

AN APPLICATION OF HYPERGEOMETRIC FUNCTIONS TO A PROBLEM IN FUNCTION THEORY

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(Received November 20, 1980 and in revised form March 9, 1981)

ABSTRACT. In some recent work in univalent function theory, Aharonov, Friedland, and Brannan studied the series, $(1 + xt)^\alpha (1 - t)^\beta = \sum_{n=0}^{\infty} A_n^{(\alpha, \beta)}(x) t^n$. Brannan posed the problem of determining $S = \{(\alpha, \beta) : |A_n^{(\alpha, \beta)}(e^{i\theta})| < |A_n^{(\alpha, \beta)}(1)|, 0 < \theta < 2\pi, \alpha > 0, \beta > 0, n = 1, 2, 3, \dots\}$. Brannan showed that if $\beta \geq \alpha \geq 0$, and $\alpha + \beta \geq 2$, then $(\alpha, \beta) \in S$. He also proved that $(\alpha, 1) \in S$ for $\alpha \geq 1$. Brannan showed that for $0 < \alpha < 1$ and $\beta = 1$, there exists a θ such that $|A_{2k}^{(\alpha, 1)}(e^{i\theta})| > |A_{2k}^{(\alpha, 1)}(1)|$ for k any integer. In this paper, we show that $(\alpha, \beta) \in S$ for $\alpha \geq 1$ and $\beta \geq 1$.

KEY WORDS AND PHRASES. Hypergeometric Functions, Jacobi Polynomials, Maximum property, and positive maximum property.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. 33 Special Functions.

1. INTRODUCTION.

Let D be a disk $\{z : |z - a| \leq r\}$ where the center a is real. Let f be a function analytic in an open neighborhood of the disk D . It is well known that the maximum modulus of F on D is attained on the boundary $\{z : |z - a| = r\}$. If the maximum modulus is attained at $a + r$ and only at $a + r$ then we say that f has the maximum property on D . If in addition $f(a + r) > 0$, then f has the positive maximum property. If the disk D is not specified then it is assumed that D is the unit disk.

Let $(1 + zt)^\alpha (1 - t)^\beta = \sum_{k=0}^{\infty} A_k^{(\alpha, \beta)}(z) t^k$ and let $MP = \{(\alpha, \beta) : \alpha \geq 0, \beta \geq 0 \text{ and } A_n^{(\alpha, \beta)}(z) \text{ satisfies the positive maximum property for } n = 1, 2, 3, \dots\}$. The main problem in this paper is to characterize the sets MP and PMP . An application to extreme point theory is given in [2].

2. SOME FUNDAMENTAL RECURRENCE RELATIONS

Starting with

$$(1 + at)^\alpha (1 - t)^{-\beta} = \sum_{n=0}^{\infty} A_n^{(\alpha, \beta)}(z) t^n, \quad (2.1)$$

one can derive a number of recurrence relations. For example

$$(1 + zt)^{\alpha+\gamma} (1 - t)^{-\beta} = \sum_{n=0}^{\infty} A_n^{(\alpha+\gamma, \beta)}(z) t^n.$$

Indeed, since $(1 + zt)^\gamma = \sum_{n=0}^{\infty} \frac{(-\gamma)_n}{n!} (-zt)^n$, by taking the Cauchy product of this last series and the series in (2.1) we obtain

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^n \frac{(-\gamma)_k (-z)^k}{k!} A_{n-k}^{(\alpha, \beta)}(z) \right) t^n = \sum_{n=0}^{\infty} A_n^{(\alpha+\gamma, \beta)}(z) t^n.$$

Hence

$$A_n^{(\alpha+\gamma, \beta)}(z) = \sum_{k=0}^n \frac{(-\gamma)_k (-z)^k}{k!} A_{n-k}^{(\alpha, \beta)}(z). \quad (2.2)$$

Similarly

$$A_n^{(\alpha, \beta+\gamma)}(z) = \sum_{k=0}^n \frac{(\gamma)_k}{k!} A_{n-k}^{(\alpha, \beta)}(z). \quad (2.3)$$

If we let $\gamma = 1$ in (2), we obtain

$$A_n^{(\alpha, \beta)}(z) + z A_{n-1}^{(\alpha, \beta)}(z) = A_{n+1}^{(\alpha+1, \beta)}(z). \quad (2.4)$$

Relations (2.3) and (2.4) are significant because if $(\alpha, \beta) \in \text{PMP}$, then $(\alpha, \beta') \in \text{PMP}$ for all $\beta' > \beta$. Also, $(\alpha, \beta) \in \text{PMP}$ implies that $(\alpha + n, \beta) \in \text{PMP}$, $n = 1, 2, 3, \dots$.

3. SOME EXPLICIT FORMULAS FOR $A_n^{(\alpha, \beta)}(z)$

Taking the Cauchy product of the series

$$\begin{aligned} (1 + zt)^\alpha &= \sum_{n=0}^{\infty} \frac{(-\alpha)_n}{n!} (-z)^n t^n, \text{ and} \\ (1 - t)^{-\beta} &= \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} t^n, \text{ we have} \\ A_n^{(\alpha, \beta)}(z) &= \sum_{k=0}^n \frac{(-\alpha)_k}{k!} \frac{(\beta)_{n-k}}{(n-k)!} (-z)^k. \end{aligned} \quad (3.1)$$

Using the fact that $(n-k)! = (1)_{n-k}$ and $(a)_{n-k} = \frac{(a)_n (-)^k}{(1-a-n)_k}$, we obtain

$$A_n^{(\alpha, \beta)}(z) = \frac{(\beta)_n}{n!} \sum_{k=0}^n \frac{(-n)_k (-\alpha)_k}{(1-\beta-n)_k k!}$$

