

A REMARK ON THE r -th MEAN DIFFERENTIABILITY

GEORGE D. STAMATELOS

Department of Mathematics
University of Patras - Greece

(Received January 11, 1983 and in revised form May 9, 1983)

ABSTRACT. This paper is concerned with the r -th mean differentiability. In the mathematical developments regarding the asymptotic expansion and the asymptotic distribution of the likelihood function, there arises the question whether the assumptions made on the model imply differentiability in the r' -th mean of the underlying random functions, for integer values $r' < r$. The present paper provides an answer to this question and also gives the explicit form of the derivatives in the r' -th mean involved.

KEY WORDS AND PHRASES. Stochastic process, derivative of r -th mean, probability measure.

1980 AMS SUBJECT CLASSIFICATION CODE. 60G99

1. INTRODUCTION AND SUMMARY.

For $n \geq 0$, integer, let X_0, X_1, \dots, X_n be the first $n+1$ r.v.'s from a stochastic process, defined on the probability space (X, A, P_θ) , whose probability law depends on a k -dimensional parameter $\theta \in \Theta$, an open subset of \mathbb{R}^k , $k \geq 1$ and satisfies certain regularity conditions. Then, for $\theta, \theta^* \in \Theta$, let

$$q_j(\theta; \theta^*) = q_j(X_j; \theta, \theta^*) = \frac{dP_{n, \theta^*}}{dP_{n, \theta}}, \quad n \geq 0$$

be specified versions of the Radon-Nikodym derivatives involved, where $X_j = (X_0, X_1, \dots, X_n)$ and $P_{n, \theta}$ is the restriction of P_θ to the σ -field $A_n = \sigma(X_0, X_1, \dots, X_n)$ induced by the r.v.'s X_0, X_1, \dots, X_n . Also, for any two parameter points θ and θ^* , let $q_j(X_j | X_{j-1}; \theta, \theta^*)$ be the quotient of the probability density functions of the random vectors $X_j = (X_0, X_1, \dots, X_j)$ and X_{j-1} . Then, as is well known, for large sample statistical inference, based on the likelihood function, the underlying conditions are set primarily on the quantities, $q_j(X_j | X_{j-1}; \theta, \theta^*)$, $j = 1, 2, \dots, n$. Typically these conditions include pointwise differentiability with respect to the "moving" parameter θ^* of up to third order. It is also known, however, that these conditions fail to be satisfied, for

instance, in such a simple and interesting case as that of the double exponential distribution.

In relatively recent asymptotic statistical work, the above mentioned type of conditions have been replaced by conditions referring to the differentiability in the r -th mean of the quantities

$$\varphi_j(\theta, \theta^*; r) = [q_j(X_j | X_{j-1}; \theta, \theta^*)]^{1/r}, \quad j \geq 1, \text{ for some integer } r \geq 1. \quad (1.1)$$

These conditions do not suffer from the kind of inadequacies mentioned above and, in addition, are of probabilistic rather than of analytical nature. (See, for example, LeCam [1,2], Johnson and Roussas [3-5], Roussas [6], Lind and Roussas [7-8] Akritas [9] and, in particular, Akritas and Roussas [10].)

2. ASSUMPTIONS AND MAIN RESULT.

In this section, the assumptions are formulated under which the main result of this note holds true. To this end, let Θ be an open subset of \mathbb{R}^k , $k \geq 1$, and for each $\theta \in \Theta$, let X_0, X_1, \dots be r.v.'s defined on the probability space $(X, \mathcal{A}, P_\theta)$ and taking values in (S, \mathcal{S}) ; here S is a Borel subset of a Euclidean space and \mathcal{S} is the σ -field of Borel subsets of S . These r.v.'s come from a certain class of stochastic processes which satisfy suitable conditions to be explicitly mentioned below. Let \mathcal{A}_n be the σ -field induced by the first $n+1$ r.v.'s, $\mathcal{A}_n = \sigma(X_0, X_1, \dots, X_n)$, and let $P_{n,0}$ be the restriction of P_θ to \mathcal{A}_n ; that is,

$$P_{n,\theta} = P_\theta | \mathcal{A}_n, \quad \mathcal{A}_n = \sigma(X_0, X_1, \dots, X_n). \quad (2.1)$$

