

ON A GENERALIZATION OF CLOSE-TO-CONVEXITY

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(Received July 9, 1979 and in revised form August 30, 1982)

ABSTRACT. A class T_k of analytic functions in the unit disc is defined in which the concept of close-to-convexity is generalized. A necessary condition for a function f to belong to T_k , radius of convexity problem and a coefficient result are solved in this paper.

KEY WORDS AND PHRASES. Close-to-convex functions, functions of bounded boundary rotation, necessary condition, radius of convexity, generalized Koebe function.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. Primary 30A32, Secondary 30A34.

1. INTRODUCTION.

This paper is directed to mathematical specialists or non-specialists familiar with multivalent functions [1], and to close-to-convex functions [2].

Let V_k be the class of functions of bounded boundary rotation and K be the class of close-to-convex functions. We generalize the concept of close-to-convexity in the following direction.

Definition. Let f with $f(z) = cz + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $E = \{z: |z| < 1\}$, $|c|=1$ and $f'(z) \neq 0$. Then $f \in T_k$, $k \geq 2$, if there exist a function $g \in V_k$ such that, for $z \in E$

$$\operatorname{Re} \frac{f'(z)}{g'(z)} > 0. \quad (1.1)$$

It is clear that $T_2 \equiv K$.

Using a method by Kaplan [2], we have

THEOREM 1. Let $f \in T_k$. Then with $z = re^{i\theta}$ and $\theta_1 < \theta_2$

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \left\{ \frac{(zf'(z))'}{f'(z)} \right\} d\theta > -\frac{k}{2} \pi \quad (1.2)$$

REMARK 1. From theorem 1, we can interpret some geometric meaning for the class T_k . For simplicity, let us suppose that the image domain is bounded by an analytic curve C . At a point on C , the outward drawn normal has an angle $\arg[e^{i\theta} f'(e^{i\theta})]$. Then from (1.2), it follows that the angle of the outward drawn normal turns back at most $\frac{k}{2}\pi$. This is a necessary condition for a function f to belong to T_k . It will be interesting to see if this condition is also sufficient.

REMARK 2. Goodman [3] defines the class $K(\beta)$ of functions as follows.

Let f with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in E and $f'(z) \neq 0$. Then for $\beta \geq 0$, $f \in K(\beta)$, if for $z = re^{i\theta}$ and $\theta_1 < \theta_2$

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \left[\frac{(zf'(z))'}{f'(z)} \right] d\theta > -\beta\pi$$

We note that $T_k \subset K(\frac{k}{2})$.

2. MAIN RESULTS

From remark 2 and results given in [3] for the class $K(\beta)$, we have at once

THEOREM 2. Let $f \in T_k$.

(i) Denote by $L(r, f)$ the length of the image of the circle $|z| = r$ under f and by $A(r, f)$ the area of $f(|z| = r)$. Then for $0 < r < 1$,

$$(a) \quad L(r, f) \leq L(r, F_k),$$

$$(b) \quad A(r, f) \leq A(r, F_k),$$

where F_k is defined by, for $z \in E$,

$$\begin{aligned} F_k(z) &= \frac{1}{(k+2)} \left[\left(\frac{1+k}{1-z} \right)^{\frac{1}{2}k+1} - 1 \right] \\ &= z + \sum_{n=2}^{\infty} A_n(k) z^n \end{aligned} \quad (2.1)$$

and clearly $F_k \in T_k$.

$$(ii) \quad |a_n| \leq A_n(k), \quad n = 2, 3, \dots, k \geq 2$$

where $A_n(k)$ is defined by (2.1). This result is sharp for each $n \geq 2$.

$$(iii) \quad \text{For } z = re^{i\theta}, \quad 0 \leq r < 1,$$

$$\frac{(1-r)^{\frac{1}{2}k}}{(1+r)^{\frac{1}{2}k+2}} \leq |f'(z)| \leq \frac{(1+r)^{\frac{1}{2}k}}{(1-r)^{\frac{1}{2}k+2}}$$

These bounds are sharp, equality being attained for the function F_k defined by (2.1).

We also need the following result.

Lemma 1 [4]. Let $g \in V_k$. Then there are two starlike functions s_1 and s_2 such that for $z \in E$

$$g'(z) = \frac{(s_1(z)/z)^{\frac{1}{2}k+\frac{1}{2}}}{(s_2(z)/z)^{\frac{1}{2}k-\frac{1}{2}}}$$

THEOREM 3. $f \in T_k$ if and only if

$$f'(z) = \frac{(k_1'(z))^{\frac{1}{2}k+\frac{1}{2}}}{(k_2'(z))^{\frac{1}{2}k-\frac{1}{2}}}, \quad k_1, k_2 \in k$$

PROOF: From definition 1, we have

$$f'(z) = g'(z)h(z), \quad g \in V_k \text{ and } \operatorname{Re} h(z) > 0.$$

Using lemma 1, we know that there are two starlike functions s_1 and s_2 such that $z \in E$,

$$g'(z) = \frac{(s_1(z)/z)^{\frac{1}{2}k+\frac{1}{2}}}{(s_2(z)/z)^{\frac{1}{2}k-\frac{1}{2}}}$$

