

## COINCIDENCE THEOREMS FOR SOME MULTIVALUED MAPPINGS

**B.E. RHOADES**

Department of Mathematics  
Indiana University

**S.L. SINGH and CHITRA KULSHRESTHA**

Department of Mathematics  
L.M.S. Government Postgraduate College  
Rishikesh, Dehra Dun 249201 INDIA

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**ABSTRACT.** Two coincidence theorems in a metric space are proved for a multi-valued mapping that commutes with a single-valued mapping and satisfies a general multi-valued contraction type condition.

**KEY WORDS AND PHRASES.** *Coincidence point, commuting mappings, multi-valued contraction.*

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### 1. INTRODUCTION.

Following the Banach contraction mapping, Nadler [1] introduced the concept of multi-valued contraction mappings and established that a multi-valued contraction mapping possesses a fixed point in a complete metric space. Subsequently a number of fixed point theorems in metric spaces have been proved for multi-valued mappings satisfying contractive type conditions; e.g. see [2]-[10], [11-17] and [18-20]. Jungck [21] generalized the Banach contraction principle by introducing a contraction condition for a pair of commuting mappings in a metric space. He also pointed out the potential of commuting mappings for generalizing fixed point theorems in [22] and [23]. One of the most general fixed point theorems for a generalized multi-valued contraction mapping appears in Ćirić [4]. In this paper we combine the ideas of Ćirić and Jungck to obtain two coincidence theorems for a multi-valued mapping.

Let  $(X, d)$  be a metric space. We shall follow the following notations and definitions.

$CL(X) = \{A : A \text{ is a nonempty closed subset of } X\}$ ,

$CB(X) = \{A : A \text{ is a nonempty closed and bounded subset of } X\}$ ,

$N(\epsilon, A) = \{x \in X : d(x, a) < \epsilon \text{ for some } a \in A, \epsilon > 0\}$ ,  $A \in CL(X)$ ,

and

$$H(A,B) = \begin{cases} \inf\{\varepsilon > 0 : A \subseteq N(\varepsilon, B) \text{ and } B \subseteq N(\varepsilon, A)\}, & \text{if the} \\ & \text{infimum exists} \\ \infty, & \text{otherwise} \end{cases}$$

for each  $A, B \in CL(X)$ .

$H$  is called the generalized Hausdorff distance function for  $CL(X)$  induced by  $d$ . If  $H(A,B)$  is defined for  $A, B \in CB(X)$  then the pair  $(X,H)$  is a metric space and  $H$  is called the Hausdorff metric induced by  $d$ .  $D(x,A)$  will denote the ordinary distance between  $x \in X$  and  $A$ , a nonempty subset of  $X$ . Let  $f$  be a single-valued mapping from  $X$  to  $X$  and  $T$  a multi-valued mapping from  $X$  to the nonempty subsets of  $X$ .

Definition 1. ([10]).  $T$  and  $f$  are said to commute if for each  $x \in X$ ,  $f(T(x)) = fTx \subseteq Tfx = T(f(x))$ .

Definition 2. ([21], [4]). An orbit for  $T$  at a point  $x_0$  is a sequence  $\{x_n : x_n \in Tx_{n-1}\}$ .

Definition 3. ([4]). A space  $X$  is said to be  $T$ -orbitally complete iff every Cauchy sequence of the form  $\{x_{n_i} : x_{n_i} \in Tx_{n_i-1}\}$  converges in  $X$ .

Definition 4. If for a point  $x_0 \in X$  there exists a sequence  $\{x_n\}$  such that  $fx_{n+1} \in Tx_n$ ,  $n = 0, 1, 2, \dots$ , then  $O_f(x_0) = \{fx_n : n = 1, 2, \dots\}$  is the orbit for  $(T,f)$  at  $x_0$ . We shall use  $O_f(x_0)$  as a set and as a sequence as the situation demands. Further  $O_f(x_0)$  is called a regular orbit for  $(T,f)$  if for each  $n$ ,

$$d(fx_{n+1}, fx_{n+2}) \leq H(Tx_n, Tx_{n+1}).$$

Definition 5. A space  $X$  is called  $(T,f)$ -orbitally complete iff every Cauchy sequence of the form  $\{fx_{n_i} : fx_{n_i} \in Tx_{n_i-1}\}$  converges in  $X$ .

An immediate consequence of this definition is that if the space  $X$  is complete then it is  $(T,f)$ -orbitally complete for any  $T$  and  $f$ . However, simple examples can be constructed to show that, if for some  $T$  and  $f$ ,  $X$  is  $(T,f)$ -orbitally complete then  $X$  need not be complete. It is also obvious from the fact that Definitions 2 and 3 are obtained from Definitions 4 and 5 when  $f$  is an identity mapping, and it is known that  $T$ -orbital completeness need not imply the completeness of  $X$ .

