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SOME UNSOLVED PROBLEMS ON MEROMORPHIC FUNCTIONS OF UNIFORMLY BOUNDED CHARACTERISTIC

SHINJI YAMASHITA

Department of Mathematics
Tokyo Metropolitan University
Fukasawa, Setagaya,
Tokyo 158, Japan

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ABSTRACT. The family UBC(R) of meromorphic functions of uniformly bounded characteristic in a Riemann surface R is defined in terms of the Shimizu-Ahlfors characteristic function. There are some natural parallels between UBC(R) and BMOA(R), the family of holomorphic functions of bounded mean oscillation in R. After a survey some open problems are proposed in contrast with BMOA(R).

KEY WORDS AND PHRASES. Riemann surface, Shimizu-Ahlfors' characteristic function, UBC(R), BMOA(R), harmonic majorant, F. Riesz' decomposition, Carleson measure, classification of Riemann surfaces.
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1. INTRODUCTION:

In a series of papers [19] ~ [21], [23] ~ [25] I have been studying functions of uniformly bounded characteristic. After a survey I propose some questions which I have been unable to answer. The adjective "unsolved" in the present title, therefore, means more precisely "unsolved by the present author".

Let R be a Riemann surface which has the Green functions $g_R(z,w)$ with poles w in R. As usual, each point of R is identified with its local-parametric image in the complex plane. By D we always mean a subdomain of R such that the closure $D \cup \partial D$ is compact and the boundary ∂D consists of a finite number of mutually disjoint, analytic, simple and closed curves in R. For a point w of D we set

$$r = \exp\{\lim_{z\to w}(g_D(z,w) + \log|z - w|)\},$$

where $z \to w$ within the parametric disk of center w.

Let $D_t = \{z \in D;\ g_D(z,w) \geq \log(r/t)\}$, $0 < t < r$. For f meromorphic in R we consider the second-order differential $f^{\#}(z)^2 dxdy$, where $f^{\#} = |f'|/(1 + |f|^2)$. The Shimizu-Ahlfors characteristic function of f is defined for a pair w, D with $w \in D$ by

$$T(D,w,f) = \pi^{-1}\int_0^r t^{-1}[\iint_{D_t} f^{\#}(z)^2 dxdy]dt.$$

In the case $R = \Delta \equiv \{|z| < 1\}$, $w = 0$, $D = \{|z| < r\}$, $0 < r < 1$, we obtain

$$T(r,f) = T(D,w,f),$$

the usual Shimizu-Ahlfors characteristic function of f. Returning to general R we now set

$$T(R,w,f) = \lim_{w \in D \uparrow R} T(D,w,f) ;$$

this means that given $\varepsilon > 0$ we may find a compact set $K \subset R$ $(w \in K)$ such that $|T(R) - T(D)| < \varepsilon$ for all $D \supset K$ with the obvious change in case $T(R) = \infty$. It is known that

$$T(R,w,f) = \pi^{-1} \iint_R f^{\#}(z)^2 g_R(z,w)dxdy, \qquad w \in R,$$

a Green potential of the measure $\pi^{-1} f^{\#}(z)^2 dxdy$ in R.

If $T(R,w,f) < \infty$ for a $w \in R$, then $T(R,w,f) < \infty$ for all $w \in R$. We call f to be of bounded characteristic, $f \in BC \equiv BC(R)$ in notation, if $T(R,w,f) < \infty$ for a (hence for each) $w \in R$. Thus, $f \in BC$ if and only if f is Lindelöfian and meromorphic in R in the sense of Maurice Heins.

By definition, f is of uniformly bounded characteristic, $f \in UBC \equiv UBC(R)$ in notation, if

$$T(R,f) \equiv \sup_{w \in R} T(R,w,f) < \infty.$$

Each meromorphic f in R can be expressed as a quotient $f = f_1/f_2$ of holomorphic functions f_1 and f_2 with no common zero in R. Therefore,

$$\Phi = \log(|f_1|^2 + |f_2|^2)$$

is a finite-valued subharmonic function in R because $\Delta\Phi(z)dxdy = 4f^{\#}(z)^2 dxdy$. If $f \in BC$, then

$$\Phi_R^{\wedge}(w) - \Phi(w) = 2T(R,w,f), \qquad w \in R, \tag{1}$$

where $\Phi_R^{\wedge}$ is the least harmonic majorant of Φ in R, the smallest among all the harmonic majorants of Φ in R. Therefore, the function $T(R,w,f)$ of w is the potential part of the F. Riesz decomposition of Φ in R. Apparently, $f \in UBC$ if and only if $\Phi_R^{\wedge}$ exists (namely, $f \in BC$) and the potential part is bounded in R.

