

LAPLACE TRANSFORM PAIRS OF N-DIMENSIONS

R. S. DAHIYA

Department of Mathematics
Iowa State University
Ames, Iowa 50011

(Received February 22, 1982)

ABSTRACT. In this paper I prove a theorem to obtain new n-dimensional Laplace transform pairs.

KEY WORDS AND PHRASES. Laplace transforms, multiple transforms.

1980 AMS SUBJECT CLASSIFICATION CODE. 44A30

1. INTRODUCTION.

The generalization of the well-known Laplace transform

$$L\{f(t) ; s\} = \int_0^\infty \exp(-st)f(t)dt \quad (1.1)$$

to n-dimensional Laplace transform is represented as follows:

$$L_n\{f(t_1, t_2, \dots, t_n; s_1, s_2, \dots, s_n)\} = L_n\{f\}$$

$$= \int_0^\infty \int_0^\infty \dots \int_0^\infty \exp\left(-\sum_{k=1}^n s_k t_k\right) f dt_1 dt_2 \dots dt_n. \quad (1.2)$$

In this paper I consider a method of computing Laplace transform pairs of n-dimensions from known one-dimensional Laplace transforms. The multi-dimensional Laplace transform pairs are useful in the solution of partial differential equations (see [1], [3] and [4]).

2. THEOREM. Let

$$(i) L_1\{f(t); s\} = \phi(s)$$

$$(ii) L_1\{\sqrt{t} \phi(\frac{1}{t}); s\} = F(s)$$

$$(iii) L_1\{t^3 f(t^4); s\} = G(s)$$

$$(iv) L_1\{t^4 f(t^4); s\} = H(s)$$

and let $f(t)$, $\sqrt{t} \phi(\frac{1}{t})$, $t^3 f(t^4)$, $t^4 f(t^4)$ be continuous and absolutely integrable in $(0, \infty)$. Then

$$\begin{aligned}
 L_n & \left\{ \frac{\left(\frac{1}{t_1} + \dots + \frac{1}{t_n}\right)^3}{(t_1 \dots t_n)^{1/2}} F\left[\frac{1}{64} \left(\frac{1}{t_1} + \dots + \frac{1}{t_n}\right)^2; s_1, \dots, s_n\right] \right\} \\
 & = 2^{10} \frac{\pi^{\frac{n+1}{2}}}{(s_1 \dots s_n)^{1/2}} \frac{G(\sqrt{s_1} + \dots + \sqrt{s_n})}{(s_1 \dots s_n)^{1/2}} + 2^9 \frac{\pi^{\frac{n+1}{2}}}{(s_1 \dots s_n)^{1/2}} \frac{\sqrt{s_1} + \dots + \sqrt{s_n}}{(s_1 \dots s_n)^{1/2}} H(\sqrt{s_1} + \dots + \sqrt{s_n}),
 \end{aligned}$$

n = 2, 3, 4, ... (2.1)

provided the integral on the left exists as an absolutely convergent in each of the variables.

PROOF: From (i), we have

$$\begin{aligned}
 \phi\left(\frac{1}{s}\right) &= \int_0^\infty e^{-t/s} f(t) dt = \int_0^\infty e^{-u/s} f(u) du, \\
 \sqrt{t} \phi\left(\frac{1}{t}\right) &= \int_0^\infty \sqrt{t} e^{-u/t} f(u) du. \quad (2.2)
 \end{aligned}$$

Let us multiply both sides of (2.2) by e^{-st} , $\operatorname{Re}(s) > 0$, and integrate between the limits $(0, \infty)$. Then on changing the order of integrations on the resulting right hand integral (permissible by Fubini's theorem, on account of absolute convergence), we obtain

$$\int_0^\infty e^{-st} \sqrt{t} \phi\left(\frac{1}{t}\right) dt = \int_0^\infty f(u) \left[\int_0^\infty \sqrt{t} e^{-st-u/t} dt \right] du.$$

