

LAPLACE TRANSFORM PAIRS OF N-DIMENSIONS

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(Received February 22, 1982)

ABSTRACT. In this paper I prove a theorem to obtain new n-dimensional Laplace transform pairs.

KEY WORDS AND PHRASES. Laplace transforms, multiple transforms.

1980 AMS SUBJECT CLASSIFICATION CODE. 44A30

1. INTRODUCTION.

The generalization of the well-known Laplace transform

$$L\{f(t) ; s\} = \int_0^{\infty} \exp(-st)f(t)dt \quad (1.1)$$

to n-dimensional Laplace transform is represented as follows:

$$\begin{aligned} L_n\{f(t_1, t_2, \dots, t_n; s_1, s_2, \dots, s_n)\} &= L_n\{f\} \\ &= \int_0^{\infty} \int_0^{\infty} \dots \int_0^{\infty} \exp\left(-\sum_{k=1}^n s_k t_k\right) f \, dt_1 dt_2 \dots dt_n. \end{aligned} \quad (1.2)$$

In this paper I consider a method of computing Laplace transform pairs of n-dimensions from known one-dimensional Laplace transforms. The multi-dimensional Laplace transform pairs are useful in the solution of partial differential equations (see [1], [3] and [4]).

2. THEOREM. Let

- (i) $L_1\{f(t); s\} = \phi(s)$
- (ii) $L_1\{\sqrt{t} \phi(\frac{1}{t}); s\} = F(s)$
- (iii) $L_1\{t^3 f(t^4); s\} = G(s)$
- (iv) $L_1\{t^4 f(t^4); s\} = H(s)$

and let $f(t)$, $\sqrt{t} \phi(\frac{1}{t})$, $t^3 f(t^4)$, $t^4 f(t^4)$ be continuous and absolutely integrable in $(0, \infty)$. Then

$$L_n \left\{ \frac{(\frac{1}{t_1} + \dots + \frac{1}{t_n})^3}{(t_1 \dots t_n)^{1/2}} F[\frac{1}{64}(\frac{1}{t_1} + \dots + \frac{1}{t_n})^2]; s_1, \dots, s_n \right\} \\ = 2^{10} \pi^{\frac{n+1}{2}} \frac{G(\sqrt{s_1} + \dots + \sqrt{s_n})}{(s_1 \dots s_n)^{1/2}} + 2^9 \pi^{\frac{n+1}{2}} \frac{\sqrt{s_1} + \dots + \sqrt{s_n}}{(s_1 \dots s_n)^{1/2}} H(\sqrt{s_1} + \dots + \sqrt{s_n}),$$

$$n = 2, 3, 4, \dots$$

$$(2.1)$$

provided the integral on the left exists as an absolutely convergent in each of the variables.

PROOF: From (i), we have

$$\phi(\frac{1}{s}) = \int_0^\infty e^{-t/s} f(t) dt = \int_0^\infty e^{-u/s} f(u) du, \\ \sqrt{t} \phi(\frac{1}{t}) = \int_0^\infty \sqrt{t} e^{-u/t} f(u) du. \quad (2.2)$$

Let us multiply both sides of (2.2) by e^{-st} , $\text{Re}(s) > 0$, and integrate between the limits $(0, \infty)$. Then on changing the order of integrations on the resulting right hand integral (permissible by Fubini's theorem, on account of absolute convergence), we obtain

$$\int_0^\infty e^{-st} \sqrt{t} \phi(\frac{1}{t}) dt = \int_0^\infty f(u) \left[\int_0^\infty \sqrt{t} e^{-st-u/t} dt \right] du.$$

We then evaluate the inner integral on the right (see [5], page 22) and use (ii) on the left to get the following result:

$$F(s) = \frac{\sqrt{\pi}}{2} \int_0^\infty (1 + 2\sqrt{us}) s^{-3/2} e^{-2\sqrt{us}} f(u) du, \\ s^{3/2} F(s) = \frac{\sqrt{\pi}}{2} \int_0^\infty (1 + 2\sqrt{us}) e^{-2\sqrt{us}} f(u) du. \quad (2.3)$$

