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AN AMENABILITY PROPERTY OF ALGEBRAS OF FUNCTIONS ON SEMIDIRECT PRODUCTS OF SEMIGROUPS

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ABSTRACT. Let S_1 and S_2 be semitopological semigroups, $S_1 \textcircled{\tau} S_2$ a semidirect product. An amenability property is established for algebras of functions on $S_1 \textcircled{\tau} S_2$. This result is used to decompose the kernel of the weakly almost periodic compactification of $S_1 \textcircled{\tau} S_2$ into a semidirect product.

KEY WORDS AND PHRASES. *Semitopological semigroup, semidirect product, compactification, amenability, strongly almost periodic, weakly almost periodic, kernel.*

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1. INTRODUCTION. Let S_1, S_2 be semitopological semigroups (in the terminology of Berglund and Hofmann (1)) with identities, each denoted by 1. That is S_1 and S_2 have (Hausdorff) topologies relative to which multiplication in S_1 and S_2 is separately continuous.

Let $\tau : S_2 \times S_1 \to S_1$ be a separately continuous map satisfying for each $s_1, t_1 \in S_1$, $s_2, t_2 \in S_2$, $\tau(s_2, s_1t_1) = \tau(s_2,s_1)\tau(s_2,t_1)$, $\tau(s_2t_2,t_1) = \tau(s_2,\tau(t_2,t_1))$, $\tau(s_2,1) = 1$, and $\tau(1,\cdot)$ is the identity map.
We shall assume the map $(s_1,s_2) \to s_1\tau(s_2,t_1) : S_1 \times S_2 \to S_1$ is continuous for each $t_1 \in S_1$. The *semidirect product* $S_1 \textcircled{\tau} S_2$ of S_1 and S_2 is the topological space

$S_1 \times S_2$ equipped with multiplication $(s_1,s_2)(t_1,t_2) = (s_1\,\tau(s_2,t_1),\ s_2t_2)$. The above conditions on τ imply that $S_1 \circledT S_2$ is a semitopological semigroup with identity $(1,1)$.

Let F be a closed translation invariant sub-C*-algebra of $C(S_1 \circledT S_2)$ (see §2 below) containing the constant functions. In previous papers (2) and (3), the author has formulated the necessary and sufficient conditions for the decomposition of the F-compactification of $(S_1 \circledT S_2)$, into a semidirect product. The decomposition may be written symbolically as

$$(S_1 \circledT S_2)^F = S_1{}^G \circledP S_2{}^H \tag{1.1}$$

where $G = \{f(\cdot,1) : f \in F\}$ and $H = \{f(1,\cdot): f \in F\}$ and equality denotes canonical isomorphism (ρ being another semidirect product).

Applications of this decomposition were then made to the almost periodic (AP), strongly almost periodic (SAP) and left-uniformly continuous (LUC) cases. The situation is less well-behaved in the weakly almost periodic (WAP) case. For example, if $S_1 = S_2$ is any commutative topological semigroup with identity for which $WAP(S_1) = AP(S_1)$, then (1.1) fails even if $S_1 \circledT S_2$ is taken to be the special case of a direct product (Junghenn (4)).

However, in the present paper we shall prove an amenability property of algebras of functions on $S_1 \circledT S_2$ which generalizes a result of Junghenn (5) and provides conditions under which the kernel of the WAP-compactification of $S_1 \circledT S_2$ can be decomposed into a semidirect product.

2. PRELIMINARIES. Throughout this section S denotes a semitopological semigroup and C(S) the C*-algebra of bounded continuous complex-valued functions on S. We define operators R_t and L_s on C(S) by

$$R_t\, f(s) = f(st) = L_s\, f(t) \qquad (s,t \in S\ ;\ f \in C(S))$$

Let F be a conjugate closed, norm closed linear subspace of C(S) containing the constant function 1. Then F is *right* (resp. *left*) *translation invariant* if $R_S\, F \subset F$ (resp. $L_S\, F \subset F$) ; *translation invariant* if it is both left and right translation invariant.

A *mean* on F is a positive linear functional μ in F^*, the dual of F, such that $\mu(1) = 1 = ||\mu||$. We denote by M(F) the set of all means on F. A mean μ

on F is *multiplicative* if $\mu(fg) = \mu(f)\mu(g)$, f, g ε F. We denote the set of all multiplicative means on F by MM(F).