We then evaluate the inner integral on the right (see [5], page 22) and use (ii) on the left to get the following result:

$$\begin{aligned}
 F(s) &= \frac{\sqrt{\pi}}{2} \int_0^\infty (1 + 2\sqrt{us}) s^{-3/2} e^{-2\sqrt{us}} f(u) du, \\
 s^{3/2} F(s) &= \frac{\sqrt{\pi}}{2} \int_0^\infty (1 + 2\sqrt{us}) e^{-2\sqrt{us}} f(u) du. \quad (2.3)
 \end{aligned}$$

Next let us write (2.3) in the form

$$\begin{aligned}
 & \left(\frac{1}{t_1} + \dots + \frac{1}{t_n}\right)^3 F\left[\frac{1}{64} \left(\frac{1}{t_1} + \dots + \frac{1}{t_n}\right)^2\right] \\
 &= 256\sqrt{\pi} \int_0^\infty e^{-\frac{\sqrt{u}}{4} \sum \frac{1}{t_i}} f(u) du + 64\sqrt{\pi} \int_0^\infty \left(\frac{1}{t_1} + \dots + \frac{1}{t_n}\right) e^{-\frac{\sqrt{u}}{4} \sum \frac{1}{t_i} \cdot \sqrt{u} f(u)} du \\
 &= 1024\sqrt{\pi} \int_0^\infty e^{-\frac{u^2}{4} \sum \frac{1}{t_i^2} u^3} f(u^4) du + 256\sqrt{\pi} \int_0^\infty \left(\frac{1}{t_1} + \dots + \frac{1}{t_n}\right) e^{-\frac{u^2}{4} \sum \frac{1}{t_i^2} u^5} f(u^4) du
 \end{aligned}$$

We multiply both sides by $(t_1 \dots t_n)^{-1/2} \exp(-\sum s_i t_i)$, integrate with respect to t_i between the limits $(0, \infty)$ and then change the order of integrations in the resulting integral on the right, permissible by Fubini's theorem, on account of absolute convergence.

This gives

$$\begin{aligned}
 & \int_0^\infty \dots \int_0^\infty \exp(-\sum s_i t_i) \frac{(\frac{1}{t_1} + \dots + \frac{1}{t_n})^3}{(t_1 \dots t_n)^{1/2}} F[\frac{1}{64}(\frac{1}{t_1} + \dots + \frac{1}{t_n})^2] dt_1 \dots dt_n \\
 &= 1024\sqrt{\pi} \int_0^\infty u^3 f(u^4) \left[\int_0^\infty \frac{1}{\sqrt{t_1}} \exp(-s_1 t_1 - \frac{u^2}{4t_1}) dt_1 \dots \int_0^\infty \frac{1}{\sqrt{t_n}} \exp(-s_n t_n - \frac{u^2}{4t_n}) dt_n \right] du \\
 &+ 256\sqrt{\pi} \int_0^\infty u^5 f(u^4) \left[\int_0^\infty \dots \int_0^\infty \left(\frac{1}{t_1^{3/2} \sqrt{t_2 \dots t_n}} + \frac{1}{t_1^{3/2} \sqrt{t_2 \dots t_n}} + \dots + \frac{1}{t_1 \dots t_{n-1} t_n^{3/2}} \right) \right. \\
 &\quad \left. \cdot \exp(-s_1 t_1 - \frac{u^2}{4} \sum \frac{1}{t_i}) dt_1 \dots dt_n \right] du. \quad (2.4)
 \end{aligned}$$

Evaluating the inner integrals on the right by (see [5], page 22, results 6 and 7)

$$\int_0^\infty \frac{1}{\sqrt{t}} \exp(-st - \frac{u^2}{4t}) dt = \sqrt{\frac{\pi}{s}} e^{-u\sqrt{s}}, \quad \int_0^\infty t^{-3/2} \exp(-st - \frac{u^2}{4}) dt = \frac{2\sqrt{\pi}}{u} e^{-u\sqrt{s}}$$