Next let us write (2.3) in the form

$$(\frac{1}{t_1} + \dots + \frac{1}{t_n})^3 F[\frac{1}{64}(\frac{1}{t_1} + \dots + \frac{1}{t_n})^2] \\ = 256\sqrt{\pi} \int_0^\infty e^{-\frac{\sqrt{u}}{4} \sum \frac{1}{t_i}} f(u) du + 64\sqrt{\pi} \int_0^\infty (\frac{1}{t_1} + \dots + \frac{1}{t_n}) e^{-\frac{\sqrt{u}}{4} \sum \frac{1}{t_i}} \sqrt{u} f(u) du \\ = 1024\sqrt{\pi} \int_0^\infty e^{-\frac{u^2}{4} \sum \frac{1}{t_i^3}} t_i^3 f(u^4) du + 256\sqrt{\pi} \int_0^\infty (\frac{1}{t_1} + \dots + \frac{1}{t_n}) e^{-\frac{u^2}{4} \sum \frac{1}{t_i^3}} t_i^5 f(u^4) du$$

We multiply both sides by $(t_1 \dots t_n)^{-1/2} \exp(-\sum s_i t_i)$, integrate with respect to t_i between the limits $(0, \infty)$ and then change the order of integrations in the resulting integral on the right, permissible by Fubini's theorem, on account of absolute convergence.

This gives

$$\begin{aligned} & \int_0^\infty \dots \int_0^\infty \exp(-\sum s_i t_i) \frac{(\frac{1}{t_1} + \dots + \frac{1}{t_n})^3}{(t_1 \dots t_n)^{1/2}} F[\frac{1}{64}(\frac{1}{t_1} + \dots + \frac{1}{t_n})^2] dt_1 \dots dt_n \\ &= 1024\sqrt{\pi} \int_0^\infty u^3 f(u^4) \left[\int_0^\infty \frac{1}{\sqrt{t_1}} \exp(-s_1 t_1 - \frac{u^2}{4t_1}) dt_1 \dots \int_0^\infty \frac{1}{\sqrt{t_n}} \exp(-s_n t_n - \frac{u^2}{4t_n}) dt_n \right] du \\ &+ 256\sqrt{\pi} \int_0^\infty u^5 f(u^4) \left[\int_0^\infty \dots \int_0^\infty \left(\frac{1}{t_1^{3/2} \sqrt{t_2} \dots t_n} + \frac{1}{\sqrt{t_1} t_2^{3/2} \sqrt{t_3} \dots t_n} + \dots + \frac{1}{\sqrt{t_1} \dots t_{n-1} t_n^{3/2}} \right) \right. \\ &\quad \left. \cdot \exp(-s_1 t_1 - \frac{u^2}{4} \sum \frac{1}{t_i}) dt_1 \dots dt_n \right] du. \quad (2.4) \end{aligned}$$

Evaluating the inner integrals on the right by (see [5], page 22, results 6 and 7)

$$\int_0^\infty \frac{1}{\sqrt{t}} \exp(-st - \frac{u^2}{4t}) dt = \sqrt{\frac{\pi}{s}} e^{-u\sqrt{s}}, \quad \int_0^\infty t^{-3/2} \exp(-st - \frac{u^2}{4t}) dt = \frac{2\sqrt{\pi}}{u} e^{-u\sqrt{s}}$$

we get

$$\begin{aligned} & L_n \left\{ \frac{(\frac{1}{t_1} + \dots + \frac{1}{t_n})^3}{(t_1 \dots t_n)^{1/2}} F[\frac{1}{64}(\frac{1}{t_1} + \dots + \frac{1}{t_n})^2]; s_1, \dots, s_n \right\} \quad (2.5) \\ &= 1024 \pi^{\frac{n+1}{2}} (s_1 \dots s_n)^{-1/2} \int_0^\infty \exp(-u \sum \sqrt{s_i}) u^3 f(u^4) du \\ &+ 512 \pi^{\frac{n+1}{2}} \frac{(\sqrt{s_1} + \dots + \sqrt{s_n})}{(s_1 \dots s_n)^{1/2}} \int_0^\infty \exp(-u \sum \sqrt{s_i}) u^4 f(u^4) du. \end{aligned}$$

The proof is complete if we use (iii) and (iv) on the right hand side of (2.5).

