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FIBONACCI POLYNOMIALS OF ORDER K, MULTINOMIAL EXPANSIONS AND PROBABILITY

ANDREAS N. PHILIPPOU
Department of Mathematics
Univeristy of Patras
Patras, Greece

COSTAS GEORGHIOU
School of Engineering
Univeristy of Patras
Patras, Greece

GEORGE N. PHILIPPOU
Department of General Studies
Higher Technical Institute
Nicosia, Cyprus

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ABSTRACT. The Fibonacci polynomials of order k are introduced and two expansions of them are obtained, in terms of the multinomial and binomial coefficients, respectively. A relation between them and probability is also established. The present work generalizes results of [2] - [4] and [5].

KEY WORDS AND PHRASES. Fibonacci polynomials of order k, expansions, multinonial and binomial coefficients, probability.

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1. INTRODUCTION.

In the sequel, k is a fixed integer greater than or equal to 2, x is a positive and finite real number, and n is a nonnegative integer unless otherwise specified. Motivated introduced the Fibonacci polynomials of order k, to be denoted by $f_n^{(k)}(x)$, and study some of their properties. First we observe that $f_n^{(k)}(x)$ are generalized polynomials, appropriate extensions for the Fibonacci and Pell numbers of order k [3], [4], and identical to the r-bonacci polynomials $R_n(x)(n\geq-(r-2))$ of [1] for k=r and n≥0. Then we state and prove a theorem, which provides two expansions of $f_n^{(k)}(x)$ (n≥1) in terms of the multinomial and binomial coefficients, respectively. Hoggatt and Bicknell [1], amoung other results, give another expansion of $f_n^{(k)}(x)$, in terms of the elements of the left - justified k-nomial triangle. The latter, however, are less widely known and used than the multinomial and binomial coefficients, and on this account our expansions may be considered better. As a corollary to our theorem, we derive several results of [2]-[4] and [5]. We also obtain a relation between $f_n^{(k)}(x)$ $(n\geq 1)$ and probability.

2. THE FIBONACCI POLYNOMIALS OF ORDER K AND MULTINOMIAL COEFFICIENTS.

In this section, we introduce the Fibonacci polynomials of order k and derive two expansions of them im terms of the multinomial and binomial coefficients, respectively. The proof is along the lines of [2] and [4].

DEFINITION. The sequence of polynomials $\{f_n^{(k)}(x)\}_{n=0}^{\infty}$ is said to be the sequel of Fibonacci polynomials of order k if $f_0^{(k)}(x)=0$, $f_1^{(k)}(x)=1$, and

$$f_n^{(k)}(x) = \begin{cases} \sum_{i=1}^{n} x^{k-i} f_{n-i}^{(k)}(x) & \text{if } 2 \leq n \leq k \\ \sum_{i=1}^{k} x^{k-i} f_{n-i}^{(k)}(x) & \text{if } n \geq k+1. \end{cases} \tag{2.1}$$

If $f_n^{(r)}(x)=0$ for $-(r-2)\leq n\leq -1$, Hoggatt and Bicknell [1] call $R_n(x)=f_n^{(r)}(x)$ $(n\geq -(r-2))$ r-bonacci polynomials.

Denoting by $F_n(x)$, $f_n^{(k)}$ and $P_n^{(k)}$, respectively, the Fibonacci polynomials [5], the Fibonacci numbers of order k [3], and the Pell numbers of order k [4], it follows from (2.1) that

$$f_n^{(2)}(x) = F_n(x), \quad f_n^{(k)}(1) = f_n^{(k)} \text{ and } F_n^{(k)}(2) = P_n^{(k)}. \tag{2.2}$$

We now proceed to show the following lemma.