we get

$$\begin{aligned}
 & L_n \left\{ \frac{(\frac{1}{t_1} + \dots + \frac{1}{t_n})^3}{(t_1 \dots t_n)^{1/2}} F[\frac{1}{64}(\frac{1}{t_1} + \dots + \frac{1}{t_n})^2]; s_1, \dots, s_n \right\} \\
 &= 1024 \frac{\pi^{\frac{n+1}{2}}}{(s_1 \dots s_n)^{1/2}} \int_0^\infty \exp(-u \sum \sqrt{s_i}) u^3 f(u^4) du \\
 &+ 512 \frac{\pi^{\frac{n+1}{2}}}{(s_1 \dots s_n)^{1/2}} \frac{(\sqrt{s_1} + \dots + \sqrt{s_n})}{\sum \sqrt{s_i}} \int_0^\infty \exp(-u \sum \sqrt{s_i}) u^4 f(u^4) du.
 \end{aligned} \quad (2.5)$$

The proof is complete if we use (iii) and (iv) on the right hand side of (2.5).

3. APPLICATIONS: n-dimensional Laplace transform pairs.

$$\text{Let } f(t) = t^v; \text{ so that } L_1\{t^v; s\} = \frac{\Gamma(v+1)}{s^{v+1}} = \phi(s). \text{ Then}$$

$$L_1\{\sqrt{t} \phi((\frac{1}{t}), s) = L_1\{\Gamma(v+1)t^{v+3/2}; s\} = \frac{\Gamma(v+1)\Gamma(v+5/2)}{s^{v+5/2}} = F(s),$$

$$L_1\{t^3 f(t^4); s\} = L_1\{t^{4v+3}; s\} = \frac{\Gamma(4v+4)}{s^{4v+4}} = G(s),$$

$$L_1\{t^4 f(t^4); s\} = L_1\{t^{4v+4}; s\} = \frac{\Gamma(4v+5)}{s^{4v+5}} = H(s). \quad \text{Hence from (2.1), we get}$$

$$\begin{aligned} L_n\left(\left(t_1 \dots t_n\right)^{-1/2}\left(\frac{1}{t_1}+\dots+\frac{1}{t_n}\right)^{-2v-2}; s_1, \dots, s_n\right) \\ = \frac{\frac{\pi^{\frac{n+1}{2}}}{8^{2v-7}} \frac{\Gamma(4v+4)}{\Gamma(v+1)\Gamma(v+5/2)}}{\left(s_1 \dots s_n\right)^{-1/2}\left(\sqrt{s_1}+\dots+\sqrt{s_n}\right)^{-4v-4}} \\ + \frac{\frac{\pi^{\frac{n+1}{2}}}{8^{2v-6}} \frac{\Gamma(4v+5)}{\Gamma(v+1)\Gamma(v+5/2)}}{\left(s_1 \dots s_n\right)^{-1/2}\left(\sqrt{s_1}+\dots+\sqrt{s_n}\right)^{-4v-4}}. \quad (3.1) \end{aligned}$$

Similarly if we take f to be the following

$$f(t) = \begin{cases} t^{c-1} {}_0F_3(a, b, c; kt) \\ t^v \exp(-\sqrt{t}) \\ J_v^2(\sqrt{2t}) \\ t^a {}_pF_q \left(\begin{matrix} (a) \\ (b) \end{matrix}; t \right) \end{cases}.$$

in the theorem, then we obtain the following n -dimensional Laplace transform pairs:

$$\begin{aligned} L_n\left\{\frac{\left(t_1 \dots t_n\right)^{-1/2}}{\left(\frac{1}{t_1}+\dots+\frac{1}{t_n}\right)^{2c}} {}_1F_2\left[a, b; \frac{64k}{\left(\frac{1}{t_1}+\dots+\frac{1}{t_n}\right)^2}\right]; s_1, \dots, s_n\right\} \\ = \frac{2\pi^{\frac{n+1}{2}} \frac{\Gamma(4c)}{\Gamma(c)\Gamma(c+3/2)} \left(s_1 \dots s_n\right)^{-1/2}}{8^{2c} \left(\sqrt{s_1}+\dots+\sqrt{s_n}\right)^{4c}} {}_4F_3\left[\begin{matrix} 2c, 2c+1/2, 2c+1, 2c+2 \\ a, b, c; \end{matrix} \frac{256k}{\left(\sqrt{s_1}+\dots+\sqrt{s_n}\right)^4}\right] \\ + \frac{\pi^{\frac{n+1}{2}} \frac{\Gamma(4c+1)}{\Gamma(c)\Gamma(c+3/2)} \left(s_1 \dots s_n\right)^{-1/2}}{8^{2c} \left(\sqrt{s_1}+\dots+\sqrt{s_n}\right)^{4c}} {}_4F_3\left[\begin{matrix} 2c+1/2, 2c+1, 2c+3/2, 2c+2 \\ a, b, c; \end{matrix} \frac{256k}{\left(\sqrt{s_1}+\dots+\sqrt{s_n}\right)^4}\right], \\ \operatorname{Re}(c) > 0. \quad (3.2) \end{aligned}$$

$$L_n \left\{ \frac{\frac{1}{t_1} + \dots + \frac{1}{t_n}}{(t_1 \dots t_n)^{1/2}} \left(\frac{1}{64} \left(\frac{1}{t_1} + \dots + \frac{1}{t_n} \right)^2 - \frac{1}{4} \right)^{-v-3/2} P_1^{-2v-3} \left(\frac{4}{\frac{1}{t_1} + \dots + \frac{1}{t_n}} \right); s_1, \dots, s_n \right\}$$

$$= \frac{\pi^{n/2} \Gamma(4v+4)}{2^{v-4} \Gamma(2v-5)} (s_1 \dots s_n)^{-1/2} \exp\left(\frac{1}{8}(\sqrt{s_1} + \dots + \sqrt{s_n})^2\right) D_{-4v-4} \left(\frac{1}{\sqrt{2}} (\sqrt{s_1} + \dots + \sqrt{s_n}) \right) \\ + \frac{\pi^{n/2} \Gamma(4v+5)}{2^{v-5/2} \Gamma(2v+1)} (s_1 \dots s_n)^{1/2} \exp\left(\frac{1}{8}(\sqrt{s_1} + \dots + \sqrt{s_n})^2\right) D_{-4v-5} \left(\frac{1}{\sqrt{2}} (\sqrt{s_1} + \dots + \sqrt{s_n}) \right),$$

$$\operatorname{Re}(v) > -1. \quad (3.3)$$

$$L_n \left\{ \frac{(t_1 \dots t_n)^{-1/2} \left(\frac{1}{t_1} + \dots + \frac{1}{t_n} \right)^3}{\left(\frac{1}{64} \left(\frac{1}{t_1} + \dots + \frac{1}{t_n} \right)^2 + 1 \right) - 1} Q_{v-1/2}^2 \left(\frac{1}{64} \left(\frac{1}{t_1} + \dots + \frac{1}{t_n} \right)^2 + 1 \right); s_1, \dots, s_n \right\}$$

$$= \frac{\frac{\pi^{n+1}}{2} \sin(v+3/2) \pi (\sqrt{s_1} + \dots + \sqrt{s_n})^{v/2}}{2^{\frac{v}{8}-9} \sin(v-1/2) \pi (s_1 \dots s_n)^{1/2}} \\ \cdot G_{50}^{03} \left\{ \frac{8\sqrt{2}}{(\sqrt{s_1} + \dots + \sqrt{s_n})^2} \left| \begin{array}{l} 2 - \frac{v}{3}, -\frac{v}{4}, -\frac{1}{2} - \frac{v}{4}, \frac{3}{2} + \frac{v}{4}, 2 + \frac{v}{4} \end{array} \right. \right\} \\ + \frac{\frac{\pi^{\frac{n}{2}+1}}{2} \sin(v+3/2) \pi (\sqrt{s_1} + \dots + \sqrt{s_n})^{\frac{v+3}{2}}}{s^{v-73/8} \sin(v-1/2) \pi (s_1 \dots s_n)^{1/2}}. \\ G_{50}^{03} \left\{ \frac{8\sqrt{2}}{(\sqrt{s_1} + \dots + \sqrt{s_n})^2} \left| \begin{array}{l} \frac{9}{4} - \frac{v}{4}, -\frac{1}{4} - \frac{v}{4}, -\frac{3}{4} - \frac{v}{4}, \frac{7}{4} + \frac{v}{4}, \frac{9}{4} + \frac{v}{4} \end{array} \right. \right\} \quad (3.4)$$