If F is left (resp. right) translation invariant, then a mean μ is *left* (resp. *right*) *invariant* if, for each f ε F, s ε S, we have $\mu(L_s f) = \mu(f)$ (resp. $\mu(R_s f) = \mu(f)$). The set of all left (resp. right) invariant means on F shall be denoted by LIM(F) (resp. RIM(F)). F is *left* (resp. *right*) *amenable* if LIM(F) $\neq \emptyset$ (resp. RIM(F) $\neq \emptyset$). If F is translation invariant and both left and right amenable, F is called *amenable*.

Now suppose F is left translation invariant. For each $\nu \in F^*$ define $T_\nu : F \to C(S)$ by $(T_\nu f)(s) = \nu(L_s f)$, f ε F, s ε S. Then F is *left introverted* if $T_\nu F \subset F$ for each $\nu \in M(F)$. If F is an algebra, then F is *left-m-introverted* if $T_\nu F \subset F$ for each $\nu \in MM(F)$. Right introversion and right-m-introversion are defined in an analogous manner.

If F is a sub-C^* - algebra of C(S) then S^F denotes the spectrum (=space of nonzero continuous complex homomorphisms) of F equipped with the relativized weak* topology, and $e: S \to S^F$ the evaluation mapping.

If F is admissible (i.e. F is translation invariant, left-m- introverted, containing the constant functions) then a binary operation $(x,y) \to xy$ may be defined on S^F relative to which the pair (S^F, e) has the following properties:

(i) S^F is a compact Hausdorff topological space and a semigroup such that for each $y \in S^F$, the mapping $x \to xy : S^F \to S^F$ is continuous;

(ii) $e : S \to S^F$ is a continuous homomorphism with range dense in S^F such that for each s ε S, the mapping $x \to e(s)x : S^F \to S^F$ is continuous; and

(iii) $e^*C(S^F) = F$.

The pair (S^F, e) is the *canonical F-compactification of* S.

Let K(S), called the *kernel* of S, denote the minimal ideal of S. We shall use the amenability property in the next section to decompose the kernel of the WAP-compactification of $S_1 Ⓣ S_2$ into a semidirect product.

3. THE AMENABILITY THEOREM. Let S_1 and S_2 denote semitopological semigroups with identities and $S_1 Ⓣ S_2$ a semidirect product as defined in §1. We shall denote by

$q_1 : S_1 \to S_1 \circledT S_2$ and $q_2 : S_2 \to S_1 \circledT S_2$ the injection mappings ($q_1(s_1) = (s_1,1)$, $q_2(s_2) = (1,s_2)$, for $s_1 \in S_1$, $s_2 \in S_2$). Let $q_i^* : C(S_1 \circledT S_2) \to C(S_i)$ denote the dual mapping of q_i, $i = 1,2$.

THEOREM 3.1

(a) Suppose F is a left translation invariant, left introverted closed subspace of $C(S_1 \circledT S_2)$ containing the constant functions, and the semigroup $D = \{s_2 \in S_2 : \overline{\tau(s_2,S_1)} = S_1\}$ is dense in S_2. Then F is left amenable if q_1^*F and q_2^*F are left amenable.

(b) Suppose F is a right translation invariant, right introverted closed subspace of $C(S_1 \circledT S_2)$ containing the constant functions. Then F is right amenable if q_1^*F and q_2^*F are right amenable.

PROOF. To prove (a) choose any $\mu_1 \in LIM(q_1^*F)$, and for each $f \in F$ define $(Uf)(s_2) = \mu_1(q_1^*(L_{(1,s_2)}f))$, $s_2 \in S_2$.

Then $U : F \to q_2^*F$: For let $f \in F$. Since F is left introverted, $T_\nu F \subset F$, $\forall \nu \in M(F)$ where $(T_\nu f)(s_1,s_2) = \nu(L_{(s_1,s_2)}f)$, $f \in F$, $(s_1,s_2) \in (S_1 \circledT S_2)$.

Observe that

$$(Uf)(s_2) = \mu_1(q_1^*(L_{(1,s_2)}f)) = T_{(\mu_1 \circ q_1^*)}f(1,s_2)$$
$$= q_2^*(T_{(\mu_1 \circ q_1^*)}f)(s_2), \text{ for any } s_2 \in S_2.$$

Then $UF \in q_2^*F$ since $T_{(\mu_1 \circ q_1^*)} f \in F$. Furthermore, $U : F \to q_2^*F$ is a positive linear operator of norm 1 since μ_1 is a mean on q_1^*F.