LEMMA. Let $\{f_n^{(k)}(x)\}_{n=0}^{\infty}$ be the sequence of Fibonacci polynomials of order k, and denote its generating function by $g_k(s;x)$. Then, for $|s|<x/(1+x^k)$,

$$g_k(s;x) = \frac{s(1-\frac{s}{x})}{1-\frac{s}{x}(1+x^k-s^k)} = \frac{s}{1-x^k[\frac{s}{x}+(\frac{s}{x})^2+\ldots+(\frac{s}{x})^k]}.$$

PROOF. We see from the definition that $f_2^{(k)}(x)=x^{k-1}$, $xf_n^{(k)}(X)-f_{n-1}^{(k)}(x)=x^k f_{n-1}^{(k)}(x)$ for $3\leq n\leq k+1$, and $xf_n^{(k)}(x)-f_{n-1}^{(k)}(x)=x^k f_{n-1}^{(k)}(x)-f_{n-1-k}^{(k)}(x)$ for $n\geq k+2$. Therefore,

$$f_n^{(k)}(x) = \begin{cases} \frac{1}{x}(1+x^k)f_{n-1}^{(k)}(x), & 3 \le n \le k+1 \\ \frac{1}{x}(1+x^k)f_{n-1}^{(k)}(x) - \frac{1}{x}f_{n-1-k}^{(k)}(x), & n \ge k+2 \end{cases}$$

$$= \begin{cases} [\frac{1}{x}(1+x^k)]^{n-2}x^{k-1}, & 2 \le n \le k+1 \\ \frac{1}{x}(1+x^k)f_{n-1}^{(k)}(x) - \frac{1}{x}f_{n-1-k}^{(k)}(x), & n \ge k+2. \end{cases} \qquad (2.3)$$

It may be seen, by means of (2.3) and induction on n, that

$$f_n^{(k)}(x) \le [\tfrac{1}{x}(1+x^k)]^{n-2}\,x^{k-1}, \quad n\ge 2, \qquad (2.4)$$

which implies the convergence of $g_k(s;x)$ for $|s|<x/(1+x^k)$. Next, by means of (2.3), we observe that

$$g_k(s;x) = \sum_{n=0}^{\infty} s^n f_n^{(k)}(x)$$

$$= s + \sum_{n=2}^{k+1} s^n[\tfrac{1}{x}(1+x^k)]^{n-2}x^{k-1} + \sum_{n=k+2}^{\infty} s^n f_n^{(k)}(x), \qquad (2.5)$$

and

$$\sum_{n=k+2}^{\infty} s^n f_n^{(k)}(x) = \tfrac{1}{x}(1+x^k)\sum_{n=k+2}^{\infty} s^n f_{n-1}^{(k)}(x) - \tfrac{1}{x}\sum_{n=k+2}^{\infty} s^n f_{n-1-k}^{(k)}(x)$$

$$= \tfrac{s}{x}(1+x^k)\left\{\sum_{n=0}^{\infty} s^n f_n^{(k)}(x) - s - \sum_{n=2}^{k+1} s^n[\tfrac{1}{x}(1+x^k)]^{n-2}x^{k-1}\right\} - \tfrac{1}{x}s^{k+1}\sum_{n=1}^{\infty} s^n f_n^{(k)}(x)$$

$$= [\tfrac{s}{x}(1+x^k) - \tfrac{s^{k+1}}{x}]g_k(s;x) - \tfrac{s^2}{x} - \sum_{n=2}^{k+1} s^n[\tfrac{1}{x}(1+x^k)]^{n-2}x^{k-1}. \qquad (2.6)$$

The last two relations give

$$g_k(s;x) = s + \tfrac{s}{x}(1 + x^k - s^k)g_k(s;x) - \tfrac{s^2}{x},$$

so that

$$g_k(s;x) = \frac{s(1-\frac{s}{x})}{1-\frac{s}{x}(1+x^k-s^k)} = \frac{s}{1-x^k[\frac{s}{x}+(\frac{s}{x})^2+\ldots+(\frac{s}{x})^k]}.$$

We will employ the above lemma to establish the following expansions of $f_n^{(k)}(x)$ $(n\ge 1)$.

