

## INTEGRABILITY OF DERIVATIONS OF CLASSICAL SOLUTIONS OF DIRICHLET'S PROBLEM FOR AN ELLIPTIC EQUATION

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**ABSTRACT.** The present work is concerned with integrability properties of derivatives of classical solutions of Dirichlet's problem for a linear second-order elliptic equation  $Lu = f$ . With the aid of special weighted Hilbert spaces of locally square integrable functions, we determine the nature of singularities that  $f$  can have near the boundary, in order that such classical solutions are in the Sobolev space  $W^1$ . By means of an example it is shown that the obtained result is exact.

**KEY WORDS AND PHRASES.** *Linear second-order elliptic equation. Dirichlet's problem. Classical solutions. Sobolev spaces. Weighted Hilbert spaces of locally square integrable functions.*

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### 1. INTRODUCTION.

The question of whether the classical solution  $u(x)$  of Dirichlet's problem for an elliptic equation  $Lu = f$  is in the Sobolev space  $W^1$  was studied in [2], [3] and [4]. The main result concerning this question is that  $u \in W^1$  provided the coefficients of  $L$  are essentially bounded and  $f \in L^2$ . Here, we prove a result showing that  $u$  may be in  $W^1$  even when  $f$  is not in the class  $L^2$ . With the aid of special weighted Hilbert spaces of functions, we determine exactly the class of functions  $f$ , for which  $u \in W^1$ .

Let  $G \subset \mathbb{R}^n$  be a bounded region, whose boundary  $\partial G$  is a closed  $(n-1)$ -dimensional surface in the class  $C^2$ . For  $p > 0$ , we let  $G^p = \{x \in G: d(x) > p\}$ , where  $d(x) = \text{dist}(x, \partial G)$ . As was shown in [5], there exist positive numbers  $m, b$ , depending only on  $G$ , and a function  $r(x) \in C^2(\bar{G})$  such that

$$r(x) = d(x), \quad x \in G \setminus G^m,$$
$$bd(x) \leq r(x) \leq b^{-1}d(x), \quad x \in G.$$

Moreover, if  $p \in [0, m]$ , then  $G^p$  is a region with boundary  $\partial G^p$  in  $C^2$ , and the relation  $x_p = x_p(x) = x - p\underline{N}(x)$ ,  $x \in \partial G$ , determines a  $C^1$ -diffeomorphism of  $\partial G$  onto  $\partial G^p$ . Here  $\underline{N}(x)$  denotes the unit outward normal to  $\partial G$  at  $x$ .

In  $G$  we consider a non-self-adjoint operator defined by

$$Lu = - D_i(a_{ij}D_j u) + a_i D_i u + au,$$

where we used summation convention, that is, we sum over an index that appears twice, and  $D_i = \partial / \partial x_i$ . It is assumed that  $a_{ij}, a_i, a \in C(G)$ , and  $L$  is strictly elliptic in  $G$ , that is

$$v|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \mu|\xi|^2; \quad v, \mu = \text{const} > 0, \quad x \in G, \tag{1.1}$$

for all real vectors  $\xi = (\xi_1, \dots, \xi_n)$ ;  $|\xi|^2 = \sum_{i=1}^n \xi_i^2$ .

Throughout this paper,  $u(x) \in C^2(G) \cap C(\bar{G})$  denotes a classical solution of the problem

$$\left. \begin{aligned} Lu &= f(x), \quad x \in G, \\ u|_{\gamma_G} &= 0, \end{aligned} \right\} \tag{1.2}$$

where  $f \in C(G)$ .

We shall employ usual notation  $W^{k,2}(G)$  and  $\overset{\circ}{W}^{k,2}(G)$  for Sobolev spaces [1], but, conventionally, without the index "2". By  $L_s^2(G)$  we shall denote the Hilbert space of all measurable functions  $v(x)$  in  $G$  for which

$$\|v\|_{L_s^2(G)}^2 = \int_G r^s(x)v^2(x)dx < \infty.$$

Lemma. If  $v \in \overset{\circ}{W}^1(G)$ , then  $v \in L_{-2}^2(G)$  and

$$\|v\|_{L_{-2}^2(G)} \leq C \|Dv\|_{L^2(G)}, \tag{1.3}$$

where  $D = (D_1, \dots, D_n)$  and  $C > 0$  depends only on  $m, b$  and the diameter of  $G^m$ .