Let $\mu_2 \in LIM(q_2^*F)$ and put $\mu = \mu_2 \circ U$. Then $\mu \in F^*$, $\mu(f) \geq 0$ for each $f \geq 0$ in F, and $\mu(1) = 1$. Thus μ is a mean on F.

We must show $\mu \in LIM(F)$. Observe that for $s_1 \in S_1$, $s_2 \in S_2$,

$$q_1^*(L_{(s_1,1)}L_{(1,s_2)}f) = q_1^*(L_{(1,s_2)(s_1,1)}f) \tag{3.1}$$
$$= q_1^*(L_{(\tau(s_2,s_1),\ s_2)}f).$$

Furthermore, for any $g \in F$, s_1, t_1, $\in S_1$,

$$q_1^*(L_{(s_1,1)}g)(t_1) = L_{(s_1,1)}g(t_1,1) = g(s_1t_1,1)$$
$$= (q_1^*g)(s_1t_1) = L_{s_1}(q_1^*g)(t_1)).$$

Thus,

$$q_1^*(L_{(s_1,1)}L_{(1,s_2)}f) = L_{s_1}(q_1^*L_{(1,s_2)}f). \tag{3.2}$$

By (3.1) and (3.2) we obtain for $d \in D$, $s_1 \in S_1$, $f \in F$,

$$\mu_1(q_1^*(L_{(\tau(d,s_1),\ d)}f)) = \mu_1(q_1^*(L_{(s_1,1)}L_{(1,d)}f)) \tag{3.3}$$

$$= \mu(L_{s_1}q_1^*(L_{(1,d)}f)) = \mu_1\ (q_1^*(L_{(1,d)}f))$$

$$= (Uf)(d).$$

By the definition of D and the continuity in the variable s_1 of the extreme left side of (3.3), we obtain,

$$\mu_1(q_1^*(L_{(s_1,d)}f)) = (Uf)(d) \quad (d \in D,\ s_1 \in S_1).$$

Since $\overline{D} = S_2$ we therefore have

$$\mu_1(q_1^*(L_{(s_1,s_2)}f)) = (Uf)(s_2) \quad (s_1 \in S_1,\ s_2 \in S_2).$$

That is,

$$UL_{(s_1,1)}f = Uf,\ \forall s_1 \in S_1. \tag{3.4}$$

Observe that for s_2, $t_2 \in S_2$,

$$U(L_{(1,s_2)}f)(t_2) = \mu_1(q_1^*(L_{(1,t_2)}L_{(1,s_2)}f))$$

$$= \mu_1(q_1^*(L_{(1,s_2t_2)}f)) = (Uf)(s_2t_2)$$

$$= (L_{s_2}Uf)(t_2).$$

Thus,

$$U(L_{(1,s_2)}f) = L_{s_2}Uf,\ s_2 \in S_2. \tag{3.5}$$

By (3.4) and (3.5) we obtain for any $s_1 \in S_1$, $s_2 \in S_2$

$$\mu(L_{(s_1,s_2)}f) = \mu(L_{(1,s_2)}L_{(s_1,1)}f)$$

$$= \mu_2[U(L_{(1,s_2)}L_{(s_1,1)}f)]$$

$$= \mu_2[L_{s_2}U(L_{(s_1,1)}f)]$$

$$= \mu_2(L_{s_2}Uf) = \mu_2(Uf) = \mu(f).$$

Thus $\mu \in LIM(F)$ and we are done.

The proof of (b) is done in an analogous manner and is, in fact, much easier. Choose any $\mu_1 \in RIM(q_1^*F)$ and for each $f \in F$, define

$(Uf)(s_2) = \mu_1(q_1^*(R_{(1,s_2)}f))$, $s_2 \in S_2$.

Then $U : F \to q_2^*F$ since F is right introverted. Furthermore, $U: F \to q_2^*F$ is a positive linear operator of norm 1 since μ_1 is a mean on q_1^*F. Let $\mu_2 \in RIM(q_2^*F)$

and put $\mu = \mu_2 \cdot U$. Then μ is a mean on F and we must show $\mu \in RIM(F)$.

