

ALMOST-CONTINUOUS PATH CONNECTED SPACES

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ABSTRACT. M. K. Singal and Asha Rani Singal have defined an almost-continuous function $f:X \rightarrow Y$ to be one in which for each $x \in X$ and each regular-open set V containing $f(x)$, there exists an open U containing x such that $f(U) \subset V$. A space Y may now be defined to be almost-continuous path connected if for each $y_0, y_1 \in Y$ there exists an almost-continuous $f:I \rightarrow Y$ such that $f(0) = y_0$ and $f(1) = y_1$. An investigation of these spaces is made culminating in a theorem showing when the almost-continuous path connected components coincide with the usual components of Y .

KEY WORDS AND PHRASES. Almost-continuous functions, path connected spaces.

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1. INTRODUCTION.

The concept of an almost continuous function $f:X \rightarrow Y$ has been defined in [1] as one in which for each $x \in X$ and each regular-open V containing $f(x)$ there exists an open set U containing x such that $f(U) \subset V$. Using this concept we make the following two definitions:

DEFINITION 1. The function $f:I \rightarrow Y$ is an almost-continuous path (a.c. path)

from y_0 to y_1 if f is almost continuous, $f(0) = y_0$ and $f(1) = y_1$.

DEFINITION 2. The space Y is a.c. path connected if for each $y_0, y_1 \in Y$, there exists an a.c. path from y_0 to y_1 .

The regular-open sets in Y may be used as a base to form the semi-regular topology T_s on Y from which $f:X \rightarrow (Y, T)$ is almost-continuous if and only if $f:X \rightarrow (Y, T_s)$ is continuous. Thus, Definition 2 may be restated as (Y, T) is a.c. path connected if for each y_0, y_1 in Y , there is a continuous $f:I \rightarrow (Y, T_s)$ such that $f(0) = y_0$ and $f(1) = y_1$. In view of this observation, many of the known results for path connected spaces in the usual sense also apply to a.c. path connected spaces. For example, if $y_0 \in Y$, then Y is a.c. path connected if and only if for each $y \in Y$, there is an a.c. path from y to y_0 . Furthermore, slight variations in known results may sometimes be made to easily produce statements concerning a.c. path connected spaces. An example is that if $f:X \rightarrow Y$ is an almost-continuous surjection and X is path connected, then Y is a.c. path connected.

2. MAIN RESULTS.

THEOREM 1. Every a.c. path connected space Y is connected.

PROOF. Assume $Y = U \cup V$ where U and V are open in (Y, T) and $U \cap V = \emptyset$. Then U and V are regular-open so that $U \cup V$ is a separation of (Y, T_2) . But (Y, T_s) is path connected, hence connected. The contradiction implies Y is connected.

Let R be the reals with the usual topology, Q the set of rational numbers and R_Q the reals with the topology generated by the usual open intervals together with Q as a subbase. Since the semi-regular topology associated with R_Q is the usual topology on R , it follows that $f:X \rightarrow R_Q$ is almost-continuous if and only if $f:X \rightarrow R$ is continuous. Similarly, $f:I \rightarrow R_Q \times R_Q$ is almost-continuous if and only if $f:I \rightarrow R \times R$ is continuous. These observations lead to the following example which shows the converse of Theorem 1 is false.

EXAMPLE 1. Let $Y = \{(x, y) : y = \sin(1/x), 0 < x \leq 1\} \subset R \times R$. Then $\bar{Y} = Y \cup \{(x, y) : -1 \leq y \leq 1\}$ in $R \times R$ as well as $R_Q \times R_Q$. Hence \bar{Y} is connected, but not a.c. path connected.

Example 1 also shows that a space may be a.c. path connected but its closure may not be a.c. path connected.

For a given space Y , define xRY to mean there is an a.c. path from x to y . We see immediately that R is an equivalence relation on Y . The resulting equivalence classes are referred to as a.c. path connected components of Y . Consideration of the semi-regular topology reveals that each a.c. path connected component in Y is open (and therefore closed) if and only if each point of Y has an a.c. path connected neighborhood.

THEOREM 2. A space Y is a.c. path connected if and only if it is connected and each $y \in Y$ has an a.c. path connected neighborhood.

PROOF. If Y is an a.c. path connected space, then Y is connected by Theorem 1 and each $y \in Y$ has an a.c. path connected neighborhood by the remarks preceding Theorem 2.

Conversely, the hypothesis and the remarks preceding Theorem 2 show that the only a.c. path connected component of Y is Y itself. Therefore, Y is a.c. path connected.

THEOREM 3. Let Y be a space. If (a) each a.c. path connected component in Y is open or (b) if each point $y \in Y$ has an a.c. path connected neighborhood, then the a.c. path connected components of Y coincide with the usual components of Y .

PROOF. The remarks preceding Theorem 2 show that conditions (a) and (b) are equivalent. So if we assume that each a.c. path connected component of Y is open, then each point of Y has an a.c. path connected neighborhood. In particular, the a.c. path connected component $[y]$ is an a.c. path connected neighborhood of y . Thus, $[y]$ is connected by Theorem 1. It follows that $[y] \subseteq C(y)$, where $C(y)$ is the usual component of $y \in Y$. Since $[y]$ is both open and closed in Y , $[y]$ is both open and closed in $C(y)$. But $C(y)$ connected implies $[y] = C(y)$.

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