The proof is similar to that of the lemma in [5].

## 2. MAIN RESULTS

Our main result is the following:

**THEOREM.** Let the operator  $L$  be strictly elliptic and have coefficients  $a_i, a \in L^\infty(G)$ . If problem (1.2) has a classical solution  $u$ , then for arbitrary  $f \in L_s^2(G)$  with  $s \leq 2$ , this solution is in  $W^1(G)$ .

Moreover,

$$\|u\|_{W^1(G)} \leq C(\|f\|_{L_s^2(G)} + \|u\|_{L^2(G)}) \tag{2.1}$$

where  $C$  is independent of  $f$  and  $u$ .

**Proof.** Let  $f^{(p)} \in L^2(G)$  be the function defined by

$$f^{(p)}(x) = \begin{cases} f(x), & x \in G^p, \\ 0, & x \in G \setminus G^p. \end{cases}$$

Recalling the properties of  $r(x)$  and using the absolute continuity of Lebesgue integral, we have

$$\lim_{p \rightarrow 0} \int_G r^s(f - f^{(p)})^2 dx = \lim_{p \rightarrow 0} \int_{G \setminus G^p} r^s f^2 dx = 0,$$

that is, as  $p \rightarrow 0$  the functions  $f^{(p)}$  converge to  $f$  in the  $L^2_S(G)$ -norm.

Since  $u \in C(\bar{G})$  and  $u|_{\partial G} = 0$ , we can choose a decreasing sequence of numbers  $\{p_k\}$  such that

$$|u| < \frac{1}{k}, \quad x \in G \setminus G^{p_k}. \tag{2.2}$$

We write  $G_k$  for  $G^{p_k}$  and  $f_k$  for  $f^{(p_k)}$ ,  $k = 1, 2, \dots$

Let  $q$  be sufficiently large positive number such that

$$q + \inf_G a - \frac{1}{4\nu} \sup_G \sum_{i=1}^n a_i^2 - \frac{1}{2} > 0 \tag{2.3}$$

and consider the problem

$$\left. \begin{aligned} Lu_k + qu_k &= f + qu, \quad x \in G_k, \\ u_k|_{\partial G_k} &= 0. \end{aligned} \right\} \tag{2.4}$$

Since  $f + qu \in L^2(G_k)$ , problem (2.4) has a weak solution in  $W^1(G_k)$  [1, p. 175]. Such solution is understood to be a function in  $\overset{\circ}{W}^1(G_k)$  satisfying the integral identity

$$\int_{G_k} [a_{ij} D_j u_k D_i v + (a_i D_i u_k + au_k + au_k + qu_k)v] dx = \int_{G_k} (f + qu)v dx \tag{2.5}$$

for all  $v \in \overset{\circ}{W}^1(G_k)$ . Taking  $v = u_k$  in (2.5), and using (1.1) and the well known Cauchy inequality  $|ab| \leq \epsilon a^2 + (1/4\epsilon)b^2$ , we obtain

$$\begin{aligned} (\nu - \epsilon) \int_{G_k} |Du_k|^2 dx + (q + \inf_G a - \frac{1}{4\epsilon} \sup_G \sum_{i=1}^n a_i^2 - \frac{1}{2}) \int_{G_k} u_k^2 dx \\ \leq \left| \int_{G_k} f u_k dx \right| + \frac{1}{2} q^2 \int_{G_k} u_k^2 dx. \end{aligned}$$

Letting  $u_k^* = \begin{cases} u_k & x \in G_k, \\ 0 & x \in G \setminus G_k \end{cases}$  and recalling the definition of  $f_k$ , the last inequality can be rewritten as

$$\begin{aligned} (\nu - \epsilon) \int_G |Du_k^*|^2 dx + (q + \inf_G a - \frac{1}{4\epsilon} \sup_G \sum_{i=1}^n a_i^2 - \frac{1}{2}) \int_G u_k^{*2} dx \\ \leq \left| \int_G f_k u_k^* dx \right| + \frac{1}{2} q^2 \int_G u_k^2 dx. \end{aligned} \tag{2.6}$$