In the special case $R = \Delta$ we can choose as D the non-Euclidean disks

$$\Delta(w,r) = \{|z - w|/|1 - \bar{w}z| < r\}$$

of center $w \in \Delta$ and the radii $\tanh^{-1} r$, $0 < r < 1$, so that

$$T(\Delta(w,r),w,f) = T(r,f_w), \tag{2}$$

where $f_w(z) = f((z + w)/(1 + \bar{w}z)), \quad z \in \Delta.$

It is not difficult to observe that $f \in UBC(R) \Leftrightarrow f \circ p \in UBC(\Delta)$, where $p: \Delta \to R$ is the universal covering projection. Actually, $T(R,p(a),f) = T(\Delta, a, f \circ p)$. This fact reduces many problems on $UBC(R)$ to those on $UBC(\Delta)$, yet, in some cases, there are subtle differences; see the problem (I) below, for example.

Hitherto we have been concerned with meromorphic functions f in R. If f is

pole-free further, then we can propose similar considerations on replacing $f^{\#}$ by $|f'|$. Thus, beginning with

$$T^*(D,w,f) = \pi^{-1}\int_0^r t^{-1}[\iint_{D_t} |f'(z)|^2 dxdy]dt,$$

we obtain $T^*(R,w,f)$ and others. If $T^*(R,w,f) < \infty$ for a $w \in R$, then $f \in H^2(R)$, that is, $|f|^2$ has the least harmonic majorant $(|f|^2)^{\wedge}_R$ in R, so that, $T^*(R,w,f)$ is finite for all $w \in R$. The converse is also true. To observe these we remember that if $f \in H^2(R)$, then

$$(|f|^2)^{\wedge}_R(w) - |f|^2(w) = 2T^*(R,w,f), \quad w \in R, \tag{3}$$

an analogue of (1), holds [21], [23]. The left-hand side of (3) coincides with

$$(|f - f(w)|^2)^{\wedge}_R(w), \quad w \in R.$$

The obvious analogue of (2), together with (3), yields in Δ the identity

$$(|f_w - f(w)|^2)^{\wedge}_{\Delta}(0) = 2T^*(\Delta,f_w). \tag{4}$$

A holomorphic function f in R is called BMOA, $f \in BMOA \equiv BMOA(R)$ in notation, if

$$T^*(R,f) \equiv \sup_{w\in R} T^*(R,w,f) < \infty;$$

the notion of $BMOA(R)$ was first introduced by Thomas A. Metzger, and the present author extended it to many-valued functions [23]. In case $R = \Delta$, this coincides with the known class $BMOA(\Delta)$. The inclusion formula $BMOA(R) \subset UBCA(R)$ is obvious, where $UBCA(R)$ is the family of all the holomorphic functions in $UBC(R)$. This is a consequence of the obvious inequality $T(R,f) \leq T^*(R,f)$, yet a better estimate

$$T(R,f) \leq 2^{-1}\log\{2T^*(R,f) + 1\}$$

can be proved. There exists $f \in UBCA(\Delta) \setminus BMOA(\Delta)$.

It is known that if $f \in BMOA(\Delta)$, then f is Bloch, $f \in B(\Delta)$ in notation, in the sense that

$$\sup_{z\in\Delta}(1 - |z|^2)|f'(z)| < \infty.$$

Similarly, it is known [19] that if $f \in UBC(\Delta)$, then f is normal in the sense of Olli Lehto and Kaarlo I. Virtanen, $f \in N(\Delta)$ in notation, that is,

$$\sup_{z\in\Delta}(1 - |z|^2)f^{\#}(z) < \infty.$$

Both $B(\Delta)$ and $N(\Delta)$ can be extended to $B(R)$ and $N(R)$ because R has the hyperbolic metric. The inclusion formulae $BMOA(R) \subset B(R)$ and $UBC(R) \subset N(R)$ are proper in the case $R = \Delta$. (See the papers [1] ~ [18],[22], [25], on normal meromorphic functions and the related topics.) Algebraically, $BMOA(R)$ is closed for summation, while $UBC(R)$ is not; UBC resembles N at this point.