It will be assumed in the following that, for each $\theta, \theta^* \in \Theta$, $P_{n,\theta} \approx P_{n,\theta^*}$ for all $n \geq 0$. Thus, the quantities $\varphi_j(\theta, \theta^*; r)$, $j \geq 1$, are well defined by (1.1). Then we have

Assumptions

(A1) For each $n \geq 0$, the probability measures $\{P_{n,\theta}; \theta \in \Theta\}$, defined by (2.1), are mutually absolutely continuous.

For θ, θ^* in Θ , define $\varphi_j(\theta, \theta^*; r)$ by (1.1)

Then

(A2) For each $\theta \in \Theta$, the random functions $\varphi_j(\theta, \cdot; r)$ are differentiable $[P_\theta]$ in the r -th mean at θ , uniformly in $j \geq 1$, for some integer $r \geq 1$.

Let $\dot{\varphi}_j(\theta; r)$, $j \geq 1$, be the r -th mean derivatives involved. Then

(A3) For each $\theta \in \Theta$ and each $h \in \mathbb{R}^k$,

$$E_\theta |h \cdot \dot{\varphi}_j(\theta; r)|^r < M_r(\theta, h) < \infty, \quad j \geq 1.$$

We may now formulate the main result of this paper, namely,

Theorem 2.1. Let the number r_k be defined by

$$r_k = \frac{r}{r-k}, \quad k=r-1, \dots, \frac{2r}{3}, \frac{r}{2}. \quad (2.2)$$

and let the random functions $\varphi_j(\theta, \cdot; r_k)$, $j \geq 1$, be defined by (1.1) with r being replaced by r_k . Then, for each $\theta \in \Theta$ and under assumptions (A1)-(A3), these random functions are differentiable in the r_k -th mean $[P_\theta]$ at θ , uniformly in $j \geq 1$. The r_k -th mean derivative $\dot{\varphi}_j(\theta; r_k)$ is given by

$$\dot{\varphi}_j(\theta; r_k) = (r-k)\dot{\varphi}_j(\theta; r) \quad (2.3)$$

and

$$E_\theta |h \dot{\varphi}_j(\theta; r_k)|^{r_k} \leq M_{r_k}(\theta, h) (< \infty), \quad j \geq 1. \quad (2.4)$$

3. PROOF OF MAIN RESULT.

In the course of the proof of Theorem 2.1, the following auxiliary result will be needed which is formulated here as a lemma. Namely,

Lemma 3.1 For $n, j \geq 1$ integers, let X_j and X_{nj} be r.v.'s defined on the probability space (Ω, \mathcal{F}, P) and suppose that, for some $r > 0$,

$$X_{nj} \xrightarrow{(r)} X_j, \text{ uniformly in } j \geq 1.$$

Then

$$E|X_{nj}|^r \longrightarrow E|X_j|^r, \text{ uniformly in } j \geq 1.$$

Proof. For the two cases $0 < r \leq 1$ and $r \geq 1$, use the c_r -inequality and the Minkowski inequality, respectively (see, for example, Loève [11] pages 1955-1956), in order to obtain

$$|E|X_{nj}|^r - E|X_j|^r| \leq E|X_{nj} - X_j|^r + E|X_{nj} - X_j|^{1/r}.$$

Since the right hand side above tends to 0, uniformly in $j \geq 1$, so does the left hand side. The proof is completed.

We may now proceed with the proof of the main result, namely,

Proof of Theorem 2.1. To show that, uniformly on bounded sets of h and as $(0 <) \lambda \rightarrow 0$,

$$E_\theta \left| \frac{1}{\lambda} [\varphi_j(\theta, \theta + \lambda h; r_k) - 1] - h \dot{\varphi}_j(\theta; r_k) \right|^{r_k} \longrightarrow 0, \text{ uniformly in } j \geq 1, \quad (3.1)$$

and that relation (2.4) holds.

