

AN APPLICATION OF HYPERGEOMETRIC FUNCTIONS TO A PROBLEM IN FUNCTION THEORY

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ABSTRACT. In some recent work in univalent function theory, Aharonov, Friedland, and Brannan studied the series, $(1 + xt)^\alpha (1 - t)^\beta = \sum_{n=0}^{\infty} A_n^{(\alpha, \beta)}(x)t^n$. Brannan posed the problem of determining $S = \{(\alpha, \beta) : |A_n^{(\alpha, \beta)}(e^{i\theta})| < |A_n^{(\alpha, \beta)}(1)|, 0 < \theta < 2\pi, \alpha > 0, \beta > 0, n = 1, 2, 3, \dots\}$. Brannan showed that if $\beta \geq \alpha \geq 0$, and $\alpha + \beta \geq 2$, then $(\alpha, \beta) \in S$. He also proved that $(\alpha, 1) \in S$ for $\alpha \geq 1$. Brannan showed that for $0 < \alpha < 1$ and $\beta = 1$, there exists a θ such that $|A_{2k}^{(\alpha, 1)} e^{i\theta}| > |A_{2k}^{(\alpha, 1)}(1)|$ for k any integer. In this paper, we show that $(\alpha, \beta) \in S$ for $\alpha \geq 1$ and $\beta \geq 1$.

KEY WORDS AND PHRASES. Hypergeometric Functions, Jacobi Polynomials, Maximum property, and positive maximum property.

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1. INTRODUCTION.

Let D be a disk $\{z : |z - a| \leq r\}$ where the center a is real. Let f be a function analytic in an open neighborhood of the disk D . It is well known that the maximum modulus of F on D is attained on the boundary $\{z : |z - a| = r\}$. If the maximum modulus is attained at $a + r$ and only at $a + r$ then we say that f has the maximum property on D . If in addition $f(a + r) > 0$, then f has the positive maximum property. If the disk D is not specified then it is assumed that D is the unit disk.

Let $(1 + zt)^\alpha (1 - t)^\beta = \sum_{k=0}^{\infty} A_k^{(\alpha, \beta)}(z)t^k$ and let $MP = \{(\alpha, \beta) : \alpha \geq 0, \beta \geq 0 \text{ and } A_n^{(\alpha, \beta)}(z) \text{ satisfies the positive maximum property for } n = 1, 2, 3, \dots\}$. The main problem in this paper is to characterize the sets MP and PMP . An application to extreme point theory is given in [2].

2. SOME FUNDAMENTAL RECURRENCE RELATIONS

Starting with

$$(1 + at)^\alpha (1 - t)^{-\beta} = \sum_{n=0}^{\infty} A_n^{(\alpha, \beta)}(z) t^n, \quad (2.1)$$

one can derive a number of recurrence relations. For example

$$(1 + zt)^{\alpha+\gamma} (1 - t)^{-\beta} = \sum_{n=0}^{\infty} A_n^{(\alpha+\gamma, \beta)}(z) t^n.$$

Indeed, since $(1 + zt)^\gamma = \sum_{n=0}^{\infty} \frac{(-\gamma)_n}{n!} (-zt)^n$, by taking the Cauchy product of this last series and the series in (2.1) we obtain

$$\sum_{n=0}^{\infty} \left(\sum_{k=0}^n \frac{(-\gamma)_k (-z)^k}{k!} A_{n-k}^{(\alpha, \beta)}(z) \right) t^n = \sum_{n=0}^{\infty} A_n^{(\alpha+\gamma, \beta)}(z) t^n.$$

Hence

$$A_n^{(\alpha+\gamma, \beta)}(z) = \sum_{k=0}^n \frac{(-\gamma)_k (-z)^k}{k!} A_{n-k}^{(\alpha, \beta)}(z). \quad (2.2)$$

Similarly

$$A_n^{(\alpha, \beta+\gamma)}(z) = \sum_{k=0}^n \frac{(\gamma)_k}{k!} A_{n-k}^{(\alpha, \beta)}(z). \quad (2.3)$$

If we let $\gamma = 1$ in (2), we obtain

$$A_n^{(\alpha, \beta)}(z) + z A_{n-1}^{(\alpha, \beta)}(z) = A_{n+1}^{(\alpha+1, \beta)}(z). \quad (2.4)$$

Relations (2.3) and (2.4) are significant because if $(\alpha, \beta) \in \text{PMP}$, then $(\alpha, \beta') \in \text{PMP}$ for all $\beta' > \beta$. Also, $(\alpha, \beta) \in \text{PMP}$ implies that $(\alpha + n, \beta) \in \text{PMP}$, $n = 1, 2, 3, \dots$.

3. SOME EXPLICIT FORMULAS FOR $A_n^{(\alpha, \beta)}(z)$

Taking the Cauchy product of the series

$$(1 + zt)^\alpha = \sum_{n=0}^{\infty} \frac{(-\alpha)_n (-z)^n}{n!} t^n, \text{ and}$$

$$(1 - t)^{-\beta} = \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} t^n, \text{ we have}$$

$$A_n^{(\alpha, \beta)}(z) = \sum_{k=0}^n \frac{(-\alpha)_k (\beta)_{n-k}}{k! (n - k)!} (-z)^k. \quad (3.1)$$

Using the fact that $(n - k)! = (1)_{n-k}$ and $(a)_{n-k} = \frac{(a)_n (-)^k}{(1-a-n)_k}$, we obtain

$$A_n^{(\alpha, \beta)}(z) = \frac{(\beta)_n}{n!} \sum_{k=0}^n \frac{(-n)_k (-\alpha)_k}{(1-\beta-n)_k k!}$$

