

INTEGRAL OPERATORS OF CERTAIN UNIVALENT FUNCTIONS

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ABSTRACT. A function f , analytic in the unit disc Δ , is said to be in the family $R_n(\alpha)$ if $\operatorname{Re}\{(z^n f(z))^{(n+1)} / (z^{n-1} f(z))^{(n)}\} > (n+\alpha)/(n+1)$ for some $\alpha (0 \leq \alpha < 1)$ and for all z in Δ , where $n \in \mathbb{N}_0$, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. The class $R_n(\alpha)$ contains the starlike functions of order α for $n \geq 0$, and the convex functions of order α for $n \geq 1$. We study a class of integral operators defined on $R_n(\alpha)$. Finally an argument theorem is proved.

KEY WORDS AND PHRASES: Univalent, convolution, starlike, convex

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1 INTRODUCTION.

Let A denote the family of functions f which are analytic in the unit disc $\Delta = \{z: |z| < 1\}$ and normalised such that $f(0) = 0 = f'(0) - 1$. The Hadamard product or convolution of two functions $f, g \in A$ is denoted by $f * g$. Let $D^n f = (z/(1-z))^{n+1} * f$, $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ which implies that

$$D^n f = z(z^{n-1} f)^{(n)} / n!, \quad n \in \mathbb{N}_0.$$

Denote by $S^*(\alpha)$ and $K(\alpha)$ the subfamilies of A whose members are, respectively, starlike of order α and convex of order α , $0 \leq \alpha < 1$. Then

$$f \in S^*(\alpha) \iff \operatorname{Re}(D^1 f / D^0 f) > \alpha, \quad z \in \Delta,$$

$$f \in K(\alpha) \iff \operatorname{Re}(D^2 f / D^1 f) > (1+\alpha)/2, \quad z \in \Delta$$

Ruscheweyh [16] introduced the classes $\{K_n\}$ of functions $f \in A$ which satisfy the condition

$$\operatorname{Re}(D^{n+1} f / D^n f) > \frac{1}{2}, \quad z \in \Delta \quad (1.1)$$

so that the definition of K_n is a natural extension of $S^*(1/2)$, and $K(0)$

He proved that $K_{n+1} \subset K_n$ for each $n \in \mathbb{N}_0$. Since $K_0 = S^*(1/2)$, the elements of K_n are univalent and starlike of order $1/2$.

In this paper, we consider the classes of functions $f \in A$ which

satisfy the condition

$$\operatorname{Re}(z(D^n f)' / D^n f) > \alpha, \quad z \in \Delta \quad (1.2)$$

for some $\alpha (0 \leq \alpha < 1)$. We denote these classes by $R_n(\alpha)$. We have $R_0(\alpha) = S^*(\alpha)$ and $R_1(\alpha) = K(\alpha)$ for $0 \leq \alpha < 1$. The classes $R_n = R_n(0)$ were considered earlier by Singh and Singh [17]. It is readily seen that for each $n \geq 0$, $R_n(\alpha) \subset R_n(0)$ and for each $n \geq 1$, $R_n(\alpha) \subset K_n$. We note that in definition (1.2), restriction $\alpha \geq 0$ can be replaced by $\alpha \geq (1-n)/2$ for each $n \geq 1$ and, further, that the negative choices of α permit us fully to partition K_n into classes $R_n(\alpha) \subset K_n$ ($n \geq 1$) such that

$$\bigcup R_n(\alpha) = K_n, \\ \frac{1-n}{2} \leq \alpha < 1$$

It can be easily seen that $R_{n+1}(\alpha) \subset R_n(\alpha)$ for each $n \in N_0$ and for all α . These inclusion relations establish that $R_n(\alpha) \subset S^*(\alpha)$ for each $n \geq 0$ and $R_n(\alpha) \subset K(\alpha)$ for each $n \geq 1$.