Suppose for a moment that (2.3) holds true. Then the fact that $r_k \leq r$ implies that

$$E_\theta |h \dot{\varphi}_j(\theta; r_k)|^{r_k} \leq (r-k)^r E_\theta^{r_k/r} |h \dot{\varphi}_j(\theta; r)|^r.$$

Thus, (2.4) is satisfied by means of assumptions (A3).

In the remaining part of the proof of this theorem, all convergences will be taken as above, that is, for $(0 <) \lambda \rightarrow 0$ and uniformly on boun-

ded sets of h . Relation (3.1) holds true for $r_k=r$ by assumption (A2). Assume it to hold true for some $r_k=1,2,\dots,r-2$ and establish it for $r_{k-1}=r_{k-\delta}$ where δ is defined by the relation

$$\delta = \frac{(r-k)^2}{k}.$$

To this end, consider the relation

$$\varphi_j(\theta, \theta+\lambda h; r_{k-1}) = \varphi_j(\theta, \theta+\lambda h; r_{k-\delta}) = \varphi_j(\theta, \theta+\lambda h; r_k) \varphi_j(\theta, \theta+\lambda h; \frac{r}{\delta}) \quad (3.2)$$

which follows from (1.1). Then by means of (3.2) and (2.3), one has the following identity

$$\begin{aligned} & \frac{1}{\lambda} \left[\varphi_j(\theta, \theta+\lambda h; r_{k-1}) - 1 \right] - h \dot{\varphi}_j(\theta; r_{k-1}) \\ &= \frac{1}{\lambda} \left[\varphi_j(\theta, \theta+\lambda h; r_{k-\delta}) - 1 \right] - h \dot{\varphi}_j(\theta; r_{k-\delta}) \\ &= \left\{ \frac{1}{\lambda} \left[\varphi_j(\theta, \theta+\lambda h; r_k) - 1 \right] - h \dot{\varphi}_j(\theta; r_k) \right\} \\ &+ \left\{ \frac{1}{\lambda} \left[\varphi_j(\theta, \theta+\lambda h; \frac{r}{\delta}) - 1 \right] - h \dot{\varphi}_j(\theta; \frac{r}{\delta}) \right\} \\ &+ \left\{ \frac{1}{\lambda} \left[\dot{\varphi}_j(\theta, \theta+\lambda h; r_k) - 1 \right] \left[\varphi_j(\theta, \theta+\lambda h; \frac{r}{\delta}) - 1 \right] \right\}. \end{aligned} \quad (3.3)$$

Hence the c_r -inequality gives

$$\begin{aligned} & E_\theta \left| \frac{1}{\lambda} \left[\varphi_j(\theta, \theta+\lambda h; r_{k-1}) - 1 \right] - h \dot{\varphi}_j(\theta; r_{k-1}) \right|^{r_{k-1}} \\ &\leq c_{r_{k-1}}^2 E_\theta \left| \frac{1}{\lambda} \left[\varphi_j(\theta, \theta+\lambda h; r_k) - 1 \right] - h \dot{\varphi}_j(\theta; r_k) \right|^{r_{k-1}} \\ &+ c_{r_{k-1}}^2 E_\theta \left| \frac{1}{\lambda} \left[\varphi_j(\theta, \theta+\lambda h; \frac{r}{\delta}) - 1 \right] - h \dot{\varphi}_j(\theta; \frac{r}{\delta}) \right|^{r_{k-1}} \\ &+ c_{r_{k-1}}^2 E_\theta \left| \frac{1}{\lambda} \left[\varphi_j(\theta, \theta+\lambda h; r_k) - 1 \right] \left[\dot{\varphi}_j(\theta, \theta+\lambda h; \frac{r}{\delta}) - 1 \right] \right|^{r_{k-1}} \\ &= I_1(\lambda, j) + I_2(\lambda, j) + I_3(\lambda, j), \end{aligned} \quad (3.4)$$

