Internat. J. Math. & Math. Sci.
Vol. 7 No. 4 (1984) 821-822

THE COMPACTUM OF A SEMI-SIMPLE COMMUTATIVE BANACH ALGEBRA

ABDULIAH H. AL-MOAJIL

GULF ORGANIZATION FOR INDUSTRIAL CONSULTING
P. O. Box 5114
Doha - Qatar.

(Received May 18, 1983)

ABSTRACT. Let A be a commutative semi-simple Banach algebra such that the set consisting of finite sums of elements from minimal left ideals coincides with that of finite sums of elements from minimal right ideals. Let S(A) (the socle of A) denote this set. Let C(A) denote the set of elements x in A such that the map $a \to xax$ is compact. It is shown that C(A) is the norm closure of S(A).

KEY WORDS AND PHRASES. *Commutative Banach algebra, semi-simple, socle, compactum, spectrum, carrier space, idempotent.*
1980 AMS MATHEMATICS SUBJECT CLASSIFICATION CODE. Primary 46H05, 46H10; Secondary 47B05.

1. INTRODUCTION

Let A be a Banach algebra. For $x \varepsilon A$ let T_x denote the operator defined by $T_x(a)=xax$. The compactum of A is defined to be the set $\{x \varepsilon A: T_x$ is a compact operator on A$\}$. A Banach algebra A in which A=C(A) is called a compact Banach algebra. Compact Banach algebras were first introducted by J. C. Alexander in [1] . The author, in [3], investigated the properties of the compactum in Banach Algebras. It was shown in [3] that if A is semi-simple and S(A) denotes the socle of A, then C(A) is non-zero if and only if S(A) is non-zero, and in this case, $S(A) \subset C(A)$. Moreover, C(A) is a closed set, therefore it contains the closure of S(A). A problem of interest is to determine sufficient conditions on A which imply that C(A) coincides with $\overline{S(A)}$. In [2] , the author proved that for a primitive B* algebra A, we have $C(A) = \overline{S(A)}$.

The purpose of this note is to prove that for a semi-simple commutative Banach algebra A, we have $C(A) = \overline{S(A)}$.

2. MAIN RESULT

To prove our theorem we use a result from [3] which states that if $x \varepsilon C(A)$ then the spectrum of $x(\sigma(x))$ is at most countable and o is its only possible accumulation point. Our terminology and notation is consistent with that of [4] , and our algebras are over the field of complex numbers.

THEOREM: Let A be a semi-simple commutative Banach algebra. If C(A) exists, then $C(A)=\overline{S(A)}$.

PROOF: We need to show that $C(A) \subset \overline{S(A)}$, as the other inclusion was already proven in [3] .

Let Φ denote the space of multiplicative linear functionals in A, i.e. Φ is the carrier space of A.

Let $x \varepsilon C(A)$. We have, from general theory of commutative Banach algebras, $\sigma(x)=\{\hat{x}(\phi):\phi\varepsilon\Phi\}\cup\{o\}$ where $\hat{x}$ is the continuous function on Φ defined by $\hat{x}(\phi)=\phi(x)$.

We claim that if ϕ is not an isolated point of Φ, then $\hat{x}(\phi)=o$.

This is true because if $\hat{x}(\phi)\neq o$ and $\{\phi_n\} \subset \Phi$ with $\lim_{n\to\infty} \phi_n = \phi$, then by the continuity of $\hat{x}$ we get $\hat{x}(\phi)= \lim_{n\to\infty} \hat{x}(\phi_n)$. But $\hat{x}(\phi_n)$ and $\hat{x}(\phi)$ belong to $\sigma(x)$. Therefore, $\hat{x}(\phi)$ is a non-zero accumulation point of $\sigma(x)$ which is impossible since $x\varepsilon C(A)$.

Now since $\sigma(x)$ is countable, let $\{\phi_n\}$ be a sequance in Φ such that $\sigma(x)=\{\hat{x}(\phi_n): n=1, 2, ..\}$ $\{o\}$, where $\hat{x}(\phi_n)\neq o$ for all n. Note that each ϕ_n is an isolated point of Φ.

Now, by Silov's idempotent theorem [4] , for each n = 1,2.. there exists an idempotent $e_n \varepsilon A$ such that $\hat{e}_n(\phi)_n = 1$ and $\hat{e}_n(\phi) = o$ if $\phi \neq \phi_n$. It is evident that e_n is a minimal idempotent for each n.

For each m, let $x_n = \sum_{i=1}^{n} \hat{x}(\phi_i) e_i$. Then by the minimality of e_n, we have $x_n \varepsilon S(A)$.

Now, if $\{\phi_n\}$ is a finite set, then $\hat{x}=\hat{x}_n$ for some n and therefore $\hat{x}=\hat{x}_n \varepsilon S(A)$. Otherwise, by the compactness of $\sigma(x)$ and the fact that o is the only possible accumulation point of $\sigma(x)$, we have $\lim_n \hat{x}(\phi_n)=o$.

Now, if $\phi\neq\phi_n$ for any n, then $\hat{x}(\phi)=o$ and $e_n(\phi) = o$ for all n, thus $\hat{x}_n(\phi) = o$ for all n. Moreover, $\hat{x}_n(\phi_m)=\hat{x}(\phi_m)$ if $n \geq m$ and o if $n < m$. Therefore, $\|\hat{x}_n - \hat{x}\| \sup_{\phi\varepsilon\Phi} |\hat{x}_n(\phi)-\hat{x}(\phi)| = \sup_m |\hat{x}_n(\phi_m)-\hat{x}(\phi_m)|$, and we get $\lim_n \|\hat{x}_n - \hat{x}\| = \lim_n \sup_{n<m} |\hat{x}(\phi_m)| = 0$.

Therefore, $\lim_n \hat{x}_n = \hat{x}$ and since the representation of A as an algebra of continuous functions on Φ is a homeomorphism, we get $\lim_n x_n = x$.

Therefore, $x\varepsilon\overline{S(A)}$.

REFERENCES

1. Alexander, J.C. Compact Banach Algebras. Proc. London Math. Soc. (3) 18, 1968, 1-18

2. Al-Moajil, A.H. The Compactum of a Primitive B* Algebra. Math. Japonica 26 (4) (1981), 385-387.

3. Al-Moajil, A.H. The Compactum and Finite Dimensionality in Banach Algebras. Internat. J. Math. & Math. Sci., 5, (2), (1982), 275-280.

4. Rickart C.E., General Theory of Banach Algebras. Von Nostrand, Princeton, N.J., MR 22 No. 5903 (1960).

Special Issue on Time-Dependent Billiards

Call for Papers

This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

Manuscript Due	March 1, 2009
First Round of Reviews	June 1, 2009
Publication Date	September 1, 2009

Guest Editors

Edson Denis Leonel, Department of Statistics, Applied Mathematics and Computing, Institute of Geosciences and Exact Sciences, State University of São Paulo at Rio Claro, Avenida 24A, 1515 Bela Vista, 13506-700 Rio Claro, SP, Brazil; edleonel@rc.unesp.br

Alexander Loskutov, Physics Faculty, Moscow State University, Vorob'evy Gory, Moscow 119992, Russia; loskutov@chaos.phys.msu.ru