z) < p(x - y) = p(fx, S)$, a contradiction. This proves the result.

As a consequence of Theorem 1, we have

COROLLARY 1. Let S be a convex and weakly compact set in E and $f: (S, \omega) \to (E, \tau)$ be a continuous multifunction with convex and ω-compact values. Then either f has a fixed point or there exists a $p \in P$ and $x \in S$ satisfying $0 < p(x, fx) = p(fx, S)$.

PROOF. For each $p \in P$, let $x_p \in S$ satisfying (1). If $p(x_p, fx_p) = 0$ for each $p \in P$, then using the implication that $f: (S,\omega) \to (S,\omega)$ is continuous, it follows that $A_p = \{x \in S: p(x, fx) = 0\}$ is nonempty, weakly compact and the family $\{A_p: p \in P\}$ has finite intersection property. Consequently, there exists $x \in S$ with $p(x, fx) = 0$ for each $p \in P$. Now, if $x \notin fx$, then since $x - fx$ is τ-closed and convex and $0 \notin x - fx$, there exists (see [3], Cor. 1, p. 30) a $x^* \in E^*$ such that $0 \notin \{x^*(x - y): y \in fx\}$. Let $p = |x^*|$. Then $p \in P$ and $p(x, fx) \neq 0$, a contradiction.

The following corollaries result from Theorem 1.

COROLLARY 2. (Reich [1]). Let S be a compact and convex in (E, τ) and $f: (S, \tau) \to (E, \tau)$ be a continuous mutifunction with convex and compact values. Then either f has a fixed point or there exists a $p \in P$ and $x \in S$ satisfying $0 < p(x, fx) = p(fx, S)$.

COROLLARY 3. (Waters [5]). Let S be a compact and convex subset of (E, τ) and $f: (S, \tau) \to (E, \tau)$ be a continuous multifunction with convex and weakly compact values. Then for each $p \in P$, there exists a $x \in S$ satisfying (1.1).

PROOF. It suffices to show that the hypotheses in Corollary 2 and Corollary 3 imply that $f: (S, \omega) \to (E, \tau)$ is a continuous multifunction. Let A be τ-closed in E. Then $f^{-1}(A)$ is τ-compact subset of S. Since S is weakly closed, it follows that $f^{-1}(A)$ is weakly closed. Thus f is u.s.c. Similarly if A is τ-open set in E then $S \setminus f^{-1}(A) = f^{-1}(E \setminus A)$ is ω-closed and hence $f^{-1}(A)$ is ω-open. Thus f is l.s.c.

In the setting **of** semi-reflexive locally convex spaces, we have

COROLLARY 4. Let S be a closed, bounded and convex subset of a semi-reflexive locally convex space E. If $f: (S, \omega) \to (E, \tau)$ is continuous multifunction with closed, bounded and convex values then for each $p \in P$, there exists $x \in S$ satisfying (1.1).

*Theorem 1 of this paper was presented at the summer meeting of the Amer. Math. Society, (1983), Albany, New York.

REFERENCES

1. REICH, S. Fixed point theorems in locally convex spaces, Math. Z. 125 (1972), pp. 17-31.
2. KY FAN Extensions of two fixed point theorems of F. E. Browder, Math. Z. 112 (1969), pp. 234-240.
3. ROBERTSON, A. P., and W. J. Topological Vector Spaces, Cambridge, University Press (1966).
4. GLICKSBERG, I. L. A further generalization of Kakutani's fixed point theorem with application to Nash equilibrium points, Proc. Amer. Math. Soc. 3 (1952), pp. 170-174.
5. WATERS, C. Ph.D. Thesis (1984), University of Wyoming.

Special Issue on Space Dynamics

Call for Papers

Space dynamics is a very general title that can accommodate a long list of activities. This kind of research started with the study of the motion of the stars and the planets back to the origin of astronomy, and nowadays it has a large list of topics. It is possible to make a division in two main categories: astronomy and astrodynamics. By astronomy, we can relate topics that deal with the motion of the planets, natural satellites, comets, and so forth. Many important topics of research nowadays are related to those subjects. By astrodynamics, we mean topics related to spaceflight dynamics.

It means topics where a satellite, a rocket, or any kind of man-made object is travelling in space governed by the gravitational forces of celestial bodies and/or forces generated by propulsion systems that are available in those objects. Many topics are related to orbit determination, propagation, and orbital maneuvers related to those spacecrafts. Several other topics that are related to this subject are numerical methods, nonlinear dynamics, chaos, and control.

The main objective of this Special Issue is to publish topics that are under study in one of those lines. The idea is to get the most recent researches and published them in a very short time, so we can give a step in order to help scientists and engineers that work in this field to be aware of actual research. All the published papers have to be peer reviewed, but in a fast and accurate way so that the topics are not outdated by the large speed that the information flows nowadays.

Before submission authors should carefully read over the journal's Author Guidelines, which are located at http://www.hindawi.com/journals/mpe/guidelines.html. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

Manuscript Due	July 1, 2009
First Round of Reviews	October 1, 2009
Publication Date	January 1, 2010

Lead Guest Editor

Antonio F. Bertachini A. Prado, Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, 12227-010 São Paulo, Brazil; prado@dem.inpe.br

Guest Editors

Maria Cecilia Zanardi, São Paulo State University (UNESP), Guaratinguetá, 12516-410 São Paulo, Brazil; cecilia@feg.unesp.br

Tadashi Yokoyama, Universidade Estadual Paulista (UNESP), Rio Claro, 13506-900 São Paulo, Brazil; tadashi@rc.unesp.br

Silvia Maria Giuliatti Winter, São Paulo State University (UNESP), Guaratinguetá, 12516-410 São Paulo, Brazil; silvia@feg.unesp.br