

ON FIXED POINTS OF SET-VALUED DIRECTIONAL CONTRACTIONS

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ABSTRACT. Using equivalent formulations of Ekeland's theorem, we improve fixed point theorems of Clarke, Sehgal, Sehgal-Smithson, and Kirk-Ray on directional contractions by giving geometric estimations of fixed points.

KEY WORDS AND PHRASES. *l.s.c. function, (weak) directional contraction, fixed point, stationary point, Hausdorff pseudometric.*

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1. INTRODUCTION AND PRELIMINARIES

In [1], [2], Sehgal and Smithson proved fixed point theorems for set-valued weak directional contractions which extend earlier results of Clarke [3], Kirk and Ray [4], and Assad and Kirk [5]. In the present paper, results in [1], [2] are substantially strengthened by giving geometric estimations of locations of fixed points.

The following equivalent formulations [6] of the well-known central result of Ekeland [7], [8] on the variational principle for approximate solutions of minimization problems is used in the proofs of the main results.

THEOREM 1. Let (V, d) be a complete metric space, and $\phi : V \rightarrow \mathbb{R} \cup \{+\infty\}$ a l.s.c. function, $\neq +\infty$, bounded from below. Let $\varepsilon > 0$ and $\lambda > 0$ be given, and a point $u \in V$ such that

$$F(u) \leq \inf_V F + \varepsilon.$$

Let $S(\lambda) = \{x \in V \mid F(x) \leq F(u) - \varepsilon \lambda^{-1} d(u, x)\}$. Then the following equivalent conditions hold:

(i) There exists a point $v \in S(\lambda)$ satisfying

$$F(w) > F(v) - \varepsilon \lambda^{-1} d(v, w) \quad \text{for } \forall w \neq v.$$

(ii) If $T : S(\lambda) \rightarrow 2^V$ is a set-valued map satisfying the condition

$$\forall x \in S(\lambda) \setminus T(x) \exists y \in V \setminus \{x\} \text{ such that}$$

$$F(y) \leq F(x) - \varepsilon \lambda^{-1} d(x, y),$$

then T has a fixed point $v \in S(\lambda)$.

(iii) If $f : S(\lambda) \rightarrow V$ satisfies

$$F(fx) \leq F(x) - \varepsilon \lambda^{-1} d(x, fx)$$

for all $x \in S(\lambda)$, then f has a fixed point $v \in S(\lambda)$.

In Theorem 1, 2^V denotes the power set of V . Note that

$$S(\lambda) \subset \{x \in V \mid F(x) \leq F(u), d(u, x) \leq \lambda\} \subset \bar{B}(u, \lambda)$$

and $FS(\lambda) \subset S(\lambda)$, where \bar{B} denotes the closed ball.

Throughout this paper, (V, d) denotes a metric space and $B(V)$ denotes the class of all nonempty bounded subsets of V with the Hausdorff pseudometric H defined by

$$H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}.$$

Also, $C(V)$ denotes the class of all nonempty compact subsets of V . For an $x \in V$ and $A \in C(V)$, we put

$$[x, A] = \{y \in A \mid d(x, y) = d(x, A)\},$$

which is nonempty. For $x, y \in V$, we denote

$$[x, y] = \{z \in V \mid d(x, z) + d(z, y) = d(x, y)\},$$

and

$$(x, y] = [x, y] \setminus \{x\}, (x, y) = (x, y] \setminus \{y\}.$$

Let S be a nonempty subset of V and $T : S \rightarrow C(V)$ be a set-valued map. For $x \in S$ and $A \in C(V)$, the weak directional derivative $DT(x, y)$ of T at x in the direction of a $y \in [x, T(x)]$ is defined by

$$DT(x, y) = \begin{cases} 0, & \text{if } x = y, \\ \inf\left\{\frac{H(Tx, Tz)}{d(x, z)} \mid z \in (x, y] \cap S\right\}, & \text{if } (x, y] \cap S \neq \emptyset, \\ \infty, & \text{if } (x, y] \cap S = \emptyset. \end{cases}$$

A map $T : S \rightarrow C(V)$ is called a weak directional contraction if there exists a $k \in [0, 1)$ such that for each $x \in S$, there exists a $y \in [x, T(x)]$ with $DT(x, y) < k$ [2].

A map $T : S \rightarrow B(V)$ is called a directional contraction if there exists a $k \in [0, 1)$ such that for each $x \in S$ and $y \in T(x)$,

$$H(T(z), T(x)) \leq kd(z, x)$$

for all $z \in [x, y] \cap S$ [7].