Definition 6. If for a point  $x_0 \in X$  there exists a sequence  $\{x_n\}$  such that the sequence  $O_f(x_0)$  converges in  $X$  then  $X$  is called  $(T,f)$ -orbitally complete with respect to  $x_0$  or simply  $(T,f,x_0)$ -orbitally complete.

Definition 7. A multivalued mapping  $T : X \rightarrow CL(X)$  is said to be asymptotically regular at  $x_0$  if, for each sequence  $\{x_n\}$ ,  $x_n \in Tx_{n-1}$ ,  $\lim d(x_n, x_{n+1}) = 0$

Let  $\psi = \{\phi : R_+ \rightarrow R_+ \mid \phi \text{ is upper semicontinuous and nondecreasing}\}$ .

## 2. MAIN THEOREMS.

THEOREM 1. Let  $T$  be a multi-valued mapping from a metric space  $X$  to  $CL(X)$ . If there exist a mapping  $f : X \rightarrow X$  such that  $TX \subseteq fX$ , for each  $x, y \in X$ ,

$$H(Tx, Ty) \leq \phi(\max\{D(fx, Tx), D(fy, Ty), D(fx, Ty), D(fy, Tx), d(fx, fy)\}), \quad (2.1)$$

$$\phi(t) < qt \text{ for each } t > 0, \text{ for some fixed} \quad (2.2)$$

$$0 < q < 1, \phi \in \psi,$$

$$\text{there exists an } x_0 \in X \text{ such that } T \text{ is asymptotically} \quad (2.3)$$

$$\text{regular at } x_0,$$

and

$$X \text{ is } (T, f, x_0)\text{-orbitally complete}, \quad (2.4)$$

then  $T$  and  $F$  have a coincidence point.

PROOF. Pick  $x_0 \in x$  satisfying (2.3). We shall construct two sequences  $\{x_n\}$  and  $\{y_n\}$  as follows. Since  $TX \subseteq fX$ , choose  $y_1 = fx_1 \in Tx_0$ . If  $Tx_0 = Tx_1$ , choose  $y_2 = fx_2 \in Tx_1$  such that  $y_1 = y_2$ . If  $Tx_0 \neq Tx_1$ , from the definition of  $H$  one can choose  $y_2 = fx_2 \in Tx_1$  such that  $d(y_1, y_2) \leq q^{-1}H(Tx_0, Tx_1)$ . In general, choose  $y_{n+2} = fx_{n+2} \in Tx_{n+1}$  such that  $y_{n+1} = y_{n+2}$  if  $Tx_n = Tx_{n+1}$ , and  $d(y_{n+1}, y_{n+2}) \leq q^{-1}H(Tx_n, Tx_{n+1})$  otherwise.

From (2.3),  $\lim d(y_n, y_{n+1}) = 0$ . We wish to show that  $\{y_n\}$  is Cauchy. It is sufficient to show that  $\{y_{2n}\}$  is Cauchy. Suppose  $\{y_{2n}\}$  is not Cauchy. Then there exists a positive  $\epsilon$  such that, for each integer  $2k$ , there exist integers  $2n(k)$ ,  $2m(k)$  satisfying  $2k \leq 2n(k) < 2m(k)$ , such that

$$d(y_{2n(k)}, y_{2m(k)}) > \epsilon. \quad (2.5)$$

For each integer  $2k$ , let  $2m(k)$  denote the smallest integer exceeding  $2n(k)$  for which (2.5) is satisfied. Thus

$$d(y_{2n(k)}, y_{2m(k)-2}) \leq \epsilon. \quad (2.6)$$

For each integer  $2k$ , with  $d_i = d(y_i, y_{i+1})$ ,

$$\epsilon < d(y_{2n(k)}, y_{2m(k)}) \leq d(y_{2n(k)}, y_{2m(k)-2}) + d_{2m(k)-2} + d_{2m(k)-1}.$$

Using (2.3) and (2.6) it follows that

$$\lim_k d(y_{2n(k)}, y_{2m(k)}) = \epsilon. \quad (2.7)$$

Using the triangular inequality,

$$|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \leq d_{2m(k)-1},$$

and

$$|d(y_{2n(k)+1}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \leq d_{2n(k)} + d_{2m(k)-1}.$$

From (2.3), (2.6) and (2.7) it follows that

$$\lim_k d(y_{2n(k)}, y_{2m(k)-1}) = \lim_k d(y_{2n(k)+1}, y_{2m(k)-1}) = \epsilon.$$