2. PROBLEMS:

Now, the problems:

(I) For $\alpha \in \mathbb{C}^* \equiv \{|z| \leq \infty\}$ we let $n(\alpha,f)$ be the number of the roots of the equation $f = \alpha$ in R. Suppose that there exists an integer $k \geq 0$ such that

$$\mathrm{cap}\{\alpha \in \mathbb{C}^*;\ n(\alpha,f) \leq k\} > 0,$$

where "cap" denotes the elliptic capacity. Is it true that $f \in UBC(R)$? This is

valid for R in case $k = 0$, and is valid for $R = \Delta$ and $k \geq 0$. *The problem is unsolved for* $k > 0$ *on general* R. Earlier and well-known conclusion is that $f \in BC(R)$.

(II) Let $A(R,f)$ be the spherical area of the image $f(R)$ of R, that is, the projection to $\mathbb{C}^*$ of the Riemannian image of R by f. *Is there a constant* $k > 0$ *such that*

$$T(R,f) \leq kA(R,f) \ ? \tag{5}$$

Since $A(R,f) \leq \pi$ always holds if the sphere $\mathbb{C}^*$ is considered to have the radius $1/2$, the answer is "no" in general. We must therefore add the condition that $A(R,f) < \pi$; in this case $T(R,f) < \infty$ is obvious; see (I). The celebrated Herbert J. Alexander-B. A. Taylor-Joseph L. Ullman inequality teaches us that for holomorphic f,

$$T^*(R,f) \leq (2\pi)^{-1}A^*(R,f),$$

where, in this case, $A^*(R,f)$ is the Euclidean area of $f(R)$. See the problem (VII) below.

(III) As usual, let O_X be the family of open Riemann surfaces R of class O_G or of those which are hyperbolic with $X(R) = \mathbb{C}^* \setminus \{\infty\}$. It is easy to observe that

$$O_{UBCA} \subset O_{BMOA}.$$

To prove that the inclusion is proper we make a few modifications to the argument in [23]. Let E be a compact set of linear measure zero, yet of positive capacity, lying on the real axis. Let $a \in E$ and let $h(z) = 1/(z - a)$, $z \in \mathbb{C}^*$. We show that $R = \mathbb{C}^* \setminus h(E)$ is of $O_H{}^2$ (hence of O_{BMOA}), yet $R \notin O_{UBCA}$. First, the function z is of $UBCA(R)$ because it omits the set $h(E)$ of positive capacity, so that $R \notin O_{UBCA}$. Next, let $f \in H^2(R)$. Then $f \circ h \in H^2(\mathbb{C}^* \setminus E)$. Since E is removable for H^2, $f \circ h$, and hence f must be a constant. Therefore, $R \in O_H{}^2$. *The problem is to find a reasonable* X *for which* $O_{UBCA} \subsetneq O_X \subsetneq O_{BMOA}$.

(IV) If $f \in UBC(R)$, then $f \in UBC(R_1)$ for each subdomain R_1 of R. Consider the converse in the special case $R = \Delta$. Let $\delta(w,r) = \Delta \cap \{|z - w| < r\}$, $w \in \partial\Delta$, $0 < r < 1$. Let f be meromorphic in Δ such that for each $w \in \partial\Delta$ we may choose $0 < r < 1$ such that $f \in UBC(\delta(w,r))$. *Is it true that* $f \in UBC(\Delta)$? The corresponding problem for $BMOA$ is solved in the positive in [21]; the extensions to the bordered Riemann surfaces under the obvious technical conditions are now easy.