2. RESULTS.

THEOREM 2. Let S be a complete subset of V and $T : S \rightarrow C(V)$ a weak directional contraction for which the function $F(x) = d(x, T(x))$, $x \in S$, is l.s.c. Then for any $u \in S$ and $\epsilon > 0$ satisfying $F(u) \leq (1 - k)\epsilon$, T has a fixed point in $S(\epsilon) \subset \bar{B}(u, \epsilon) \cap S$.

PROOF. Choose a point $u \in S$ satisfying $F(u) \leq \inf_S F + (1 - k)\epsilon$. Suppose $x \in S(\epsilon) \setminus T(x)$. Since T is a weak directional contraction, there exists a $y \in [x, T(x)]$, $x \neq y$, with $DT(x, y) < k$. Hence, there exists a $z \in (x, y] \cap S$ such that

$$H(T(x), T(z)) < k d(x, z).$$

Since

$$d(x, z) + d(z, T(x)) \leq d(x, y) = d(x, T(x)),$$

we have

$$\begin{aligned} d(z, T(z)) &\leq d(z, T(x)) + H(T(x), T(z)) \\ &\leq d(x, T(x)) - d(x, z) + k d(x, z) \\ &\leq d(x, T(x)) - (1 - k)d(x, z). \end{aligned}$$

Hence, $F(z) \leq F(x) - (1 - k)d(x, z)$. Therefore, by Theorem 1(iii), T has a fixed point $v \in S(\epsilon)$.

Theorem 2 is an improved version of Theorem (a) of [2] with much simpler proof. In fact, for suitable values of ϵ and k , the conclusion gives geometric estimations of locations of fixed points. However, for Theorem (b) of [2], such estimation seems to be hard to get.

Note also that for Theorem 1 of Clarke [3], we can apply our Theorem 2.

The following improves Corollary of [2] and a result of Kirk and Ray [4].

COROLLARY 1. Let S be a closed convex subset of a Banach space X and $T : S \rightarrow C(S)$ a map for which the function $F(x) = d(x, T(x))$, $x \in S$, is l.s.c. Suppose there exists a $k \in [0, 1)$ such that for each $x \in S$ there correspond a $y = y(x) \in [x, T(x)]$ and a $\delta \in (0, 1)$ satisfying

$$H(T(x), T(x + \delta(y - x))) \leq k\delta \|y - x\|.$$

Then the conclusion of Theorem 1 follows.

PROOF. As in the proof in [2], T is a weak directional contraction with the constant k .

THEOREM 3. Let S be a closed subset of a complete metric space V and $T : S \rightarrow B(V)$ a directional contraction with the constant α . If T satisfies

(a) for each $x \in S$, $y \in T(x) \setminus S$, there exists a $z \in (x, y) \setminus S$ with $T(z) \subset S$, and

(b) $g(x) = d(x, T(x))$ is l.s.c.,

then, for any $u \in S$, $\epsilon > 0$ and β , $\alpha < \beta < 1$ satisfying $g(u) \leq (1 - \beta)\epsilon$, there exists a fixed point v of T in $S(\epsilon) \cap S$.

LEMMA [4]. Under the hypothesis of Theorem 3, there exists a map $A : S \rightarrow B(X)$ with the following properties

- i) for each $x \in S$, $A(x) \neq \emptyset$ and $A(x) \subset T(x)$
- ii) if $y \in A(x)$, then $d(x, y) \leq (1 - \beta + \alpha)^{-1}d(x, T(x))$,
- iii) if $A(x) \cap S = \emptyset$ for some $x \in S$, then there exists a $y = y(x) \in A(x)$ and a $z = z(x, y) \in (x, y) \cap S$ such that

$$d(x, y) \leq d(x, T(x)) + (\beta - \alpha)d(x, z). \quad (2.1)$$

PROOF OF THEOREM 3. Define a map $f : S \rightarrow S$ as follows: for $x \in S$ such that $A(x) \cap S \neq \emptyset$, let $f(x)$ be any element of $A(x) \cap S$; and for $x \in S$ such that $A(x) \cap S = \emptyset$, since there exist $y = y(x) \in A(x)$ and $z = z(x, y) \in (x, y) \cap S$ satisfying (2.1) by Lemma, let $f(x) = z$. We claim that for any $x \in S$,

$$H(T(x), T(f(x))) \leq \alpha d(x, f(x)). \quad (2.2)$$

This is clear if $A(x) \cap S = \emptyset$. If $A(x) \cap S \neq \emptyset$, since $f(x) \in T(x)$ and $f(x) \in [x, f(x)] \cap S$, the definition of T implies (2.2). Set $F(x) = (1 - \beta)^{-1}g(x)$. We know that for any $x \in S$ and $y = f(x)$,