3. APPLICATIONS: n -dimensional Laplace transform pairs.

Let $f(t) = t^v$; so that $L_1\{t^v; s\} = \frac{\Gamma(v+1)}{s^{v+1}} = \phi(s)$. Then

$$L_1\{\sqrt{t} \phi(\frac{1}{t}, s)\} = L_1\{\Gamma(v+1)t^{v+3/2}; s\} = \frac{\Gamma(v+1)\Gamma(v+5/2)}{s^{v+5/2}} = F(s),$$

$$L_1\{t^3 f(t^4); s\} = L_1\{t^{4v+3}; s\} = \frac{\Gamma(4v+4)}{s^{4v+4}} = G(s),$$

$$L_1\{t^4 f(t^4); s\} = L_1\{t^{4v+4}; s\} = \frac{\Gamma(4v+5)}{s^{4v+5}} = H(s). \quad \text{Hence from (2.1), we get}$$

$$\begin{aligned} L_n\{(t_1 \dots t_n)^{-1/2} (\frac{1}{t_1} + \dots + \frac{1}{t_n})^{-2v-2}; s_1, \dots, s_n\} \\ = \frac{\pi^{\frac{n+1}{2}} \Gamma(4v+4)}{8^{2v-7} \Gamma(v+1) \Gamma(v+5/2)} (s_1 \dots s_n)^{-1/2} (\sqrt{s_1} + \dots + \sqrt{s_n})^{-4v-4} \\ + \frac{\pi^{\frac{n+1}{2}} \Gamma(4v+5)}{8^{2v-6} \Gamma(v+1) \Gamma(v+5/2)} (s_1 \dots s_n)^{-1/2} (\sqrt{s_1} + \dots + \sqrt{s_n})^{-4v-4}. \quad (3.1) \end{aligned}$$

Similarly if we take f to be the following

$$f(t) = \begin{cases} t^{c-1} {}_0F_3(a, b, c; kt) \\ t^v \exp(-\sqrt{t}) \\ J_v^2(\sqrt{2t}) \\ t^\alpha {}_pF_q\left(\begin{matrix} (a); \\ (b); \end{matrix} t\right) \end{cases}$$

in the theorem, then we obtain the following n -dimensional Laplace transform pairs:

$$\begin{aligned} L_n \left\{ \frac{(t_1 \dots t_n)^{-1/2}}{(\frac{1}{t_1} + \dots + \frac{1}{t_n})^{2c}} {}_1F_2 \left[\begin{matrix} c + 3/2; \\ a, b; \end{matrix} \frac{64k}{(\frac{1}{t_1} + \dots + \frac{1}{t_n})^2} \right]; s_1, \dots, s_n \right\} \\ = \frac{2\pi^{\frac{n+1}{2}} \Gamma(4c)}{8^{2c} \Gamma(c) \Gamma(c+3/2) (\sqrt{s_1} + \dots + \sqrt{s_n})^{4c}} (s_1 \dots s_n)^{-1/2} {}_4F_3 \left[\begin{matrix} 2c, 2c+1/2, 2c+1, 2c+2; \\ a, b, c; \end{matrix} \frac{256k}{(\sqrt{s_1} + \dots + \sqrt{s_n})^4} \right] \\ + \frac{\pi^{\frac{n+1}{2}} \Gamma(4c+1)}{8^{2c} \Gamma(c) \Gamma(c+3/2) (\sqrt{s_1} + \dots + \sqrt{s_n})^{4c}} (s_1 \dots s_n)^{-1/2} {}_4F_3 \left[\begin{matrix} 2c+1/2, 2c+1, 2c+3/2, 2c+2; \\ a, b, c; \end{matrix} \frac{256k}{(\sqrt{s_1} + \dots + \sqrt{s_n})^4} \right], \end{aligned}$$

$$\operatorname{Re}(c) > 0.$$

(3.2)

$$\begin{aligned}
& L_n \left\{ \frac{\frac{1}{t_1} + \dots + \frac{1}{t_n}}{\left(\frac{1}{t_1} \dots \frac{1}{t_n}\right)^{1/2}} \left(\frac{1}{64} \left(\frac{1}{t_1} + \dots + \frac{1}{t_n}\right)^2 - \frac{1}{4}\right)^{-v-3/2} p_1^{-2v-3} \left[\frac{4}{\frac{1}{t_1} + \dots + \frac{1}{t_n}} \right]; s_1, \dots, s_n \right\} \\
&= \frac{\pi^{n/2} \Gamma(4v+4)}{2^{v-4} \Gamma(2v-5)} (s_1 \dots s_n)^{-1/2} \exp\left(\frac{1}{8}(\sqrt{s_1} + \dots + \sqrt{s_n})^2\right) D_{-4v-4} \left(\frac{1}{\sqrt{2}}(\sqrt{s_1} + \dots + \sqrt{s_n})\right) \\
&+ \frac{\pi^{n/2} \Gamma(4v+5)}{2^{v-5} \Gamma(2v+5)} (\sqrt{s_1} + \dots + \sqrt{s_n}) \exp\left(\frac{1}{8}(\sqrt{s_1} + \dots + \sqrt{s_n})^2\right) D_{-4v-5} \left(\frac{1}{\sqrt{2}}(\sqrt{s_1} + \dots + \sqrt{s_n})\right), \\
&\text{Re}(v) > -1. \tag{3.3}
\end{aligned}$$

