

ALMOST-CONTINUOUS PATH CONNECTED SPACES

LARRY L. HERRINGTON

Department of Mathematics
Louisiana State University at Alexandria
Alexandria, Louisiana 71402 U.S.A.

and

PAUL E. LONG

Department of Mathematics
The University of Arkansas at Fayetteville
Fayetteville, Arkansas 72701 U.S.A.

(Received October 31, 1980)

ABSTRACT. M. K. Singal and Asha Rani Singal have defined an almost-continuous function $f:X \rightarrow Y$ to be one in which for each $x \in X$ and each regular-open set V containing $f(x)$, there exists an open U containing x such that $f(U) \subset V$. A space Y may now be defined to be almost-continuous path connected if for each $y_0, y_1 \in Y$ there exists an almost-continuous $f:I \rightarrow Y$ such that $f(0) = y_0$ and $f(1) = y_1$. An investigation of these spaces is made culminating in a theorem showing when the almost-continuous path connected components coincide with the usual components of Y .

KEY WORDS AND PHRASES. Almost-continuous functions, path connected spaces.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. 54C10.

1. INTRODUCTION.

The concept of an almost continuous function $f:X \rightarrow Y$ has been defined in [1] as one in which for each $x \in X$ and each regular-open V containing $f(x)$ there exists an open set U containing x such that $f(U) \subset V$. Using this concept we make the following two definitions:

DEFINITION 1. The function $f:I \rightarrow Y$ is an almost-continuous path (a.c. path)

from y_0 to y_1 if f is almost continuous, $f(0) = y_0$ and $f(1) = y_1$.

DEFINITION 2. The space Y is a.c. path connected if for each $y_0, y_1 \in Y$, there exists an a.c. path from y_0 to y_1 .

The regular-open sets in Y may be used as a base to form the semi-regular topology T_s on Y from which $f: X \rightarrow (Y, T)$ is almost-continuous if and only if $f: X \rightarrow (Y, T_s)$ is continuous. Thus, Definition 2 may be restated as (Y, T) is a.c. path connected if for each y_0, y_1 in Y , there is a continuous $f: I \rightarrow (Y, T_s)$ such that $f(0) = y_0$ and $f(1) = y_1$. In view of this observation, many of the known results for path connected spaces in the usual sense also apply to a.c. path connected spaces. For example, if $y_0 \in Y$, then Y is a.c. path connected if and only if for each $y \in Y$, there is an a.c. path from y to y_0 . Furthermore, slight variations in known results may sometimes be made to easily produce statements concerning a.c. path connected spaces. An example is that if $f: X \rightarrow Y$ is an almost-continuous surjection and X is path connected, then Y is a.c. path connected.

2. MAIN RESULTS.

THEOREM 1. Every a.c. path connected space Y is connected.

PROOF. Assume $Y = U \cup V$ where U and V are open in (Y, T) and $U \cap V = \emptyset$. Then U and V are regular-open so that $U \cup V$ is a separation of (Y, T_2) . But (Y, T_s) is path connected, hence connected. The contradiction implies Y is connected.

Let R be the reals with the usual topology, Q the set of rational numbers and R_Q the reals with the topology generated by the usual open intervals together with Q as a subbase. Since the semi-regular topology associated with R_Q is the usual topology on R , it follows that $f: X \rightarrow R_Q$ is almost-continuous if and only if $f: X \rightarrow R$ is continuous. Similarly, $f: I \rightarrow R_Q \times R_Q$ is almost-continuous if and only if $f: I \rightarrow R \times R$ is continuous. These observations lead to the following example which shows the converse of Theorem 1 is false.

EXAMPLE 1. Let $Y = \{(x, y) : y = \sin(1/x), 0 < x \leq 1\} \subset R \times R$. Then $\bar{Y} = Y \cup \{(x, y) : -1 \leq y \leq 1\}$ in $R \times R$ as well as $R_Q \times R_Q$. Hence \bar{Y} is connected, but not a.c. path connected.

Example 1 also shows that a space may be a.c. path connected but its closure may not be a.c. path connected.

For a given space Y , define xRY to mean there is an a.c. path from x to y . We see immediately that R is an equivalence relation on Y . The resulting equivalence classes are referred to as a.c. path connected components of Y . Consideration of the semi-regular topology reveals that each a.c. path connected component in Y is open (and therefore closed) if and only if each point of Y has an a.c. path connected neighborhood.

THEOREM 2. A space Y is a.c. path connected if and only if it is connected and each $y \in Y$ has an a.c. path connected neighborhood.

PROOF. If Y is an a.c. path connected space, then Y is connected by Theorem 1 and each $y \in Y$ has an a.c. path connected neighborhood by the remarks preceeding Theorem 2.

Conversely, the hypothesis and the remarks preceeding Theorem 2 show that the only a.c. path connected component of Y is Y itself. Therefore, Y is a.c. path connected.

THEOREM 3. Let Y be a space. If (a) each a.c. path connected component in Y is open or (b) if each point $y \in Y$ has an a.c. path connected neighborhood, then the a.c. path connected components of Y coincide with the usual components of Y .

PROOF. The remarks preceeding Theorem 2 show that conditions (a) and (b) are equivalent. So if we assume that each a.c. path connected component of Y is open, then each point of Y has an a.c. path connected neighborhood. In particular, the a.c. path connected component $[y]$ is an a.c. path connected neighborhood of y . Thus, $[y]$ is connected by Theorem 1. It follows that $[y] \subset C(y)$, where $C(y)$ is the usual component of $y \in Y$. Since $[y]$ is both open and closed in Y , $[y]$ is both open and closed in $C(y)$. But $C(y)$ connected implies $[y] = C(y)$.

REFERENCES

1. SINGAL, M. K. and SINGAL, Asha Rani Almost-Continuous Mappings, The Yokohama Math. J. 16 (1968) pp. 63-73.

Special Issue on Singular Boundary Value Problems for Ordinary Differential Equations

Call for Papers

The purpose of this special issue is to study singular boundary value problems arising in differential equations and dynamical systems. Survey articles dealing with interactions between different fields, applications, and approaches of boundary value problems and singular problems are welcome.

This Special Issue will focus on any type of singularities that appear in the study of boundary value problems. It includes:

- Theory and methods
- Mathematical Models
- Engineering applications
- Biological applications
- Medical Applications
- Finance applications
- Numerical and simulation applications

Before submission authors should carefully read over the journal's Author Guidelines, which are located at <http://www.hindawi.com/journals/bvp/guidelines.html>. Authors should follow the Boundary Value Problems manuscript format described at the journal site <http://www.hindawi.com/journals/bvp/>. Articles published in this Special Issue shall be subject to a reduced Article Processing Charge of €200 per article. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/> according to the following timetable:

Manuscript Due	May 1, 2009
First Round of Reviews	August 1, 2009
Publication Date	November 1, 2009

Lead Guest Editor

Juan J. Nieto, Departamento de Análisis Matemático,
Facultad de Matemáticas, Universidad de Santiago de

Compostela, Santiago de Compostela 15782, Spain;
juanjose.nieto.roig@usc.es

Guest Editor

Donal O'Regan, Department of Mathematics, National
University of Ireland, Galway, Ireland;
donal.oregan@nuigalway.ie