Thus

$$\begin{aligned} f'(z) &= \frac{(s_1(z)/z)^{\frac{1}{2}k+\frac{1}{2}}}{(s_2(z)/z)^{\frac{1}{2}k-\frac{1}{2}}} h(z) = \frac{((s_1(z)h(z))/z)^{\frac{1}{2}k+\frac{1}{2}}}{((s_2(z)h(z))/z)^{\frac{1}{2}k-\frac{1}{2}}} \\ &= \frac{(k_1'(z))^{\frac{1}{2}k+\frac{1}{2}}}{(k_2'(z))^{\frac{1}{2}k-\frac{1}{2}}} \end{aligned}$$

where k_1 and k_2 are two suitable selected close-to-convex functions.

Lemma 2. Let H be analytic and be defined as

$$H(z)g'(z) = (zg'(z))', \quad g \in V_k \text{ and } H(z) = \left(\frac{k}{4} + \frac{1}{2}\right)h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)h_2(z),$$

$$\operatorname{Re} h_i(z) > 0, \quad i=1,2, \quad h_i(0)=1$$

Then

$$\frac{1}{2\pi} \int_0^{2\pi} |H(z)|^2 d\theta \leq \frac{1 + (k^2-1)r^2}{1-r^2} \quad (z = re^{i\theta})$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} |H'(z)| d\theta \leq \frac{k}{1-r^2}$$

PROOF: By the representation formula due to Paatero [5], we can write

$$H(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1+ze^{it}}{1-ze^{-it}} d\mu(t),$$

where

$$\int_0^{2\pi} d\mu(t) = 2\pi, \text{ and } \int_0^{2\pi} |d\mu(t)| \leq k\pi$$

$$\text{Let } H(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

Then

$$c_n = \frac{1}{\pi} \int_0^{2\pi} e^{-int} d\mu(t), \text{ and so for } n \geq 1,$$

$$|c_n| \leq \frac{1}{\pi} \int_0^{2\pi} |d\mu(t)| \leq k$$

Thus

$$\frac{1}{2\pi} \int_0^{2\pi} |H(z)|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} \leq (1+k)^2 \sum_{n=1}^{\infty} r^{2n} = \frac{1+(k^2-1)r^2}{1-r^2}$$

Also

$$H'(z) = \frac{1}{\pi} \int_0^{2\pi} \frac{e^{it}}{(1-ze^{-it})^2} d\mu(t)$$

Thus

$$\frac{1}{2\pi} \int_0^{2\pi} |H'(z)| d\theta \leq \frac{1}{\pi} \int_0^{2\pi} \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1-re^{i(\theta+t)}|^2} d\theta |d\mu(t)| \leq \frac{1}{1-r^2} \frac{1}{\pi} \int_0^{2\pi} |d\mu(t)| \leq \frac{k}{1-r^2}$$

THEOREM 4: Let $f \in T_k$. Then for $n \geq 1$,

$$\left| |a_{n+1}| - |a_n| \right| \leq c(k) n^{\frac{k}{2} - 1},$$

where $c(k)$ is a constant and depends only on k .

PROOF: Since $f \in T_k$, we have for $z \in E$,

$$f'(z) = g'(z)h(z), \quad g \in V_k \text{ and } \operatorname{Re} h(z) > 0$$

Set

$$F(z) = z(zf'(z))' = zg'(z)[H(z)h(z) + zh'(z)], \quad (2.2)$$

where $\operatorname{Re} h(z) > 0$ and $H(z)g'(z) = (zg'(z))'$, with

$$H(z) = \left(\frac{k}{4} + \frac{1}{2}\right)h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)h_2(z), \quad \operatorname{Re} h_i(z) > 0, \quad i=1,2, \quad h_1(0)=1$$

Thus, for $\xi \in E$ and $n \geq 1$;

$$|(n+1)^2 \xi a_{n+1} - n^2 a_n| \leq \frac{1}{2\pi r^{n+1}} \int_0^{2\pi} |z-\xi| |F(z)| d\theta,$$

and by using lemma 1 and (2.2), we obtain

$$|(n+1)^2 \xi a_{n+1} - n^2 a_n| \leq \frac{1}{2\pi r^{n+1}} \int_0^{2\pi} |z-\xi| \frac{|s_1(z)|^{\frac{1}{2}k+\frac{1}{2}}}{|s_2(z)|^{\frac{1}{2}k-\frac{1}{2}}} |H(z)h(z) + zh'(z)| d\theta, \quad (2.3)$$

where s_1, s_2 are starlike functions.