Observe that for any $s_1, t_1 \in S_1, s_2, t_2 \in S_2, f \in F$,

$$q_1{*}R_{(1,t_2)}R_{(s_1,s_2)}f(t_1) = f[(t_1,1)(1,t_2)(s_1,s_2)]$$
$$= f[(t_1\tau(t_2,s_1), t_2s_2)]$$
$$= f[(t_1\tau(t_2,s_1),1)(1,t_2s_2)]$$
$$= q_1{*}R_{(1,t_2s_2)}f(t_1\tau(t_2,s_1))$$
$$= R_{\tau(t_2,s_1)}q_1{*}R_{(1,t_2s_2)}f(t_1).$$

Thus, $q_1{*}R_{(1,t_2)}R_{(s_1,s_2)}f = R_{\tau(t_2,s_1)}q_1{*}R_{(1,t_2s_2)}f$,
and therefore since $\mu_1 \in RIM(q_1{*}F)$,

$$U(R_{(s_1,s_2)}f)(t_2) = \mu_1(q_1{*}R_{(1,t_2)}R_{(s_1,s_2)}f)$$
$$= \mu_1(R_{\tau(t_2,s_1)}q_1{*}R_{(1,t_2s_2)}f) = \mu_1(q_1{*}R_{(1,t_2s_2)}f)$$
$$= R_{s_2}\ Uf(t_2).$$

Then,

$$\mu(R_{(s_1,s_2)}f) = \mu_2[U(R_{(s_1,s_2)}f)] = \mu_2(R_{s_2}Uf)$$
$$= \mu_2(Uf) = \mu(f).$$

Hence $\mu \in RIM(F)$ and we are done.

4. Application to $K(S_1 \circledT S_2)^{WAP}$

Let S be a semitopological semigroup and let SAP(S) denote the closed linear span in C(S) of the coefficients of all finite-dimensional continuous unitary representation of S. SAP(S) is called the space of *strongly almost periodic* functions on S. Let WAP(S) = $\{f \in C(S) : R_S f$ is relatively weakly compact$\}$. WAP(S) is called the space of *weakly almost periodic* functions on S. (See Berglund, Junghenn and Milnes (6) for properties of SAP(S), WAP(S)).

In (3) it was shown that if S_1, S_2 are semitopological semigroups with identities then

$$(S_1 \circledT S_2)^{SAP} = X \circledP S_2^{SAP} \tag{4.1}$$

where $(S_1 \circledT S_2)^{SAP}$ and S_2^{SAP} are the canonical SAP-compactifications of $S_1 \circledT S_2$ and S_2 respectively, X is a compact topological group which is a homomorphic image of the canonical SAP-compactification of S_1, and equality denotes canonical isomorphism.

We now prove the following lemma which shall be used in the decomposition of the kernel.

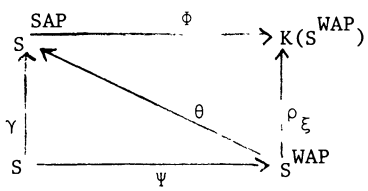
LEMMA. 4.1 Let S be a semitopological semigroup such that WAP(S) is amenable. Let (S^{WAP},Ψ) be a WAP-compactification of S, ξ the identity of $K(S^{WAP})$, ρ_ξ: $S^{WAP}\to K(S^{WAP})$ be right multiplication. Then $(K(S^{WAP}), \rho_\xi\Psi)$ is an SAP-compactification of S.

PROOF. Since WAP(S) is amenable, $K(S^{WAP}) = S^{WAP}\xi$ and is a compact topological group (deLeeuw and Glicksberg (7)). Then ρ_ξ maps S^{WAP} onto $K(S^{WAP})$ and $\rho_\xi\Psi$: $S \to K(S^{WAP}) = S^{WAP}\ \xi$ is a continuous homomorphism with range dense in $S^{WAP}\xi$. Observe that $\rho_\xi|_{K(S^{WAP})}$ is the identity mapping on $K(S^{WAP})$.

Let (S^{SAP}, γ) be the canonical SAP-compactification of S. By the universal mapping property of SAP and WAP compactifications, there exist continuous homomorphisms Φ and θ such that $\Phi:S^{SAP}\to K(S^{WAP})$, $\theta : S^{WAP}\to S^{SAP}$ and $\Phi\gamma = \rho_\xi\Psi, \theta\Psi = \gamma$.