An important problem in univalent functions is the following: Given a compact family F and an operator J defined on F , is $J(f) \in F$ for every $f \in F$? Libera [11] established that the operator

$$J(f) = \frac{2}{z} \int_0^z f(t) dt \quad (1.3)$$

preserves convexity, starlikeness, and close-to-convexity. Bernardi [5] greatly generalised Libera's results. Many authors [1,2,7,8,12,15,17] studied operators of the form

$$J(f) = \frac{1+\gamma}{z^\gamma} \int_0^z t^{\gamma-1} f(t) dt, \quad (1.4)$$

where γ is a real (or complex) constant and f belongs to some favoured class of univalent functions from A . Recently, operators (1.4) have been studied in more general form by Causey and White [6], Miller, Mocanu and Reade [14], Barnard and Kellogg [3], and Bajpai [2].

In this paper, we study a class of integral operators of the form (1.4) defined on our family $R_n(\alpha)$. We also obtain an argument theorem for the class $R_n(\alpha)$.

2. INTEGRAL OPERATORS.

Let γ be a complex number with $\operatorname{Re} \gamma \neq -1$. We define h_γ by

$$h_{\gamma}(z) = \sum_{j=1}^{\infty} \frac{\gamma+1}{\gamma+j} z^j, \quad z \in \Delta. \quad (2.1)$$

Let the operator $J: A \rightarrow A$ be defined by $F = J(f)$, where

$$F(z) = \frac{1+\gamma}{z^{\gamma}} \int_0^z f(t) t^{\gamma-1} dt \quad (2.2)$$

Then the function F can also be written in the form

$$F(z) = f(z) * h_{\gamma}(z).$$

We need the following result of Jack [9] which is also due to Suffridge [18]

LEMMA. Let w be nonconstant and analytic in $|z| < r < 1, w(0) = 0$. If $|w|$ attains its maximum value on the circle $|z| = r$ at z_0 , then $z_0 w'(z_0) = k w(z_0)$, where k is a real number and $k \geq 1$.

We first give a condition on $f \in A$ for which the function $J(f)$ belongs to $R_n(\alpha)$.

THEOREM 1. Let $0 \leq \alpha < 1$, and $\gamma \neq -1$ be a complex constant such that $\operatorname{Re} \gamma \geq -\alpha$, $\operatorname{Im} \gamma \geq 0$, and $|\gamma|^2 + 2\alpha(1 + \operatorname{Re} \gamma) \geq 1$. If for a given $n \in \mathbb{N}$, $f \in A$ satisfies the condition

$$\operatorname{Re} \frac{z(D^n f(z))'}{D^n f(z)} > \alpha - \frac{(1-\alpha)(\alpha + \operatorname{Re} \gamma)}{2\{|\gamma|^2 + 2\alpha \operatorname{Re} \gamma + \alpha^2 + (1-\alpha)\operatorname{Im} \gamma\}} \quad (2.3)$$

for all $z \in \Delta$, then $F(z)$ given by (2.2) belongs to $R_n(\alpha)$.

PROOF From (2.2), we obtain

$$z(D^n F(z))' + \gamma D^n F(z) = (\gamma+1)D^n f(z). \quad (2.4)$$

Define w in Δ by

$$\frac{z(D^n F(z))'}{D^n F(z)} = \frac{1+(2\alpha-1)w(z)}{1+w(z)}. \quad (2.5)$$

Here $w(z)$ is analytic in Δ with $w(0) = 0$ and $w(z) \neq -1$, $z \in \Delta$.

We need to show that $|w(z)| < 1$ for all $z \in \Delta$. In view of (2.4),

(2.5) yields

$$\frac{D^n f(z)}{D^n F(z)} = \frac{(1+\gamma) + (2\alpha-1+\gamma)w(z)}{(1+\gamma)(1+w(z))} \quad (2.6)$$

Differentiating (2.6) logarithmically and simplifying, we obtain

$$\frac{z(D^n f(z))'}{D^n f(z)} = \alpha + (1-\alpha) \frac{1-w(z)}{1+w(z)} - \frac{2(1-\alpha)zw'(z)}{(1+w(z))(1+\gamma+(2\alpha-1+\gamma)w(z))} \quad (2.7)$$