It is well-known [1] that for starlike function $s \in S$,

$$\frac{r}{(1+r)^2} \leq |s(z)| \leq \frac{r}{(1-r)^2} \quad (2.4)$$

Let $0 < r < 1$. Then by a result of Golusin [6, p162], there exists a z_1 with

$|z_1| = r$ such that for all z , $|z| = r$,

$$|z - z_1| |s_1(z)| \leq \frac{2r^2}{1-r^2}. \quad (2.5)$$

From (2.3)-(2.5), we have

$$|(n+1)^2 \xi a_{n+1} - n^2 a_n| \leq \frac{1}{2\pi r^{n+1}} \left(\frac{4}{r}\right)^{\frac{1}{2}k-\frac{1}{2}} \left(\frac{2r^2}{1-r^2}\right) \left(\frac{r}{(1-r)^2}\right)^{\frac{1}{2}k-\frac{1}{2}} \int_0^{2\pi} |H(z)h(z) + zh'(z)| d\theta \quad (2.6)$$

Now as in [7], we have with $z = re^{i\theta}$

$$\left. \begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |h(z)|^2 d\theta &\leq \frac{1+3r^2}{1-r^2} \\ \text{and} \\ \frac{1}{2\pi} \int_0^{2\pi} |zh'(z)| d\theta &\leq \frac{2r}{1-r^2}, \quad \text{where } \operatorname{Re} h(z) > 0. \end{aligned} \right\} \quad (2.7)$$

Also

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |H(z)h(z) + zh'(z)| d\theta &\leq \frac{1}{2\pi} \int_0^{2\pi} |H(z)h(z)| d\theta + \frac{1}{2\pi} \int_0^{2\pi} |zh'(z)| d\theta \\ &\leq \frac{(1+(k^2-1)r^2)^{\frac{1}{2}}(1+3r^2)^{\frac{1}{2}}}{1-r^2} + \frac{2r}{1-r^2} \end{aligned} \quad (2.8)$$

by using Schwarz's inequality, lemma 2 and (2.7).

Hence from (2.6) and (2.8), we have

$$|(n+1)^2 \xi a_{n+1} - n^2 a_n| \leq \frac{1}{r^{n+1}} 2^{\frac{1}{2}k} \left[(1+(k^2-1)r^2)^{\frac{1}{2}} + 1 \right] \frac{1}{(1-r)^{\frac{1}{2}k+1}},$$

and so choosing $|\xi| = r = \left(\frac{n}{n+1}\right)^2$, we obtain for $n \geq 1$

$$n^2 ||a_{n+1}| - |a_n|| \leq \left[(1 + (k^2 - 1)r^2)^{\frac{1}{2}} + 1 \right] e^2 2^{\frac{1}{2}k+2} \left(\frac{4}{3}\right)^{\frac{1}{2}k+1} n^{\frac{1}{2}k+1}$$

Thus

$$||a_{n+1}| - |a_n|| \leq c(k)n^{\frac{1}{2}k-1}.$$

The function F_k defined by (2.1) shows that the index $\left(\frac{k}{2} - 1\right)$ is best possible.

We now evaluate the radius of convexity for the class T_k .

THEOREM 5: Let $f \in T_k$. Then the radius R of the circle which f maps onto a convex domain is given by

$$R = \frac{1}{2} \left[(k+2) - \sqrt{(k+2)^2 - 4k} \right].$$

The function F_k defined by (2.1) shows that this result is best possible. In particular, when $k = 2$, $R = 2\sqrt{3}$, which is well known. This result also follows from the remark in [3, p.23].

PROOF: By definition

$$zf'(z) = ag'(z)h(z) \quad g \in V_k; \operatorname{Re} h(z) > 0.$$

Thus

$$\frac{(zf'(z))'}{f'(z)} = \frac{(zg'(z))'}{g'(z)} + \frac{zh'(z)}{h(z)}$$

and so

$$\operatorname{Re} \frac{(zf'(z))'}{f'(z)} \geq \operatorname{Re} \frac{(zg'(z))'}{g'(z)} - \left| \frac{zh'(z)}{h(z)} \right|$$

For $g \in V_k$, it is well known [9] that, for $z = re^{i\theta}$, $0 < r < 1$,

$$\operatorname{Re} \frac{(zg'(z))'}{g'(z)} \geq \frac{r^2 - kr + 1}{1 - r^2}$$

Hence

$$\operatorname{Re} \frac{(zf'(z))'}{f'(z)} \geq \frac{r^2 - kr + 1}{1 - r^2} - \frac{2r}{1 - r^2} = \frac{r^2 - (k+2)r + 1}{1 - r^2}$$

This gives the required result.

REMARKS 3.

(i). We also note that the extremal function $F_k(z)$ defined by (2.1) is the same function as $F_\beta(z)$ defined by equation (2.6) in [3]. As A. W. Goodman has pointed out that this function is sometime referred to as the generalized Koebe function.

(ii). We conjecture that the class T_k is a proper subclass of the class $K(\beta)$ as defined in [3], since in the definition of T_k , $g \in V_k$ and we know that $g \in V_k$, $2 \leq k \leq 4$, is convex in one direction and all the functions in one direction form a proper subclass of the class of close-to-convex functions.

(iii). It remains open whether T_k is a linear invariant family.

ACKNOWLEDGEMENT. The author wishes to thank Prof. A. W. Goodman and the referee for their interesting and helpful comments, which influenced the final version of this paper.

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