where the quantities $I_i(\lambda, j)$ stand for the i -th term, $i=1,2,3$ on the right hand side of relation (3.4). Since for all values of k for which r_k is integer $r_{k-1} < r_k < r$,

$$I_1(\lambda, j) \leq c_{r_{k-1}}^2 E_\theta^{(r_{k-1})/r_k} \left| \frac{1}{\lambda} \left[\varphi_j(\theta, \theta+\lambda h; r_k) - 1 \right] - h \dot{\varphi}_j(\theta; r_k) \right|^{r_k} \rightarrow 0, \quad (3.5)$$

uniformly in $j \geq 1$, by induction hypothesis.

Next,

$$I_2(\lambda, j) \leq c_{r_k-1} E_{\theta}^{\delta(r_k-1)/r} \left| \frac{1}{\lambda} \left[\varphi_j(\theta, \theta + \lambda h; \frac{r}{\delta}) - 1 \right] - h \dot{\varphi}_j(\theta; \frac{r}{\delta}) \right| \frac{r}{\delta} \rightarrow 0, \quad (3.6)$$

uniformly in $j \geq 1$, by hypothesis (A2).

Finally to the term $I_3(\lambda, j)$ apply the Hölder inequality with

$$s = \frac{r}{k} \quad \text{and} \quad t = \frac{r}{r-k}$$

in order to get

$$\begin{aligned} I_3(\lambda, j) &\leq c_{r_k-1} E_{\theta}^{k/r} \left| \frac{1}{\lambda} \varphi_j(\theta, \theta + \lambda h; r_k) - 1 \right|^{[(r-1)/k]} \left[\frac{r}{k} \right] \\ &\quad E_{\theta}^{(r-k)/r} \left| \varphi_j(\theta, \theta + \lambda h; \frac{r}{\delta}) - 1 \right|^{(r_k-1) \cdot r/(r-k)} \\ &= c_{r_k-1} E_{\theta}^{k/r} \left| \frac{1}{\lambda} \left[\varphi_j(\theta, \theta + \lambda h; r_k) - 1 \right] \right|^{r_k} \cdot E_{\theta}^{(r-k)/r} \left| \varphi_j(\theta, \theta + \lambda h; \frac{r}{\delta}) - 1 \right|^{r/\delta}. \end{aligned} \quad (3.7)$$

By the induction hypothesis and Lemma 3.1,

$$E_{\theta} \left| \frac{1}{\lambda} \varphi_j(\theta, \theta + \lambda h; r_k) - 1 \right|^{r_k} \rightarrow E_{\theta} |h \dot{\varphi}_j(\theta; r_k)|^{r_k}, \quad (3.8)$$

uniformly in $j \geq 1$.

Also, by assumptions (A2), (A3) and Lemma 3.1,

$$E_{\theta} \left| \varphi_j(\theta, \theta + \lambda h; \frac{r}{\delta}) - 1 \right|^{r/\delta} \rightarrow 0, \quad \text{uniformly in } j \geq 1. \quad (3.9)$$

Relations (3.7)-(3.9) imply that $I_3(\lambda, j) \rightarrow 0$, uniformly in $j \geq 1$.

Hence, from (3.5), (3.6) and (3.9), relation (3.4) gives

$$E_{\theta} \left| \frac{1}{\lambda} \left[\varphi_j(\theta, \theta + \lambda h; r_k) - 1 \right] - h \dot{\varphi}_j(\theta; r_k) \right|^{r_k-1} \rightarrow 0,$$

uniformly in $j \geq 1$, which completes the proof of the theorem.

ACKNOWLEDGEMENT. The author wishes to express his sincere thanks to referee for his useful comments.