For each integer  $2k$  define  $p(2k) = d(y_{2n(k)}, y_{2m(k)})$ ,  $q(2k) = d(y_{2n(k)+1}, y_{2m(k)-1})$ , and  $r(2k) = d(y_{2n(k)}, y_{2m(k)-1})$ . Then

$$\begin{aligned} p(2k) &\leq d_{2n(k)} + d(y_{2n(k)+1}, y_{2m(k)}) \\ &\leq d_{2n(k)} + q^{-1}H(Tx_{2n(k)}, Tx_{2m(k)-1}) \\ &\leq d_{2n(k)} + q^{-1}\phi(\max\{D(fx_{2n(k)}, Tx_{2n(k)}), D(fx_{2m(k)-1}, Tx_{2m(k)-1}), \\ &\quad D(fx_{2n(k)}, Tx_{2m(k)-1}), D(fx_{2m(k)-1}, Tx_{2n(k)}), \\ &\quad d(fx_{2n(k)}, fx_{2m(k)-1})\}) \\ &\leq d_{2n(k)} + q^{-1}\phi(\max\{d_{2n(k)}, d_{2m(k)-1}, p(2k), q(2k), r(2k)\}). \end{aligned}$$

Since  $\phi$  is upper semicontinuous, taking the limit as  $k \rightarrow \infty$  yields

$$\epsilon \leq q^{-1}\phi(\max\{0, 0, \epsilon, \epsilon, \epsilon\}) = q^{-1}\phi(\epsilon) < \epsilon,$$

a contradiction.

Thus  $\{y_n\}$  is Cauchy, and since  $fX$  is  $(T, f, x_0)$ -orbitally complete,  $\{y_n\}$  converges to a point  $u$  in  $X$ . Hence there exists a point  $z$  in  $fX$  such that  $u = fz$ . Then

$$\begin{aligned} D(fz, Tz) &\leq d(fz, fx_{n+1}) + D(fx_{n+1}, Tz) \\ &\leq d(fz, fx_{n+1}) + H(Tx_n, Tz) \\ &\leq d(fz, fx_{n+1}) + \phi(\max\{D(fx_n, Tx_n), D(fz, Tz), \\ &\quad D(fx_n, Tz), D(fz, Tx_n), d(fx_n, fz)\}) \\ &\leq d(fz, fx_{n+1}) + \phi(\max\{d(fx_n, fx_{n+1}), D(fz, Tz), d(fx_n, fz) \\ &\quad + D(fz, Tz), d(fz, fx_{n+1}), d(fx_n, fz)\}). \end{aligned}$$

Taking the limit as  $n \rightarrow \infty$  yields

$$D(fz, Tz) \leq \phi(\max\{0, D(fz, Tz), D(fz, Tz), 0, 0\}) < qD(fz, Tz) ,$$

which implies  $fz \in Tz$  .

If, in (2.1) the terms  $D(fx, Ty)$  ,  $D(fy, Tx)$  are replaced by  $[D(fx, Ty) + D(fy, Tx)]/2$  , then  $\{fx_n\}$  can be proved to be a Cauchy sequence without the assumption of the asymptotic regularity of  $T$  .

Replacing the condition  $TX \subseteq fX$  by orbital regularity one obtains the following.

THEOREM 2. Let  $T : X \rightarrow CL(X)$  . If there exists a selfmap  $f$  of  $X$  such that (2.1),

(2')  $\psi(t) < t$  for each  $t > 0$  ,  $\phi \in \psi$  , and

(3') there exists a sequence  $\{x_n\}$  such that the orbit  $O_f(x_0)$  is regular and asymptotically regular, and  $X$  is  $(T, f, x_0)$ -orbitally complete,

then  $T$  and  $f$  have a coincidence point.

PROOF. Examining the proof of Theorem 1, the only change is to note that the regularity of the orbit  $O_f(x_0)$  allows one to replace the inequality  $d(y_n, y_{n+1}) \leq g^{-1}H(Tx_n, Tx_{n+1})$  with the stronger inequality  $d(y_n, y_{n+1}) \leq H(Tx_n, Tx_{n+1})$  .

If  $f$  is not the identity mapping, then a commuting  $T$  and  $f$  need not have a common fixed point. An example illustrating this fact appears in [19], where the commutativity of  $T$  and  $f$  is defined by  $fTx = Tfx$  ,  $X$  not necessarily a metric space.

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The theorems of this paper generalize the corresponding results in [21], and the open question of [21] still remains; namely, what additional conditions will guarantee the existence of a common fixed point for  $T$  and  $f$  ?

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