THEOREM. Let $\{f_n^{(k)}(x)\}_{n=0}^{\infty}$ be the Fibonacci polynomials of order k. Then

(a) $$f_{n+1}^{(k)}(x) = \sum_{n_1,\ldots,n_k} \binom{n_1+\ldots+n_k}{n_1,\ldots,n_k} x^{k(n_1+\ldots+n_k)-n}, \quad n\ge 0,$$

where the summation is over all non-negative integers $n_1,\dots,n_k$ such that $n_1+2n_2+..+kn_k=n$;

$$\text{(b)}\quad f_{n+1}^{(k)}(x) = \left(\frac{1+x^k}{x}\right)^n \sum_{i=0}^{[n/(k+1)]} (-1)^i \binom{n-ki}{i} x^{ki}(1+x^k)^{-(k+1)i}$$

$$- \frac{1}{x}\left(\frac{1+x^k}{x}\right)^{n-1} \sum_{i=0}^{[(n-1)/k+1)]} (-1)^i \binom{n-1-ki}{i} x^{ki}(1+x^k)^{-(k+1)i},\ n\ge 1,$$

$n\ge 1,$

where, as usual, $[x]$ denotes the greatest interger in x.

PROOF. First we show (a). Let $|s| < x/(1+x^k)$, so that $|x^k[\frac{s}{x}+(\frac{s}{x})^2+\dots+(\frac{s}{x})^k]|<1$. Let n_i $(1\le i\le k)$ be non-negative integers as specified below. Then, using the lemma and the multinomial theorem, and replacing n by $n-\sum_{i=1}^{k}(i-1)n_i$, we get,

$$\sum_{n=0}^{\infty} s^n f_{n+1}^{(k)}(x) = \{1-x^k[\frac{s}{x} + (\frac{s}{x})^2+\dots+(\frac{s}{x})^k]\}^{-1}$$

$$= \sum_{n=0}^{\infty} \{x^k[\frac{s}{x} + (\frac{s}{x})^2+\dots+(\frac{s}{x})^k]\}^n$$

$$= \sum_{n=0}^{\infty} x^{kn} \sum_{\substack{n_1,\dots,n_k \ni \\ n_1+\dots+n_k=n}} \binom{n}{n_1,\dots,n_k} (\frac{s}{x})^{n_1+2n_2+\dots+kn_k}$$

$$= \sum_{n=0}^{\infty} s^n \sum_{\substack{n_1,\dots,n_k \ni \\ n_1+2n_2+\dots+kn_k=n}} \binom{n_1+\dots+n_k}{n_1,\dots,n_k} n^{k(n_1+\dots+n_k)-n}, \tag{2.8}$$

from which (a) follows.

We now proceed to establish (b). Let $0 < s < x/(1 + x^k)$, so that $|\frac{s}{x}(1+x^k-s^k)| < 1$. Then, using the lemma and the binomial theorem, replacing n by $n-ki$, and setting

$$B_n^{(k)}(x) = (\frac{1+x^k}{x})^n \sum_{i=0}^{[n/(k+1)]} (-1)^i \binom{n-ki}{i} x^{ki}(1+x^k)^{-(k+1)i},\ n\ge 0, \tag{2.9}$$

we get

$$\sum_{n=0}^{\infty} s^n f_{n+1}^{(k)}(x) = (1-\frac{s}{x})[1-\frac{s}{x}(1+x^k-s^k)]^{-1}$$

$$= (1-\frac{s}{x}) \sum_{n=0}^{\infty} [\frac{s}{x}(1+x^k-s^k)]^n$$

$$= (1-\frac{s}{x})^n \sum_{i=0}^{\infty} (\frac{s}{x})^n \sum_{i=0}^{\infty} (-1)^i \binom{n}{i} (1+x^k)^{n-i} s^{ki}$$

$$= (1-\frac{s}{x}) \sum_{n=0}^{\infty} s^n \sum_{i=0}^{[n/(k+1)]} (-1)^i \binom{n-ki}{i} (1+x^k)^{n-(k+1)i} x^{-(n-ki)}$$

$$= (1-\frac{s}{x}) \sum_{n=0}^{\infty} s^n B_n^{(k)}(x), \text{ by } (2.9)$$

$$= 1+ \sum_{n=1}^{\infty} s^n [B_n^{(k)}(x) - \frac{1}{x} B_{n-1}^{(k)}(x)], \tag{2.10}$$

since $B_0^{(k)}(x) = 1$ from (2.9). The last two relations show part (b) of the theorem.