$$L_n \left\{ \frac{(t_1 \dots t_n)^{-1/2}}{\left(\frac{1}{t_1} + \dots + \frac{1}{t_n} \right)^{2\alpha+2}} {}_{p+2}F_q \left[\begin{array}{l} (a), \alpha+1, \alpha+5/2; \\ (b); \end{array} \frac{64}{\left(\frac{1}{t_1} + \dots + \frac{1}{t_n} \right)^2} \right]; s_1, \dots, s_n \right\} \\ = \frac{\pi^{\frac{n+1}{2}} \Gamma(4\alpha+4) (s_1 \dots s_n)^{-1/2}}{4(8)^{2\alpha+1} \Gamma(\alpha+1) \Gamma(\alpha+5/2) (\sqrt{s_1} + \dots + \sqrt{s_n})^{4\alpha+4}} \\ \cdot {}_{p+4}F_q \left[\begin{array}{l} (a), 2\alpha+2, 2\alpha+5/2, 2\alpha+3, 2\alpha+7/2; \\ (b); \end{array} \frac{256}{(\sqrt{s_1} + \dots + \sqrt{s_n})^4} \right] \\ + \frac{\pi^{\frac{n+1}{2}} \Gamma(4\alpha+5) (s_1 \dots s_n)^{-1/2}}{8^{2\alpha+2} \Gamma(\alpha+1) \Gamma(\alpha+5/2) (\sqrt{s_1} + \dots + \sqrt{s_n})^{4\alpha+4}} \\ \cdot {}_{p+4}F_q \left[\begin{array}{l} (a), 2\alpha+3, 2\alpha+7/2, 2\alpha+4, 2\alpha+9/2; \\ (b); \end{array} \frac{256}{(\sqrt{s_1} + \dots + \sqrt{s_n})^4} \right],$$

$$\operatorname{Re}(\alpha) > 1. \quad (3.5)$$

REFERENCES

- [1] Ditkin, V.A., and Prudnikov, A. P., Operational calculus in two variables and its application (English translation Pergmon Press. London (1962)).
- [2] Erdelyi, A. W. Magnus, F. Oberhettinger, F. Tricomi, Tables of Integral Transforms. Vol.1, McGraw-Hill, New York (1954).
- [3] Estrin, T. A. and Higgins, T. J., Solutions of boundary value problem by multiple Laplace transformation. Jour. Frank. Inst. 252 (1951), 152-167.
- [4] Jaeger, J., The solution of boundary value problems by a double Laplace transformation. Bull. Amer. Math. Soc. 46 (1940), 687-693.
- [5] Roberts, G. E. and Kaufman, H., Table of Laplace transforms. W. B. Saunders Company (1966).

Special Issue on Time-Dependent Billiards

Call for Papers

This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

Authors should follow the Mathematical Problems in Engineering manuscript format described at <http://www.hindawi.com/journals/mpe/>. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/> according to the following timetable:

Manuscript Due	March 1, 2009
First Round of Reviews	June 1, 2009
Publication Date	September 1, 2009

Guest Editors

Edson Denis Leonel, Department of Statistics, Applied Mathematics and Computing, Institute of Geosciences and Exact Sciences, State University of São Paulo at Rio Claro, Avenida 24A, 1515 Bela Vista, 13506-700 Rio Claro, SP, Brazil; edleonel@rc.unesp.br

Alexander Loskutov, Physics Faculty, Moscow State University, Vorob'evy Gory, Moscow 119992, Russia; loskutov@chaos.phys.msu.ru