Now (2.7) should yield $|w(z)| < 1$ for all $z \in \Delta$ for otherwise, there exists a point $z_0 \in \Delta$ at which $|w(z_0)| = 1$ and by Lemma, we have $z_0 w'(z_0) = kw(z_0)$, $k \geq 1$. For this value of $z = z_0$, we find that (2.7) yields

$$\operatorname{Re} \frac{z_0(D^n f(z_0))'}{D^n f(z_0)} = \alpha - \frac{2k(1-\alpha)(\alpha+\operatorname{Re} \gamma)}{|(1+\gamma)+(2\alpha-1+\gamma)w(z_0)|^2} \quad (2.8)$$

$$\leq \alpha - \frac{(1-\alpha)(\alpha+\operatorname{Re} \gamma)}{2\{|\gamma|^2 + 2\alpha\operatorname{Re} \gamma + \alpha^2 + (1-\alpha)\operatorname{Im} \gamma\}}$$

which contradicts (2.3). Hence $|w(z)| < 1$ for all $z \in \Delta$ and by (2.5), it follows that $F(z) \in R_n(\alpha)$.

COROLLARY. If for a given $n \in \mathbb{N}_0$, $f \in A$ satisfies the condition

$$\operatorname{Re} \frac{z(D^n f(z))'}{D^n f(z)} > \frac{2\alpha(\gamma+\alpha)-(1-\alpha)}{2(\gamma+\alpha)}, \quad z \in \Delta, \quad (2.9)$$

where (α, γ) is any point in the set

$$D = \{(\alpha, \gamma) : \gamma + 2\alpha \geq 1, 0 \leq \alpha < 1, \gamma > -1\},$$

then $F(z)$ given by (2.2) belongs to $R_n(\alpha)$.

PROOF. If $\gamma \neq -1$ is a real constant such that $\gamma + \alpha \geq 0$, then

$$|\gamma|^2 + 2\alpha(1+\operatorname{Re} \gamma) \geq 1 \text{ implies } (\gamma+1)(\gamma+2\alpha-1) \geq 0. \text{ The result follows}$$

from Theorem 1

It is easy to show that if $f \in R_n(\alpha)$, then f satisfies the condition

(2.3). Thus it follows from Theorem 1 that $J(R_n(\alpha)) \subset R_n(\alpha)$. More precisely,

we state the result in

THEOREM 2. If $f \in R_n(\alpha)$, then the function

$$J(f) = \frac{\gamma+1}{z^\gamma} \int_0^z f(t) t^{\gamma-1} dt$$

is again an element of $R_n(\alpha)$, where $\gamma \neq -1$ is a complex constant with restrictions as stated in Theorem 1.

REMARK 1. Letting $n = 0 = \gamma - 1$ and $n = 1 = \gamma$, in Theorem 1, we get $L(S^*(\beta)) \subset S^*(\alpha)$ and $L(K(\beta)) \subset K(\alpha)$ respectively, where L is the Libera transform defined in (1.3), and

$$\beta = ((2\alpha^2 + 3\alpha - 1)/2(1+\alpha)) < \alpha.$$

These results improve the earlier results due to Libera [11] and Bernardi [5] in the sense that their results hold under much weaker conditions

In [2], Bajpai has established that $J(S^*) \subset S^*(\alpha)$ for some α . We generalize this result in

THEOREM 3. Let $J:A \rightarrow A$ be defined as in (2.2), where γ is a complex constant. If $f \in R_n$, then $J(f) \in R_n(\alpha)$, where α satisfies the inequality

$$\alpha(|1+\gamma|+|2\alpha-1+\gamma|)^2 \leq 2(1-\alpha)(\alpha+\operatorname{Re}\gamma), \quad \text{and } 0 \leq \alpha < 1$$

PROOF. Proceeding as in Theorem 1 and applying Lemma, we have

$$\begin{aligned} \operatorname{Re} \frac{z_0(D^n f(z_0))'}{D^n f(z_0)} &\leq \alpha - \frac{2(1-\alpha)(\alpha+\operatorname{Re}\gamma)}{(|1+\gamma|+(2\alpha-1+\gamma)w(z_0))^2} \\ &\leq \alpha - \frac{2(1-\alpha)(\alpha+\operatorname{Re}\gamma)}{(|1+\gamma|+|2\alpha-1+\gamma|)^2}, \end{aligned}$$

where $\operatorname{Re}\gamma \geq -\alpha$. Since the right hand side is ≤ 0 , we have a contradiction for $f \in R_n \equiv R_n(0)$. Thus we must have $|w(z)| < 1$ for all z in Δ and by (2.5), it follows that $J(f) \in R_n(\alpha)$.