$$F(y) \leq F(x) - d(x, y)$$

holds as in the proof of [1, Theorem 1]. Therefore, by Theorem 1 (iii), for any $u \in S$ and $\epsilon > 0$ satisfying $F(u) \leq \inf_S F + \epsilon$, there exists a fixed point v of f in $S(\epsilon) \cap S$. This implies that $v \in T(v)$ for otherwise $f(v) \notin A(v) \cap S$ and hence by the definition of f , $A(v) \cap S = \emptyset$. Thus, $f(v) \in (v, y(v))$ for some $y(v) \in A(v)$. This contradicts $v \neq f(v)$. Consequently, $v \in T(v)$. Since $\inf_S F = 0$, u can be

chosen so that $F(u) \leq \epsilon$, that is, $d(u, T(u)) \leq (1 - \beta)\epsilon$. This completes our proof.

Note that Theorem 3 is a strengthened form of [1, Theorem 1].

A metric space is said to be convex if for each $x, y \in X$, $x \neq y$, there exists a $z \in (x, y)$. It is known that if S is a closed subset of a complete convex metric space V and $x \in S$ and $y \notin S$, then there exists a $z \in [x, y) \cap \delta S$ where δ is the boundary.

Now, we obtain the following improved version of [1, Corollary 1] as an immediate consequence of Theorem 3.

COROLLARY 2. Let S be a closed subset of a complete convex metric space V . Let $T : S \rightarrow B(V)$ be a directional contraction with the constant α such that $T(\delta S) \subset S$. If $g(x) = d(x, T(x))$ is l.s.c. on S , then for any $u \in S$, $\epsilon > 0$, and β , $\alpha < \beta < 1$, satisfying $g(u) \leq (1 - \beta)\epsilon$, there exists a fixed point v of T in $S(\epsilon) \cap S$.

Also, the following improves [1, Corollary 2] and an earlier result of Assad-Kirk [5].

COROLLARY 3. Let S be a closed subset of a complete convex metric space V . Suppose $T : S \rightarrow B(X)$ is a contraction, that is, there exists an $\alpha \in [0, 1)$ such that for all $x, y \in S$,

$$H(T(x), T(y)) \leq \alpha d(x, y).$$

If $T(\delta S) \subset S$, then for any $u \in S$, $\epsilon > 0$, and β , $\alpha < \beta < 1$, satisfying $d(u, T(u)) \leq (1 - \beta)\epsilon$, either u is a fixed point of T or there exists a fixed point v of T in

$$S(\epsilon) \cap S \setminus B(u, s)$$

where $s = d(u, T(u)) (1 + \alpha)^{-1}$.

PROOF. Since a contraction is a directional contraction and $g(x) = d(x, T(x))$ is continuous, by Corollary 2, T has a fixed point $v \in S(\epsilon) \cap S$. Suppose u is not fixed under T . Then for any $y \in B(u, s) \cap S$ we have

$$d(u, T(u)) \leq d(u, y) + d(y, T(u))$$

$$< s + d(y, T(u)),$$

that is,

$$\alpha(1 + \alpha)^{-1}d(u, T(u)) < d(y, T(u)).$$

Hence,

$$d(y, T(u)) > \alpha s > \alpha d(y, u).$$

Suppose $y \in T(y)$. Then we have

$$H(T(y), T(u)) > \alpha d(y, u),$$

a contradiction. This completes our proof.

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REFERENCES

1. SEHGAL, V. M. Some fixed point theorems for set valued directional contraction mappings, Internat. J. Math. & Math. Sci. 3(1980), 455-460.
2. SEHGAL, V. M. and SMITHSON, R. E. A fixed point theorem for weak directional contraction multifunctions, Math. Japonica 25(1980), 345-348.
3. CLARKE, F. H. Pointwise contraction criteria for the existence of fixed points, Canad. Math. Bull. 21(1978), 7-11.
4. KIRK, W. A. and RAY, W. O. A remark on directional contractions, Proc. Amer. Math. Soc. 66(1977), 279-283.
5. ASSAD, N. A. and KIRK, W. A. Fixed point theorems for set valued mappings, Pacific J. Math. 43(1972), 553-561.
6. PARK, S. Equivalent formulations of Ekeland's variational principle for approximate solutions of minimization problems and their applications, The MSRI-Korea Publ. 1(1985), to appear.
7. EKELAND, I. On the variational principle, J. Math. Anal. Appl. 47(1974), 324-353.
8. EKELAND, I. Nonconvex minimization problems, Bull. Amer. Math. Soc., 1(1979), 443-474.

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