Using ${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k$, we obtain

$$A_n^{(\alpha, \beta)}(z) = \frac{(\beta)_n}{n!} {}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-z}{2}). \quad (3.2)$$

The Jacobi polynomials are defined as

$$P_n^{(\alpha, \beta)}(z) = \frac{(\alpha+1)_n}{n!} {}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-z}{2}).$$

Hence

$$A_n^{(\alpha, \beta)}(z) = (-1)^n P_n^{(-\beta-n, \beta-\alpha-1)}(2z+1). \quad (3.3)$$

Replacing k by $n-k$ in (3.1), and following the same procedure we get

$$A_n^{(\alpha, \beta)}(z) = z^n P_n^{(\alpha-n, \beta-\alpha-1)}(1 + \frac{2}{z}). \quad (3.4)$$

Using Pfaff's transformation [1, p. 64]

$${}_2F_1(a, b; c; z) = (1-z)^{-a} {}_2F_1(a, c-b; c; \frac{z}{z-1}), \quad c \neq 0, -1, -2, \dots$$

we can write $A_n^{(\alpha, \beta)}(z)$ as

$$A_n^{(\alpha, \beta)}(z) = \frac{(\beta)_n}{n!} (1+z)^\alpha {}_2F_1(-\alpha, 1-\beta; 1-\beta-n; \frac{z}{z+1}). \quad (3.5)$$

Setting $\beta \rightarrow 1$, we get

$$A_n^{(\alpha, 1)}(z) = (1+z)^\alpha (1 + \frac{(-1)^n (-\alpha)_{n+1}}{(n+1)!} (\frac{z}{z+1})^{n+1}) {}_2F_1(\frac{n+1, n+1-\alpha}{n+2}; \frac{z}{z+1}). \quad (3.6)$$

4. SOME MAXIMALITY PROPERTIES FOR $A_n^{(\alpha, \beta)}(z)$

It has been proven in 3 that $(\alpha, \beta) \in \text{MP}$ for $\beta = 1$ and $\alpha \geq 1$. We can now strengthen that result.

THEOREM 1. $(\alpha, \beta) \in \text{PMP}$ for $\alpha \geq 1$ and $\beta \geq 1$.

PROOF: It is evident from (3.2) that all coefficients of $A_n^{(\alpha, \beta)}(z)$ are positive for $\alpha \geq n$. So clearly $A_n^{(\alpha, 1)}(z)$ will satisfy the positive maximum property for $\alpha \geq n$.

The theorem follows from (2.3) upon showing that $A_n^{(\alpha, 1)}(1) > 0$ for $1 < \alpha < n$.

Assume that $1 < \alpha < n$. Then it follows from (3.6) that if

$$\left| \frac{(-\alpha)_{n+1}}{(n+1)!} \left(\frac{1}{2}\right)^{n+1} {}_2F_1 \left(\begin{matrix} n+1, n+1-\alpha \\ n+2 \end{matrix}; \frac{1}{2} \right) \right| < 1, \quad (3.7)$$

then $A_n^{(\alpha, 1)}(1) > 0$.

Note that all terms of the ${}_2F_1$ in (3.7) are positive. Moreover

$${}_2F_1 \left(\begin{matrix} n+1, n+1-\alpha \\ n+2 \end{matrix}; \frac{1}{2} \right) < {}_1F_0 \left(\begin{matrix} n+1-\alpha \\ - \end{matrix}; \frac{1}{2} \right) = 2^{n+1-\alpha},$$

by the binomial theorem.

Hence the left side of (3.7) is less than $|(-\alpha)_{n+1} 2^{-\alpha} / (n+1)!|$. Let m be an integer such that $m-1 < \alpha \leq m$. Then

$$\left| \frac{(-\alpha)_{n+1}}{(n+1)!} \right| = \frac{|(-\alpha)(1-\alpha)\cdots(m-\alpha-1)(m-\alpha)\cdots(n-\alpha)|}{(n+1)!} =$$

$$\frac{\alpha(\alpha-1)\cdots(\alpha-m+1)(m-\alpha)\cdots(n-\alpha)}{(n+1)!} \leq$$

$$\frac{m(m-1)\cdots 2 \cdot 1 \cdot 1 \cdot 2 \cdots (n-m+1)}{(n+1)!} = \binom{n+1}{m}^{-1} < 1.$$

Consequently $|(-\alpha)_{n+1} 2^{-\alpha} (n+1)!| < 1$, and (3.7) is established. Brannan [3], showed that $(\alpha, 1) \in \text{MP}$ for $\alpha \geq 1$. Hence $(\alpha, 1) \in \text{PMP}$ for all $\alpha > 1$, and by (2.3), $(\alpha, \beta) \in \text{PMP}$ for all $\alpha > 1$ and $\beta \geq 1$.

The author feels that the properties of Jacobi polynomials as given in (3.3) and (3.4) will be useful in answering other questions of Brannan's regarding the series (2.1).

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REFERENCES

1. BATEMAN, H. Higher Transcendental Functions, McGraw-Hill, (1953).
2. BRANNAN, D.A.; CLUNIE, J.G.; KIRWAN, M.E. On the coefficient problem for the functions of bounded boundary rotation, Ann. Acad. Sci. Fenn. Ser. A1. Math.,
3. CLUNIE, J.; HAYMAN, W.H., Symposium on Complex Analysis, Canterbury, 1973, pp. 17-27.

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