REMARK 1. If we let $n=0=\gamma-1$ in the above theorem, then

$L(S^*) \subset S^*(\frac{\sqrt{17}-3}{4})$, where $L(f) = (2/z) \int_0^z f(t)dt$. Thus we have recovered a result of Miller, Mocanu and Reade ([14], pp 162-163).

REMARK 2. If $n=1$, γ is a real constant such that $\gamma+\alpha \geq 0$, and $f \in K$, then it follows from Theorem 3 that the function $F(z)$ in (2.2) is an element of $K(\alpha)$, where

$$\alpha = \frac{-(2\gamma+1) + \sqrt{(2\gamma-1)^2+8(1+\gamma)}}{4}.$$

This result was proved by Miller, Mocanu and Reade ([14], pp 165).

Further, this is an improvement of an earlier result due to Bernardi [5],

who proved that $f \in K$ implies $F \in K$.

For $\gamma = n$, where $n \in \mathbb{N}_0$, we have an improvement over Theorem 2

THEOREM 4. Let

$$F(z) = f(z) * h_n(z) = \frac{n+1}{z^n} \int_0^z f(t)t^{n-1}dt \quad (2.10)$$

If $f \in R_n(\alpha)$, then $F \in R_{n+1}(\alpha)$

PROOF. From (2.10), we obtain

$$z(D^{n+1}F(z))' + nD^{n+1}F(z) = (n+1)D^{n+1}f(z) \quad (2.11)$$

and

$$z(D^n F(z))' + nD^n F(z) = (n+1)D^n f(z) \quad (2.12)$$

Using the identity

$$z(D^n f(z))' = (n+1)D^{n+1} f(z) - nD^n f(z) \quad (2.13)$$

in (2.11) and (2.12), we obtain

$$(n+1)D^{n+1} f(z) = (n+2)D^{n+2} F(z) - D^{n+1} F(z) \quad (2.14)$$

and

$$D^n f(z) = D^{n+1} F(z) \quad (2.15)$$

In view of the identity (2.13) and the relations (2.14) and (2.15),

$f \in R_n(\alpha)$ yields

$$\operatorname{Re} \left\{ \frac{(n+2)D^{n+2} F(z) - (n+1)D^{n+1} F(z)}{D^{n+1} F(z)} \right\} > \alpha$$

which implies that

$$\operatorname{Re} \left\{ \frac{z(D^{n+1} F(z))'}{D^{n+1} F(z)} \right\} > \alpha, \quad z \in \Delta$$

This proves that $F \in R_{n+1}(\alpha)$.

REMARK For $n = 0$, Theorem 4 gives the well known result:

$$J(S^*(\alpha)) \subset K(\alpha), \text{ where } J(f) = \int_0^z (f(t)/t) dt$$

We now investigate the converse of Theorem 2. In fact, we find the sharp radius of the disc in which $f \in R_n(\beta)$ when F , defined in (2.2), is in $R_n(\alpha)$ for $0 \leq \alpha < 1$, $0 < \beta \leq 1$: In [12], Libera and Livingston have solved this converse problem for the case $n = 0$, $\gamma = 1$ when $\alpha \leq \beta < 1$. These authors were not able to obtain suitable results for the complementary case when $\beta < \alpha$. However, the method used in the next theorem gives results that are more general and also covers both $\beta \geq \alpha$ and $\beta < \alpha$.