(V) Each $f \in UBC(\Delta)$ has, as a member of $BC(\Delta)$, the expression: $f = (B_1/B_2)F$, where B_1 and B_2 are Blaschke products with no common zero and F is holomorphic and zero-free in Δ. We observe that $F \in UBC(\Delta)$. Conversely suppose that $F \in UBC(\Delta)$ for $f \in BC(\Delta)$. Is it true that $f \in UBC(\Delta)$? The answer is "no" [19]. Actually, there exists a nonnormal Blaschke quotient B_1/B_2 ; we have only to let $F = 1$. *What is a reasonable condition for* $B_1/B_2 \in UBC(\Delta)$? An attempt is proposed in [25] in terms of uniformly separated sequences.

(VI) For $w \in W$ we set

$$\Sigma(w) = \{z \in \Delta;\ |w| \leq |z| < 1,\ |\arg(z/w)| < \pi(1 - |w|)\}, \qquad \text{if } w \neq 0;$$

$$= \Delta, \qquad \text{if } w = 0.$$

The length of the arc $\partial\Delta \cap \partial\Sigma(w)$ is $\ell(w) = 2\pi(1 - |w|)$. A measure $\mu \geq 0$ in Δ is called a Carleson measure if $\sup \mu(\Sigma(w))/\ell(w) < \infty$, where w ranges over Δ. It is known that $f \in BMOA(\Delta)$ if and only if $(1 - |z|^2)|f'(z)|^2 dxdy$ is a Carleson measure (in the differential form). I proved [24] that if $f \in UBC(\Delta)$, then

$$d\mu_f(z) = (1 - |z|^2)f^{\#}(z)^2 dxdy$$

is a Carleson measure, and gave a partial answer for the converse. *The problem is:* $d\mu_f$ *is a Carleson measure.* $\Rightarrow$ $f \in UBC(\Delta)$? I note that a difficulty lies in the fact that $f^{\#2}$ is not subharmonic in general.

(VII) Let $UBC_0(R)$ be the family of meromorphic functions f in R such that $T(R,w,f) \to 0$ as $w \to \partial R$, namely, given $\varepsilon > 0$ we may find a compact $K \subset R$ such that $T(R,w,f) < \varepsilon$ for $w \in R \setminus K$; this can be read in case $R = \Delta$, $T(1,f_w) \to 0$ as $|w| \to 1$; the holomorphic analogue is $VMOA(R)$ obtained on replacing T by T^* in the above. It is not difficult to prove that $UBC_0(R) \subset UBC(R)$ and $VMOA(R) \subset BMOA(R)$. Furthermore, we see

$$f \circ p \in UBC_0(\Delta) \Rightarrow f \in UBC_0(R) \text{ ; and } f \circ p \in VMOA(\Delta) \Rightarrow f \in VMOA(R),$$

because the projection of a closed disk in Δ is compact in R. *Are the converses valid ?* We remark that the following are known:

$$\iint_\Delta f^{\#}(z)^2 dxdy < \infty \Rightarrow f \in UBC_0(\Delta); \text{ and } \iint_\Delta |f'(z)|^2 dxdy < \infty \Rightarrow f \in VMOA(\Delta);$$

see [19] and [24]. *Are the extensions true for* R *?* Metzger, in a communication, informed me some partial answers on the VMOA part. I do not know further information.

Finally we note that the hyperbolic analogue of $UBC(R)$ is possible; see [21], [26], and a forthcoming paper [27]. Let f be holomorphic and bounded, $|f| < 1$, in R, and let $\Psi = -\log(1 - |f|^2)$. Then Ψ is subharmonic with $\Delta\Psi(z)dxdy = 4|f'(z)|^2/(1 - |f(z)|^2)^2 dxdy$. The analogue of (1), and others, are true on replacing $f^{\#}$ by the hyperbolic derivative $|f'|/(1 - |f|^2)$. We note that $\sigma(f,0) = \tanh^{-1}|f|$ is the non-Euclidean hyperbolic distance of f and 0, and $\sigma(f,0)$ is subharmonic with $\Psi \leq 2\sigma(f,0) \leq \Psi + \log 4$. It is a future task to find some problems on this family.

The present article depends on the lecture on February 4, 1985, at Mie University, Tsu, Mie, Japan, where slightly abstract treatment beginning with the F. Riesz decomposition of subharmonic function was discussed.

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