We have the following obvious corollary to the theorem, by means of relation (2.2).

COROLLARY 2.1. Let $F_n(x)$, $f_n^{(k)}$ and $P_n^{(k)}$ denote the Fibonacci polynomials, the Fibonacci numbers of order k and the Pell numbers of order k, respectively. Then,

(a) $$F_{n+1}(x) = \sum_{i=0}^{[n/2]} \binom{n-i}{i} x^{n-2i}, \quad n\geq 0;$$

(b)(i) $$f_{n+1}^{(k)} = \sum_{\substack{n_1,\ldots,n_k \ni \\ n_1+2n_2+\ldots+kn_k=n}} \binom{n_1+\ldots+n_k}{n_1,\ldots,n_k}, \quad n\geq 0;$$

(b)(ii) $$f_{n+1}^{(k)} = 2^n \sum_{i=0}^{[n/(k+1)]} (-1)^i \binom{n-ki}{i} 2^{-(k+1)i}$$

$$-2^{n-1} \sum_{i=0}^{[(n-1)/(k+1)]} (-1)^i \binom{n-1-ki}{i} 2^{-(k+1)i}, \quad n\geq 1;$$

(c)(i) $$P_{n=1}^{(k)} = \sum_{\substack{n_1,\ldots,n_k \ni \\ n_1+2n_2+\ldots+kn_k=n}} \binom{n_1+\ldots+n_k}{n_1,\ldots,n_k} 2^{k(n_1+\ldots+n_k)-n}, \quad n\geq 0;$$

(c)(ii) $$P_{n+1}^{(k)} = (\frac{1+2^k}{2})^n \sum_{i=0}^{[n/(k+1)]} (-1)^i \binom{n-ki}{i} 2^{ki} (1+2^k)^{-(k+1)i}$$

$$- \frac{1}{2}(\frac{1+2^k}{2})^{n-1} \sum_{i=0}^{[(n-1)/(k+1)]} (-1)^i \binom{n-1-ki}{i} 2^{ki} (1+2^k)^{-(k+1)i}, \quad n\geq 1.$$

REMARK. Part (a) of Corollary 2.1 was proposed by Swamy [5], who appears to be the first to introduce the Fibonacci polynomials. Part (b)(i) was first shown in [3], while (b) and (c), respectively, were later proved by a different method in [2] and [4].

The following corollary relates the Fibonacci polynomials of order k to probability.

COROLLARY 2.2. Let $\{f_n^{(k)}(x)\}_{n=0}^{\infty}$ be the sequence of Fibonacci polynomials of order k, and denote by N_k the number of trials until the occurrence of the kth consecutive success in independent trials with success probability p $(0<p<1)$. Then,

$$P(N_k=n+k) = p^{n+k}\left(\frac{1-p}{p}\right)^{n/k} f_{n+1}^{(k)}\left(\left(\frac{1-p}{p}\right)^{1/k}\right), \quad n\geq 0.$$

PROOF. It follows directly from Theorem 3.1 of [3] and part (a) of the present theorem.

In particular, Corollary 2.2. reduces to the following results of [2] and [4], respectively, by means of (2.2).

Let N_k be as above, and set $p=(1+2^k)^{-1}$. Then,

$$P(N_k=n+k) = \frac{2^n}{(1+2^k)^{n+k}} P_{n+1}^{(k)}, \quad n\geq 0. \tag{2.11}$$

Let N_k be as above, and set $p=1/2$. Then,

$$P(N_k=n+k) = \frac{1}{2^{n+k}} f_{n+1}^{(k)}, \quad n\geq 0. \tag{2.12}$$

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