THEOREM 5. If F is an element of $R_n(\alpha)$ for $n \geq 0$ and $0 \leq \alpha < 1$,

$$F(z) = \frac{1+\gamma}{z^\gamma} \int_0^z f(t) t^{\gamma-1} dt \quad (2.16)$$

with $z \in \Delta$, $\operatorname{Re} \gamma \geq -\alpha$, and $0 \leq \beta < 1$, then the function f is an element of $R_n(\beta)$ for $|z| < r_0$, where r_0 is the smallest positive root in $(0, 1)$ of the equation

$$(\gamma+2\alpha-1)(2\alpha-\beta-1)r^2 + 2((\gamma+\alpha)(\alpha-\beta) - (1-\alpha)(2-\alpha))r + (\gamma+1)(1-\beta) = 0 \quad (2.17)$$

The result is sharp

PROOF Since $F \in R_n(\alpha)$, we can write

$$\frac{z(D^n F(z))'}{D^n F(z)} = \alpha + (1-\alpha)P_n(z), \quad (2.18)$$

where $P_n(z)$ is analytic in Δ and satisfies the conditions $P_n(0) = 1$

$\operatorname{Re} P_n(z) > 0$ for $z \in \Delta$. Using the identity

$$z(D^n F(z))' = (n+1)D^{n+1}F(z) - nD^n F(z) \quad (2.19)$$

in (2.18) and then taking logarithmic derivative, we obtain

$$z(D^{n+1}F(z))' = D^{n+1}F(z) \left[\alpha + (1-\alpha)P_n(z) + \frac{(1-\alpha)zP_n'(z)}{n+\alpha+(1-\alpha)P_n(z)} \right] \quad (2.20)$$

From (2.16) we obtain

$$z(D^{n+1}F(z))' + \gamma D^{n+1}F(z) = (\gamma+1)D^{n+1}f(z). \quad (2.21)$$

From (2.20) and (2.21) we have

$$(\gamma+1)D^{n+1}f(z) = D^{n+1}F(z) \left[\alpha + \gamma + (1-\alpha)P_n(z) + \frac{(1-\alpha)zP_n'(z)}{n+\alpha+(1-\alpha)P_n(z)} \right] \quad (2.22)$$

Also (2.18) together with the identity (2.4) yields

$$(1+\gamma)D^n f(z) = D^n F(z)(\alpha + \gamma + (1-\alpha)P_n(z)). \quad (2.23)$$

Now from the relations (2.22), (2.23), and (2.18) we conclude that

$$\frac{z(D^n f(z))'}{D^n f(z)} - \beta = \alpha - \beta + (1-\alpha)P_n(z) + \frac{(1-\alpha)zP_n'(z)}{\alpha + \gamma + (1-\alpha)P_n(z)}. \quad (2.24)$$

Using the well known estimates

$$\left| zP_n'(z) \right| \leq (2r/(1-r^2))\operatorname{Re} P_n(z)$$

and

$$\operatorname{Re} P_n(z) \geq (1-r)/(1+r), \quad |z| = r$$

in (2.24), we obtain

$$\operatorname{Re} \left[\frac{z(D^n f(z))'}{D^n f(z)} - \beta \right] \geq (\alpha - \beta) + \frac{(1-\alpha)((1-r)(\gamma+1+(\gamma+2\alpha-1)r)-2r)}{(1-r)((\gamma+2\alpha-1)r+\gamma+1)} \quad (2.25)$$

where $\operatorname{Re} \gamma \geq -\alpha$. Therefore,

$$\operatorname{Re} \left\{ \frac{z(D^n f(z))'}{D^n f(z)} \right\} > \beta$$

if the right side of (2.25) is positive, which is satisfied provided that

$r < r_0$, where r_0 is the smallest positive root in $(0,1)$ of (2.17).

The result in the theorem is sharp with the function f defined by

$$f(z) = (1/(1+c))z^{1-c}(z^c F(z))', \quad (2.26)$$

where $c = \operatorname{Re} \gamma \geq -\alpha$, and F is given by

$$z \frac{(D^n F(z))'}{D^n F(z)} = \frac{1-(2\alpha-1)z}{1-z} \quad (2.27)$$

REMARK. By specializing choices of α, β, γ , and n , theorem 5 gives rise to the corresponding results obtained earlier in [3,4,8,12,13,15] and by many others

3 AN ARGUMENT THEOREM.

THEOREM 6 If $f \in R_n(\alpha)$, then

$$\left| \arg \frac{D^k f(z)}{z} \right| \leq 2(1-\alpha) \sin^{-1} r + \sum_{m=0}^{k-1} \sin^{-1} \left(\frac{2(1-\alpha)r}{m+1-(m+2\alpha-1)r^2} \right)$$

for each $k (0 \leq k \leq n+1)$.