The first term on the right of (2.6) we estimate by using Cauchy inequality and (1.3) as follows:

$$\begin{aligned} \left| \int_G f_k u_k^* dx \right| &\leq \frac{1}{4\epsilon} \int_G r^2 f_k^2 dx + \epsilon_1 \int_G (u_k^{*2}/r^2) dx \\ &\leq \frac{1}{4\epsilon_1} (\max_G r^{2-s}) \int_G r^s f^2 dx + \epsilon_1 C \int_G |Du_k^*|^2 dx. \end{aligned} \tag{2.7}$$

Since  $2 - s \geq 0$ , the function  $r^{2-s}$  is bounded, and it follows from (2.3), (2.6) and (2.7) with sufficiently small  $\epsilon, \epsilon_1$  that

$$\int_G (|Du_k^*|^2 + u_k^{*2}) dx \leq C \left( \int_G r^s f^2 dx + \int_G u^2 dx \right) \leq K, \tag{2.8}$$

where  $C$  depends only on  $m, b, s, G$  and the coefficients of  $L$ . Hence,  $K$  is independent of  $k$ . Consequently, there is a subsequence of  $\{u_k^*\}$  weakly converging

in the metric of  $W^1(G)$  to some function  $w \in \overset{\circ}{W}^1(G)$ . Without loss of generality, we can assume that the sequence itself weakly converges to  $w$ . In view of (2.8), we have

$$\|w\|_{W^1(G)}^2 \leq C(\|f\|_{L_s^2(G)}^2 + \|u\|_{L^2(G)}^2). \tag{2.9}$$

Since  $u \in C^2(\overline{G}_k)$  and  $u_k$  is a weak solution of problem (2.4), the function  $u - u_k$  is a weak solution in  $W^1(G_k)$  of the problem

$$\left. \begin{aligned} Lv_k + qv_k &= 0, \quad x \in G_k, \\ v|_{\partial G_k} &= u|_{\partial G_k} \end{aligned} \right\}$$

The conditions imposed in our theorem and the fact that  $q + a > 0$ ; cf.(2.3), are sufficient to apply the weak maximum principle [1,p. 168] to the function  $u - u_k$ . Hence,

$$|u - u_k^*| < \max_{\partial G_k} |u|$$

almost everywhere (a.e.) in  $G_k$ . Taking (2.2) into consideration, we find that a.e. in  $G$

$$|u - u_k^*| < \frac{1}{k},$$

that is, the sequence  $\{u_k^*\}$  uniformly converges to  $u$  a.e. in  $G$ . But, as was shown above, the same sequence weakly converges to  $w$  in the metric of  $W^1(G)$ . Hence,  $u = w$  a.e. on  $G$ , which completes the proof of the theorem.

Now we show that the condition  $f \in L_s^2(G)$ ,  $s \leq 2$ , is exact in the sense that, it cannot be weakened to allow functions  $f$  with singularities near the boundary of degree higher than the second. If  $f$  is in  $L_{2+h}^2(G)$  for any  $h > 0$ , but it is not in  $L_2^2(G)$ , then the inclusion  $u \in W^1(G)$  may be false. This may be seen from the following example.

Let  $B$  be the unit disk  $\{|x| < 1\}$  in  $R^2$ . In  $B$  consider the function  $u(x) = |x|^2(1 - |x|)^{\frac{1}{2}}$ . It is easily verified that  $u \in C^2(B) \cap C(\overline{B})$ ,  $u|_{|x|=1} = 0$  and  $u \notin W^1(B)$ . At the same time  $\Delta u \in L_{2+h}^2(B)$  for any  $h > 0$ ;  $\Delta = D_{11} + D_{22}$ , while  $\Delta u \notin L_2^2(B)$ .

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