Observe further that since $\Phi\theta\Psi = \rho_\xi\Psi$, then $\Phi\theta = \rho_\xi$ by the continuity of $\Phi\theta$, ρ_ξ and the fact that $\overline{\Psi(S)} = S^{WAP}$.

All of the above relations are illustrated in the following commutative diagram:



It suffices to show that Φ is one-to-one so that $K(S^{WAP})$ will be an SAP-compactification of S. Let y_1, $y_2 \in S^{SAP}$. Then there exist x_1, $x_2 \in S^{WAP}$ such that $\theta(x_1) = y_1$ and $\theta(x_2) = y_2$. Suppose $\Phi(y_1) = \Phi(y_2)$. Then $\Phi(\theta(x_1)) = \Phi(\theta(x_2))$. Since S^{SAP} is a compact topological group and $\theta(\xi)$ is an idempotent in S^{SAP}, $\theta(\xi)$ is the identity of S^{SAP}. Thus $\theta(x_i) = \theta(x_i)\theta(\xi) = \theta(x_i\xi)(i=1,2)$, so $\Phi(\theta(x_1\xi)) = \Phi(\theta(x_2\xi))$. On the other hand,

$$\Phi(\theta(x_i\xi)) = \rho_\xi(x_i\xi) = x_i\xi \ (i = 1, 2),$$

so, $x_1\xi = x_2\xi$, and hence $y_1 = \theta(x_1\xi) = \theta(x_2\xi) = y_2$. //

We shall use the relation (4.1), the above lemma and the results in the following discussion to establish conditions under which we may express $K[(S_1 \circledτ S_2)^{WAP}]$ as a semidirect product

$$K[(S_1 \circledτ S_2)^{WAP}] = X \circledρ K(S_2^{WAP})$$

where equality denotes canonical isomorphism and X is a compact topological group.

We shall assume that $WAP(S_1)$ and $WAP(S_2)$ are amenable. By (deLeeuw and Glicksberg (7), Lemma 5.2) since $q_i : S_i \to S_1 \circledτ S_2$ is a continuous homomorphism for $i = 1,2$, then $F_1 = q_1^*WAP(S_1 \circledτ S_2) \subset WAP(S_1)$, and $F_2 = q_2^*WAP(S_1 \circledτ S_2) \subset WAP(S_2)$. (In fact, equality holds in the latter.) Thus, $q_1^*WAP(S_1 \circledτ S_2)$ and $q_2^*WAP(S_1 \circledτ S_2)$ are amenable and if we assume $D = \{s_2 \in S_2: \overline{\tau(s_2,S_1)} = S_2\}$ is dense in S_1, then $WAP(S_1 \circledτ S_2)$ is amenable by Theorem 3.1.

By ((7), Theorem 4.11) $K[(S_1 \circledτ S_2)^{WAP}]$ and $K(S_2^{WAP})$ are compact topological groups. Furthermore, by Lemma 4.1, $K[(S_1 \circledτ S_2)^{WAP}]$ is a SAP-compactification of $S_1 \circledτ S_2$, and $K(S_2^{WAP})$ is a SAP-compactification of S_2 (symbollically denoted by $K[(S_1 \circledτ S_2)^{WAP}] = (S_1 \circledτ S_2)^{SAP}$, and $K(S_2^{WAP}) = S_2^{SAP}$, respectively, where equality denotes canonical isomorphism). Thus, we have proved the following

PROPOSITION 4.2 Let S_1, S_2 be semitopological semigroups with identities and $S_1 \circledτ S_2$ a semidirect product. Suppose $WAP(S_1)$, $WAP(S_2)$ are amenable, and $D = \{s_2 \in S_2 : \overline{\tau(s_2,S_1)} = S_1\}$ is dense in S_2. Then $WAP(S_1 \circledτ S_2)$ is amenable. Furthermore, we may represent $K[(S_1 \circledτ S_2)^{WAP}]$ as a semidirect product $K[(S_1 \circledτ S_2)^{WAP}] = X \circledρ K(S_2^{WAP})$, where equality denotes canonical isomorphism, $(S_1 \circledτ S_2)^{WAP}$ and S_2^{WAP} are canonical WAP-compactifications of $S_1 \circledτ S_2$ and S_2 respectively, and X is a compact topological group which is a continuous homomorphic image of the canonical SAP-compactification of S_1.//

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