REFERENCES

1. LECAM, L. Locally asymptotically normal families of distributions. Univ. California Publ. Statist. 3, (1960) 57-98.
2. LECAM, L. On the assumptions used to prove asymptotic normality of maximum likelihood estimates. Ann. Math. Statist. 41, (1970) 802-828.
3. JOHNSON, R. A. and ROUSSAS, G.G. Asymptotically most powerful tests in Markov processes. Ann. Math. Statist. 40, (1969) 1207-1215.

4. JOHNSON, R.A. and ROUSSAS, G.G. Asymptotically tests in Markov processes. Ann. Math. Statist. 41 (1970) 918-938.
5. JOHNSON, R. A. and ROUSSAS, G.G. Applications of contiguity to multiparameter hypothesis testing. Proc. 6th. Berkeley Symp. of Math. Statist. and Prob. 1 (1972) 195-226.
6. ROUSSAS, G.G. Contiguity of Probability Measures: Some Applications in Statistics. Cambridge University Press, 1972.
7. LIND, B. and ROUSSAS, G.G. A remark on quadratic mean differentiability. Ann. Math. Statist. 43, (1972) 1030-1034.
8. LIND, B. and ROUSSAS, G.G. Cramér-type conditions and quadratic mean differentiability. Ann. Inst. Statist. Math. 29, (1977) Part A, 189-201.
9. AKRITAS, M.G. Contiguity of probability measures associated with continuous time stochastic processes. Ph. D. dissertation, Univ. of Wisconsin, 1978.
10. AKRITAS, M.G. and ROUSSAS, G.G. (1978). A characterization of contiguity by sample size and parameter rates. Symposia Mathematica, Istituto Nazionale di Alta Matematica, Vol. 25, (1980) 11. 155-170.
11. LOÈVE, M. Probability Theory (3rd ed.) Van Nostrand, New York, 1963.

Special Issue on Space Dynamics

Call for Papers

Space dynamics is a very general title that can accommodate a long list of activities. This kind of research started with the study of the motion of the stars and the planets back to the origin of astronomy, and nowadays it has a large list of topics. It is possible to make a division in two main categories: astronomy and astrodynamics. By astronomy, we can relate topics that deal with the motion of the planets, natural satellites, comets, and so forth. Many important topics of research nowadays are related to those subjects. By astrodynamics, we mean topics related to spaceflight dynamics.

It means topics where a satellite, a rocket, or any kind of man-made object is travelling in space governed by the gravitational forces of celestial bodies and/or forces generated by propulsion systems that are available in those objects. Many topics are related to orbit determination, propagation, and orbital maneuvers related to those spacecrafts. Several other topics that are related to this subject are numerical methods, nonlinear dynamics, chaos, and control.

The main objective of this Special Issue is to publish topics that are under study in one of those lines. The idea is to get the most recent researches and published them in a very short time, so we can give a step in order to help scientists and engineers that work in this field to be aware of actual research. All the published papers have to be peer reviewed, but in a fast and accurate way so that the topics are not outdated by the large speed that the information flows nowadays.

Before submission authors should carefully read over the journal's Author Guidelines, which are located at <http://www.hindawi.com/journals/mpe/guidelines.html>. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/> according to the following timetable:

Manuscript Due	July 1, 2009
First Round of Reviews	October 1, 2009
Publication Date	January 1, 2010

Lead Guest Editor

Antonio F. Bertachini A. Prado, Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, 12227-010 São Paulo, Brazil; prado@dem.inpe.br

Guest Editors

Maria Cecilia Zanardi, São Paulo State University (UNESP), Guaratinguetá, 12516-410 São Paulo, Brazil; cecilia@feg.unesp.br

Tadashi Yokoyama, Universidade Estadual Paulista (UNESP), Rio Claro, 13506-900 São Paulo, Brazil; tadashi@rc.unesp.br

Silvia Maria Giuliatti Winter, São Paulo State University (UNESP), Guaratinguetá, 12516-410 São Paulo, Brazil; silvia@feg.unesp.br