PROOF We may write

$$\frac{D^k f(z)}{z} = \frac{f(z)}{z} \prod_{m=0}^{k-1} \frac{D^{m+1} f(z)}{D^m f(z)}, \quad 0 \leq k \leq n+1,$$

which yields

$$\left| \arg \frac{D^k f(z)}{z} \right| \leq \left| \arg \frac{f(z)}{z} \right| + \sum_{m=0}^{k-1} \left| \arg \frac{D^{m+1} f(z)}{D^m f(z)} \right|. \quad (3.1)$$

Since $R_{n+1}(\alpha) \subset R_n(\alpha) \quad \forall n \in N_0$, it follows that $f \in R_m(\alpha)$ for each

$m (0 \leq m \leq n)$ Setting

$$\frac{D^{m+1} f(z)}{D^m f(z)} = q_m(z), \quad (0 \leq m \leq n), \quad (3.2)$$

we note that $\operatorname{Re}(q_m(z)) \geq (m+\alpha)/(m+1)$

Therefore, the function

$$\begin{aligned} w(z) &= \frac{(q_m(z) - \frac{m+\alpha}{m+1}) - (1 - \frac{m+\alpha}{m+1})}{(q_m(z) - \frac{m+\alpha}{m+1}) + (1 - \frac{m+\alpha}{m+1})} \\ &= \frac{q_m(z) - 1}{q_m(z) - (\frac{2(m+\alpha)}{m+1} - 1)} \end{aligned}$$

is analytic with $w(0) = 0$ and $|w(z)| < 1$ in Δ Hence by Schwarz's

Lemma,

$$\left| \frac{q_m(z) - 1}{q_m(z) + 1 - 2(m+\alpha)/(m+1)} \right| < |z|$$

for z in Δ . Now it is easy to see that the values of $q_m(z)$ are contained in the circle of Apollonius whose centre is at the point $(m+1-(m+2\alpha-1)r^2)/((1+m)(1-r^2))$ and has radius $2(1-\alpha)r/((m+1)(1-r^2))$.

Thus $\max_{z \in \Delta} |\arg q_m(z)|$ is attained at the points where

$$\arg q_m(z) = \pm \sin^{-1} \left(\frac{2(1-\alpha)r}{m+1-(m+2\alpha-1)r} \right)$$

which gives

$$\left| \arg \frac{D^{m+1}f(z)}{D^m f(z)} \right| \leq \sin^{-1} \left(\frac{2(1-\alpha)r}{m+1-(m+2\alpha-1)r} \right), \quad (3.3)$$

for $0 \leq m \leq n$, and $z \in \Delta$.

Next, note that $R_n(\alpha) \subset S^*(\alpha)$, $n \geq 0$, and $f \in S^*(\alpha)$ if and only if $F(z) = \int (f(z)/z) dz$ is in $K(\alpha)$. But for $F \in K(\alpha)$, we have

$$|\arg F'(z)| \leq 2(1-\alpha)\sin^{-1} r \quad (|z| = r)$$

Thus $f \in R_n(\alpha)$ implies

$$\left| \arg \frac{f(z)}{z} \right| \leq 2(1-\alpha)\sin^{-1} r \quad (3.4)$$

Applying (3.3) and (3.4) to (3.1) we obtain the result.

For $n = 0$, we obtain

COROLLARY If $f \in S^*(\alpha)$, then (3.4)

and

$$|\arg f'(z)| \leq 2(1-\alpha)\sin^{-1} r + \sin^{-1} \left(\frac{2(1-\alpha)r}{1-(2\alpha-1)r^2} \right)$$

REMARK The case $n = 0$, $\alpha = 0$ was proved by Krzyz [10].

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This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

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