Using ${}_2F_1\left(\begin{smallmatrix} a, b \\ c \end{smallmatrix}; z\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k$, we obtain

$$A_n^{(\alpha, \beta)}(z) = \frac{(\beta)_n}{n!} {}_2F_1\left(\begin{smallmatrix} -n, -\alpha \\ 1-\beta-n \end{smallmatrix}; -z\right). \quad (3.2)$$

The Jacobi polynomials are defined as

$$P_n^{(\alpha, \beta)}(z) = \frac{(\alpha+1)_n}{n!} {}_2F_1\left(\begin{smallmatrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{smallmatrix}; \frac{1-z}{2}\right).$$

Hence

$$A_n^{(\alpha, \beta)}(z) = (-1)^n P_n^{(-\beta-n, \beta-\alpha-1)}(2z+1). \quad (3.3)$$

Replacing k by $n-k$ in (3.1), and following the same procedure we get

$$A_n^{(\alpha, \beta)}(z) = z^n P_n^{(\alpha-n, \beta-\alpha-1)}\left(1+\frac{2}{z}\right). \quad (3.4)$$

Using Pfaff's transformation [1, p. 64]

$${}_2F_1\left(\begin{smallmatrix} a, b \\ c \end{smallmatrix}; z\right) = (1-z)^{-a} {}_2F_1\left(\begin{smallmatrix} a, c-b \\ c \end{smallmatrix}; \frac{z}{z-1}\right), \quad c \neq 0, -1, -2, \dots$$

we can write $A_n^{(\alpha, \beta)}(z)$ as

$$A_n^{(\alpha, \beta)}(z) = \frac{(\beta)_n}{n!} (1+z)^\alpha {}_2F_1\left(\begin{smallmatrix} -\alpha, 1-\beta \\ 1-\beta-n \end{smallmatrix}; \frac{z}{z+1}\right). \quad (3.5)$$

Setting $\beta \rightarrow 1$, we get

$$A_n^{(\alpha, 1)}(z) = (1+z)^\alpha \left(1 + \frac{(-1)^n (-\alpha)_{n+1}}{(n+1)!} \left(\frac{z}{z+1}\right)^{n+1} {}_2F_1\left(\begin{smallmatrix} n+1, n+1-\alpha \\ n+2 \end{smallmatrix}; \frac{z}{z+1}\right)\right). \quad (3.6)$$

4. SOME MAXIMALITY PROPERTIES FOR $A_n^{(\alpha, \beta)}(z)$

It has been proven in [3] that $(\alpha, \beta) \in \text{MP}$ for $\beta = 1$ and $\alpha \geq 1$. We can now strengthen that result.

THEOREM 1. $(\alpha, \beta) \in \text{PMP}$ for $\alpha \geq 1$ and $\beta \geq 1$.

PROOF: It is evident from (3.2) that all coefficients of $A_n^{(\alpha, \beta)}(z)$ are positive for $\alpha \geq n$. So clearly $A_n^{(\alpha, 1)}(z)$ will satisfy the positive maximum property for $\alpha \geq n$.

The theorem follows from (2.3) upon showing that $A_n^{(\alpha, 1)}(1) > 0$ for $1 < \alpha < n$.

Assume that $1 < \alpha < n$. Then it follows from (3.6) that if

$$\left| \frac{(-\alpha)_{n+1}}{(n+1)!} \left(\frac{1}{2}\right)^{n+1} {}_2F_1\left(\begin{matrix} n+1, n+1-\alpha \\ n+2 \end{matrix}; \frac{1}{2}\right) \right| < 1, \quad (3.7)$$

then $A_n^{(\alpha, 1)}(1) > 0$.

Note that all terms of the ${}_2F_1$ in (3.7) are positive. Moreover

$${}_2F_1\left(\begin{matrix} n+1, n+1-\alpha \\ n+2 \end{matrix}; \frac{1}{2}\right) < {}_1F_0\left(\begin{matrix} n+1-\alpha \\ - \end{matrix}; \frac{1}{2}\right) = 2^{n+1-\alpha},$$

by the binomial theorem.

Hence the left side of (3.7) is less than $|(-\alpha)_{n+1} 2^{-\alpha} / (n+1)!|$. Let m be an integer such that $m - 1 < \alpha \leq m$. Then

$$\left| \frac{(-\alpha)_{n+1}}{(n+1)!} \right| = \frac{|(-\alpha)(1-\alpha) \cdots (m-\alpha-1)(m-\alpha) \cdots (n-\alpha)|}{(n+1)!} =$$

$$\frac{\alpha(\alpha-1) \cdots (\alpha-m+1)(m-\alpha) \cdots (n-\alpha)}{(n+1)!} \leq$$

$$\frac{m(m-1) \cdots 2 \cdot 1 \cdot 1 \cdot 2 \cdots (n-m+1)}{(n+1)!} = \binom{n+1}{m}^{-1} < 1.$$

Consequently $|(-\alpha)_{n+1} 2^{-\alpha} / (n+1)!| < 1$, and (3.7) is established. Brannan [3], showed that $(\alpha, 1) \in \text{MP}$ for $\alpha \geq 1$. Hence $(\alpha, 1) \in \text{PMP}$ for all $\alpha > 1$, and by (2.3), $(\alpha, \beta) \in \text{PMP}$ for all $\alpha > 1$ and $\beta \geq 1$.

The author feels that the properties of Jacobi polynomials as given in (3.3) and (3.4) will be useful in answering other questions of Brannan's regarding the series (2.1).

ACKNOWLEDGEMENTS

The author is thankful to the referee for his valuable suggestions.

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