$$\begin{aligned}
& L_n \left\{ \frac{(t_1 \dots t_n)^{-1/2} \left(\frac{1}{t_1} + \dots + \frac{1}{t_n}\right)^3}{\left(\frac{1}{64} \left(\frac{1}{t_1} + \dots + \frac{1}{t_n}\right)^2 + 1\right)^{-1}} Q_{v-1/2}^2 \left(\frac{1}{64} \left(\frac{1}{t_1} + \dots + \frac{1}{t_n}\right)^2 + 1\right); s_1, \dots, s_n \right\} \\
&= \frac{\pi^{\frac{n}{2}+1} \sin(v+3/2) \pi (\sqrt{s_1} + \dots + \sqrt{s_n})^{v/2}}{2^{\frac{v}{2}-9} \sin(v-1/2) \pi (s_1 \dots s_n)^{1/2}} \\
&\cdot G_{50}^{03} \left[\frac{8\sqrt{2}}{(\sqrt{s_1} + \dots + \sqrt{s_n})^2} \left| 2 - \frac{v}{3}, -\frac{v}{4}, -\frac{1}{2} - \frac{v}{4}, \frac{3}{2} + \frac{v}{4}, 2 + \frac{v}{4} \right. \right] \\
&+ \frac{\pi^{\frac{n}{2}+1} \sin(v+3/2) \pi (\sqrt{s_1} + \dots + \sqrt{s_n})^{\frac{v+3}{2}}}{s^{v-73/8} \sin(v-1/2) \pi (s_1 \dots s_n)^{1/2}} \cdot \\
&G_{50}^{03} \left[\frac{8\sqrt{2}}{(\sqrt{s_1} + \dots + \sqrt{s_n})^2} \left| \frac{9}{4} - \frac{v}{4}, -\frac{1}{4} - \frac{v}{4}, -\frac{3}{4} - \frac{v}{4}, \frac{7}{4} + \frac{v}{4}, \frac{9}{4} + \frac{v}{4} \right. \right] \tag{3.4}
\end{aligned}$$

$$\begin{aligned}
& L_n \left\{ \frac{(t_1 \dots t_n)^{-1/2}}{\left(\frac{1}{t_1} + \dots + \frac{1}{t_n}\right)^{2\alpha+2}} p+2 F_q \left[\begin{matrix} (a), \alpha+1, \alpha+5/2; \\ (b); \end{matrix} \frac{64}{\left(\frac{1}{t_1} + \dots + \frac{1}{t_n}\right)^2} \right]; s_1, \dots, s_n \right\} \\
&= \frac{\pi^{\frac{n+1}{2}} \Gamma(4\alpha+4) (s_1 \dots s_n)^{-1/2}}{4(8)^{2\alpha+1} \Gamma(\alpha+1) \Gamma(\alpha+5/2) (\sqrt{s_1} + \dots + \sqrt{s_n})^{4\alpha+4}} \\
&\cdot p+4 F_q \left[\begin{matrix} (a), 2\alpha+2, 2\alpha+5/2, 2\alpha+3, 2\alpha+7/2; \\ (b); \end{matrix} \frac{256}{(\sqrt{s_1} + \dots + \sqrt{s_n})^4} \right] \\
&+ \frac{\pi^{\frac{n+1}{2}} \Gamma(4\alpha+5) (s_1 \dots s_n)^{-1/2}}{8^{2\alpha+2} \Gamma(\alpha+1) \Gamma(\alpha+5/2) (\sqrt{s_1} + \dots + \sqrt{s_n})^{4\alpha+4}} \\
&\cdot p+4 F_q \left[\begin{matrix} (a), 2\alpha+3, 2\alpha+7/2, 2\alpha+4, 2\alpha+9/2; \\ (b); \end{matrix} \frac{256}{(\sqrt{s_1} + \dots + \sqrt{s_n})^4} \right], \\
&\text{Re}(\alpha) > 1. \tag{3.5}
\end{aligned}$$

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